

Performance Evaluation

Lecture 2: Epidemics

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There is more: Independence

□ Theorem 2

– Under the assumptions of Theorem 1, and that the collection of objects at time 0 is exchangeable

$$(X_1^N(0), X_2^N(0), \dots, X_N^N(0)),$$

then for any fixed n and t :

$$\begin{aligned} \lim_{N \rightarrow \infty} \text{Prob}(X_1^N(t)=i_1, X_2^N(t)=i_2, \dots, X_n^N(t)=i_n) &= \\ &= \mu_{i_1}(t) \mu_{i_2}(t) \dots \mu_{i_n}(t) \end{aligned}$$

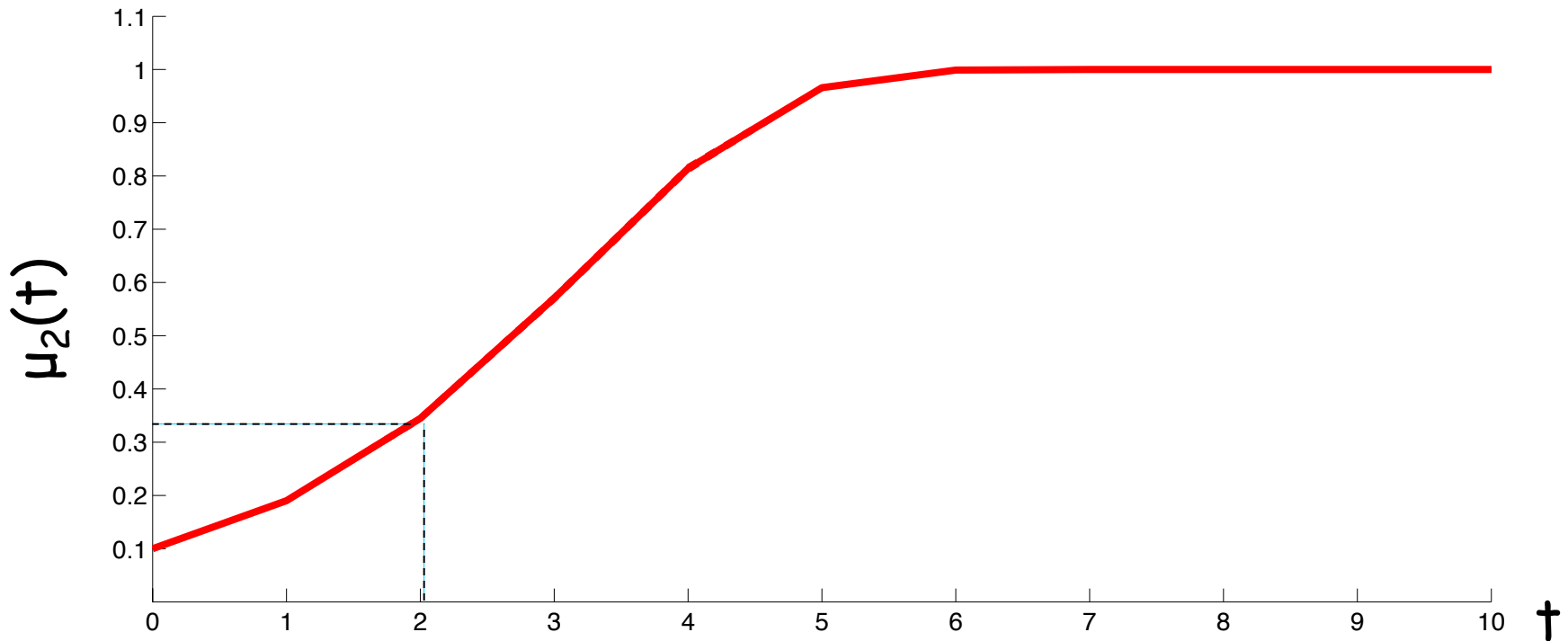
□ MF Independence Property, a.k.a.

Decoupling Property, Propagation of Chaos

Remarks

- $(X_1^N(0), X_2^N(0), \dots, X_N^N(0))$ exchangeable
 - Means that all the states that have the same occupancy measure m_0 have the same probability
- $\underline{X}^{(N)}(k \varepsilon(N)) = X^{(N)}(k)$ for k integer
- $\underline{X}^{(N)}(t)$ is constant on $[k \varepsilon(N), (k+1)\varepsilon(N))$
- $\lim_{N \rightarrow \infty} \text{Prob}(\underline{X}_1^N(t) = i_1, \underline{X}_2^N(t) = i_2, \dots, \underline{X}_n^N(t) = i_n) = \mu_{i_1}(t) \mu_{i_2}(t) \dots \mu_{i_n}(t)$
 - Application
$$\text{Prob}(X_1^N(k) = i_1, X_2^N(k) = i_2, \dots, X_n^N(k) = i_n) \approx \mu_{i_1}(k\varepsilon(N)) \mu_{i_2}(k\varepsilon(N)) \dots \mu_{i_n}(k\varepsilon(N))$$

Probabilistic interpretation of the occupancy measure (SI model with $p=10^{-4}$, $N=100$)

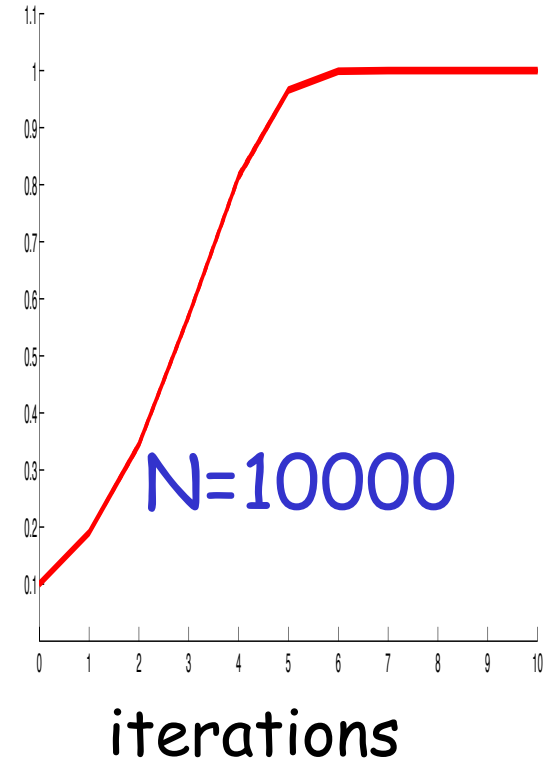
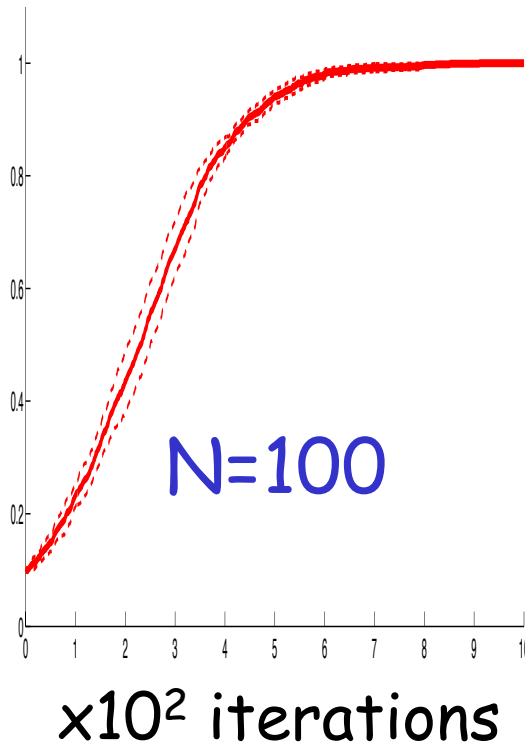
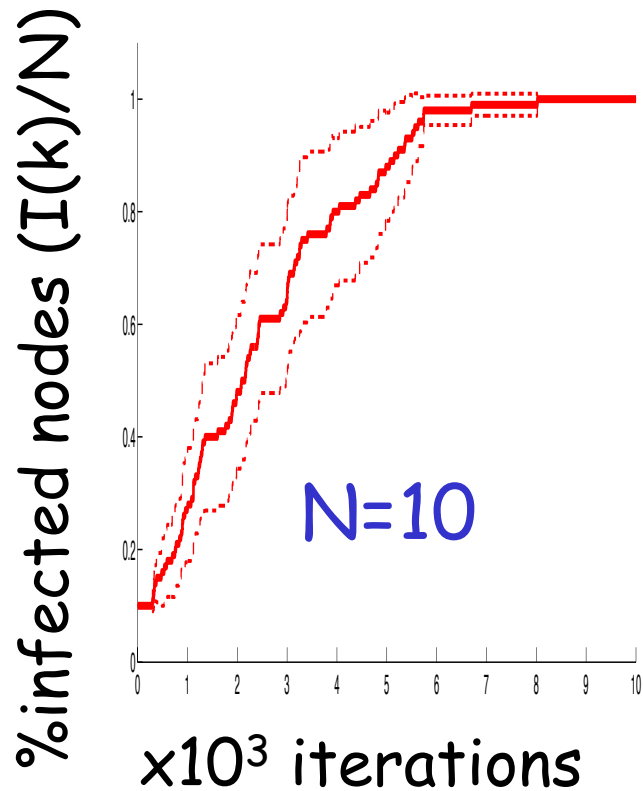


Prob(nodes 1,17,21 and 44 infected at $k=200$)=
 $=\mu_2(k p N)^4 = \mu_2(2)^4 \approx (1/3)^4$

What if 1,17,21 and 44 are surely infected at $k=0$

On approximation quality

$p=10^{-4}$, $I(0)=N/10$, 10 runs

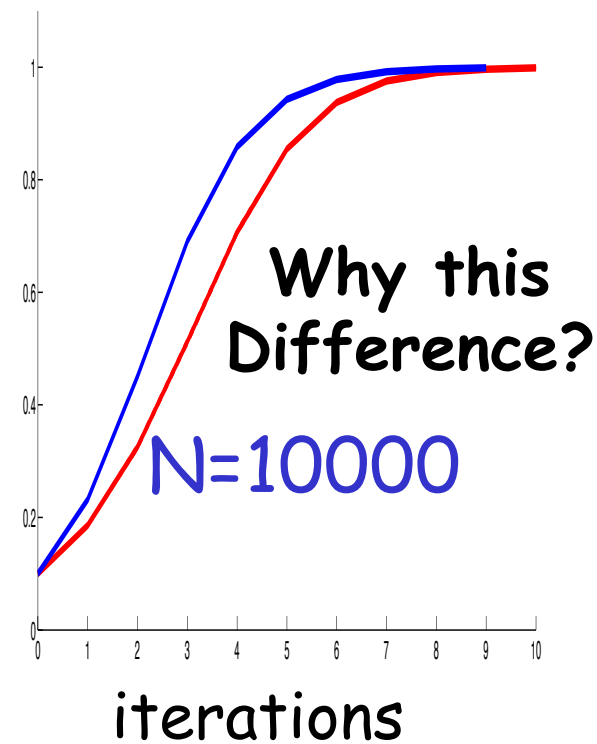
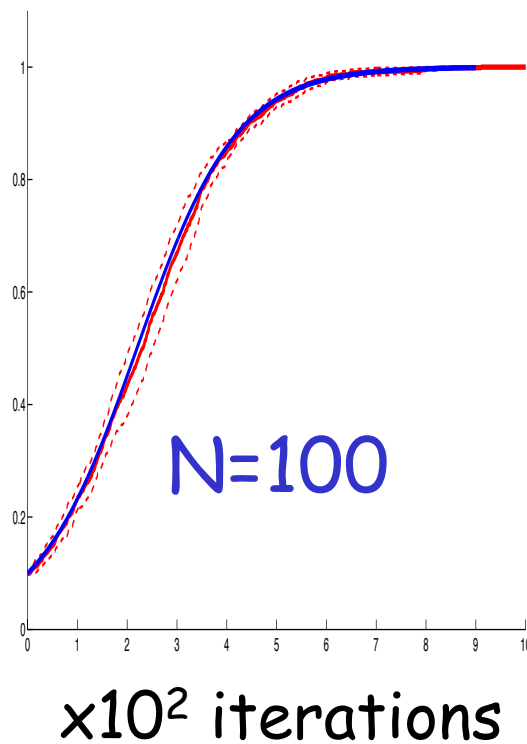
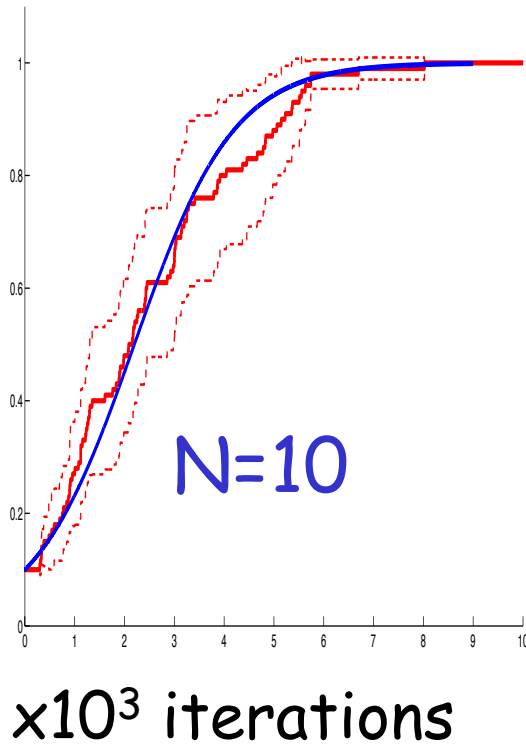


On approximation quality

$p=10^{-4}$, $I(0)=N/10$, 10 runs

Model vs Simulations

%infected nodes ($I(k)/N$)



Why the difference?

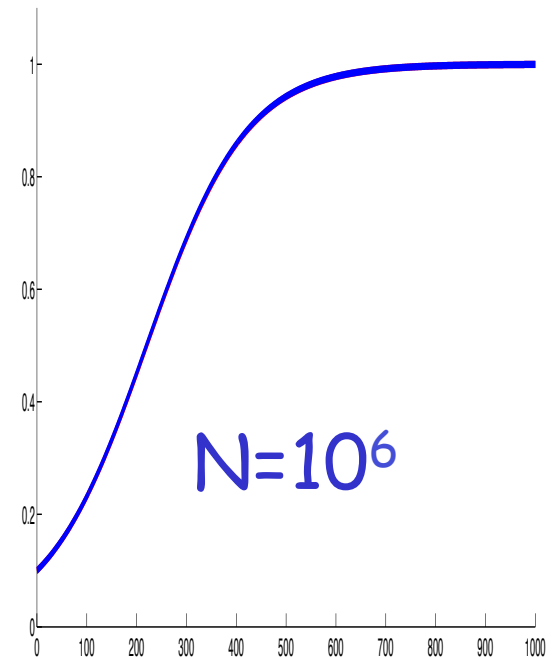
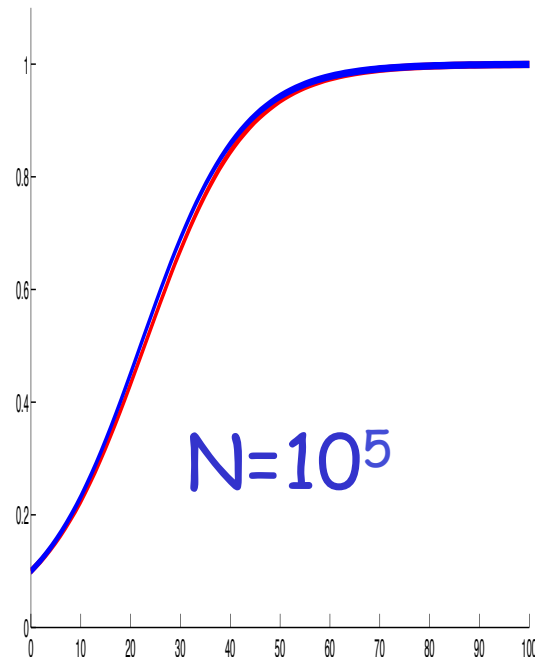
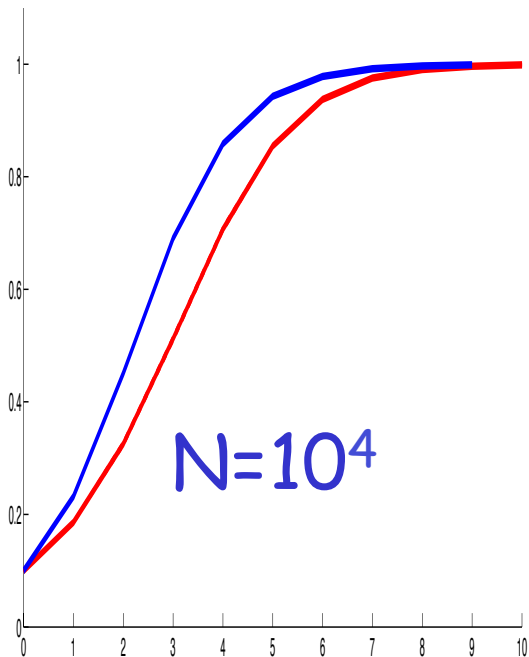
- ❑ N should be large (the larger the better)
- ❑ p should be small
 - $p^{(N)} = p_0 / N^2$
- ❑ For $N = 10^4$ $p = 10^{-4}$ is not small enough!
- ❑ What if we do the correct scaling?

On approximation quality

$p=10^4/N^2$, $I(0)=N/10$, 10 runs

Model vs Simulations

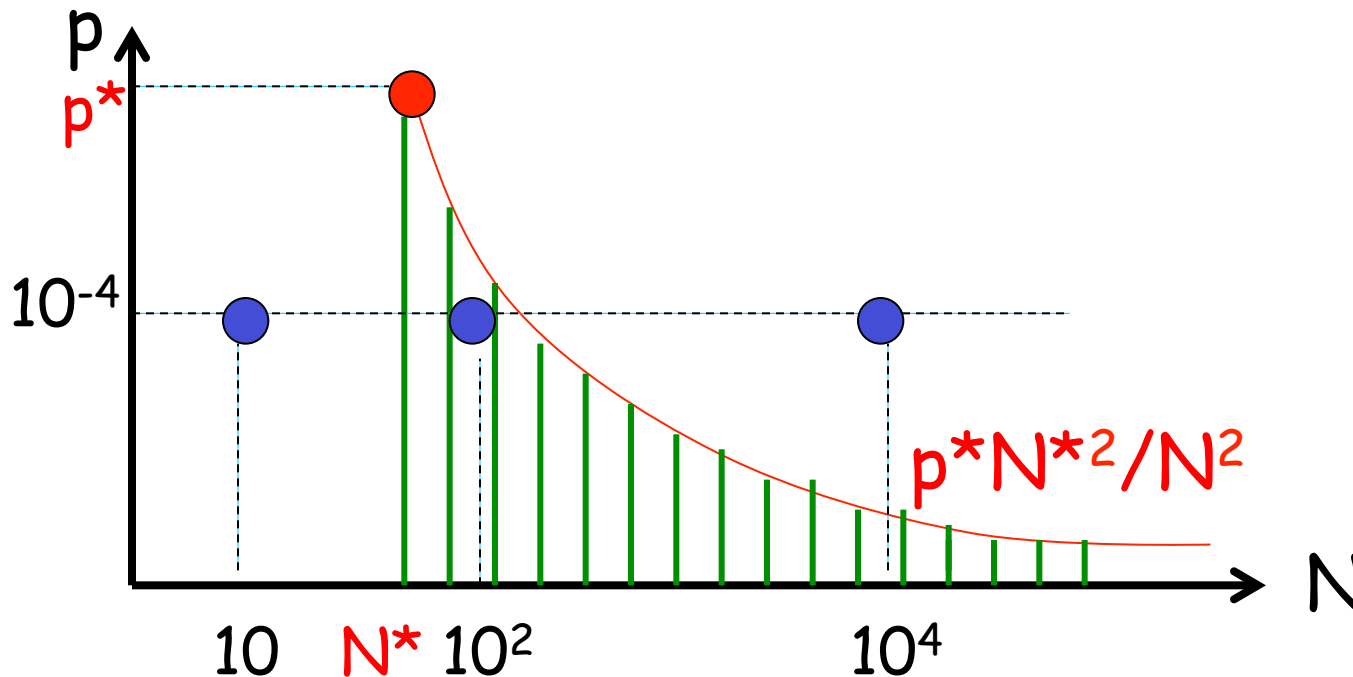
%infected nodes ($I(k)/N$)



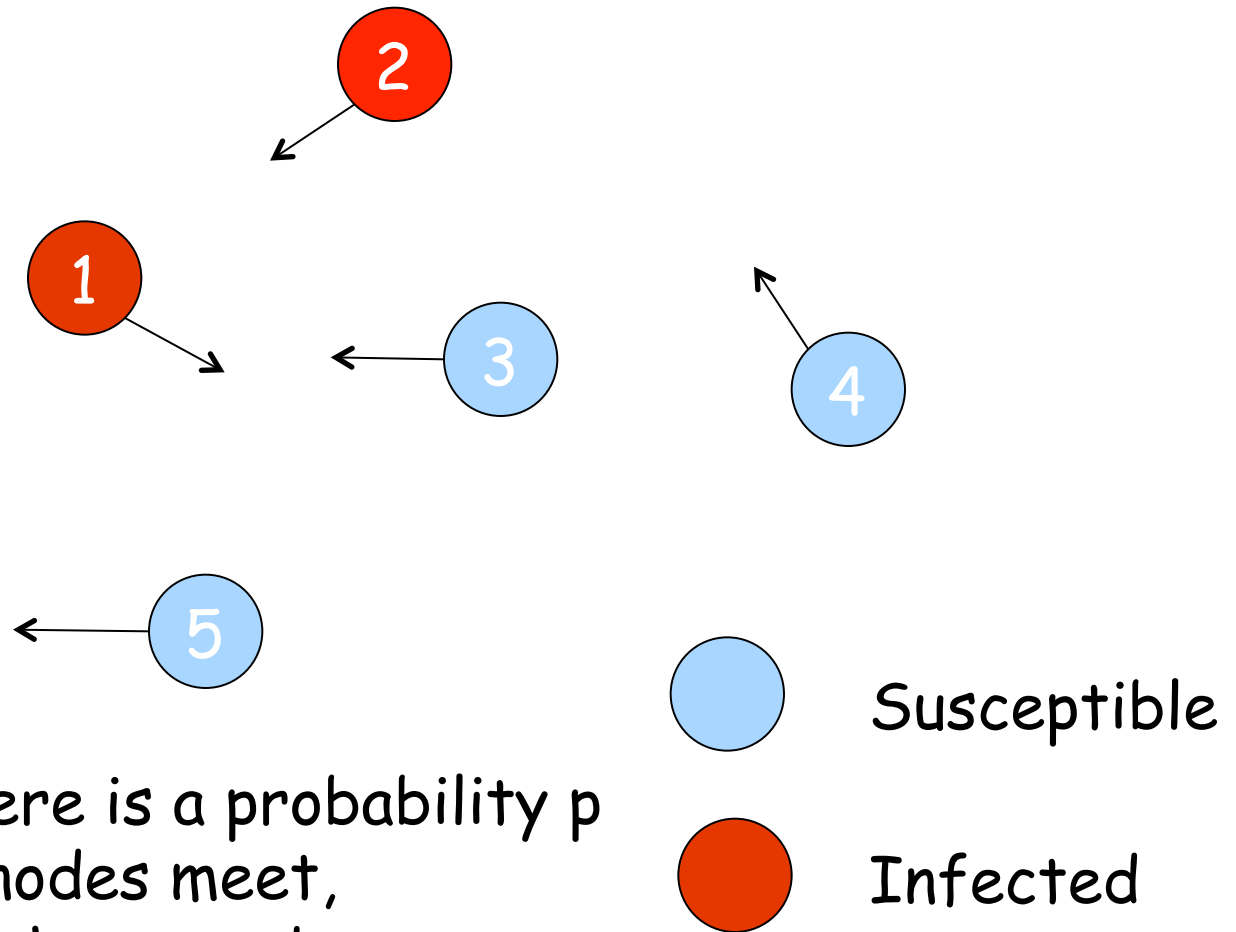
iterations

Lesson

- You need to check (usually by simulation) in which parameter region the fluid model is a good approximation.
 - e.g. $N > N^*$ $p < p^*/N^2$

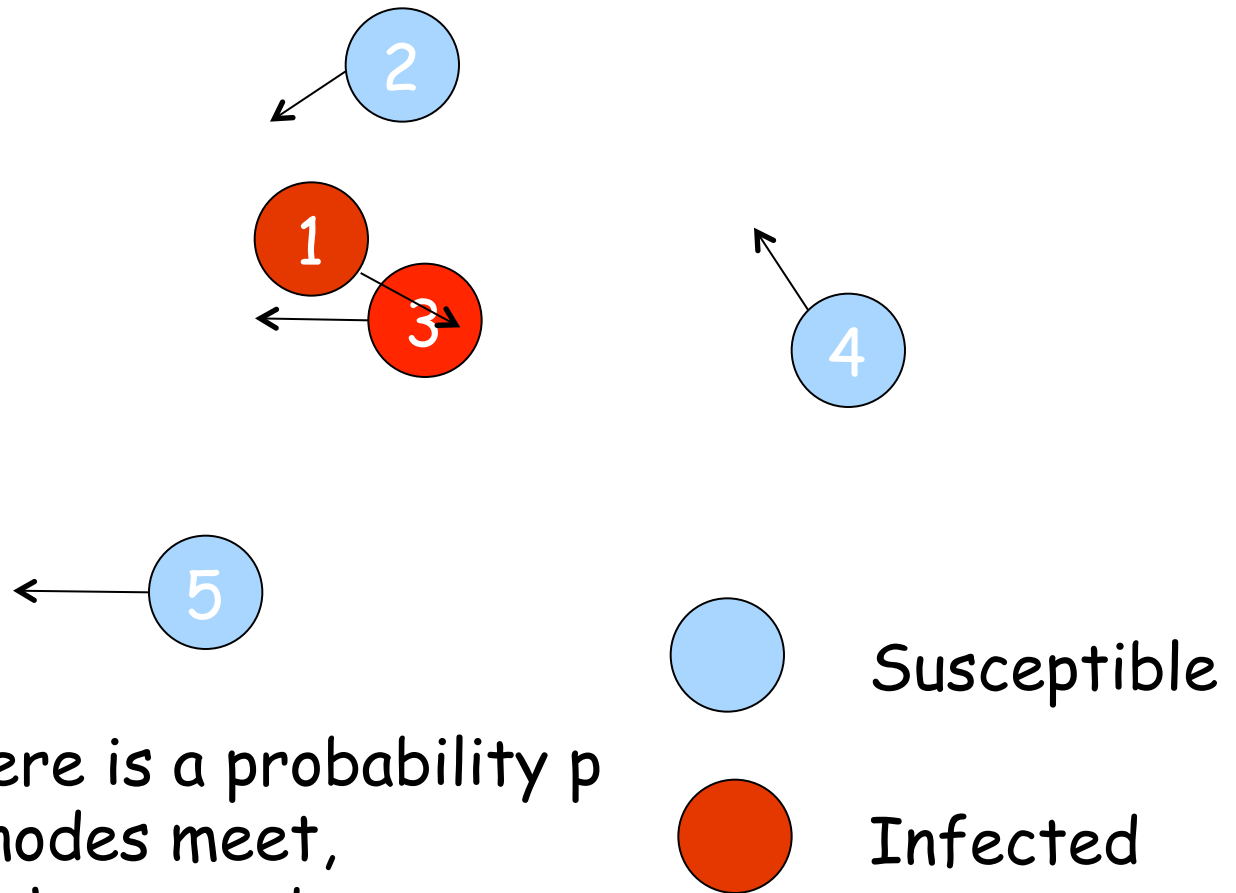


SIS model



At each slot there is a probability p that two given nodes meet, a probability r that a node recovers.

SIS model



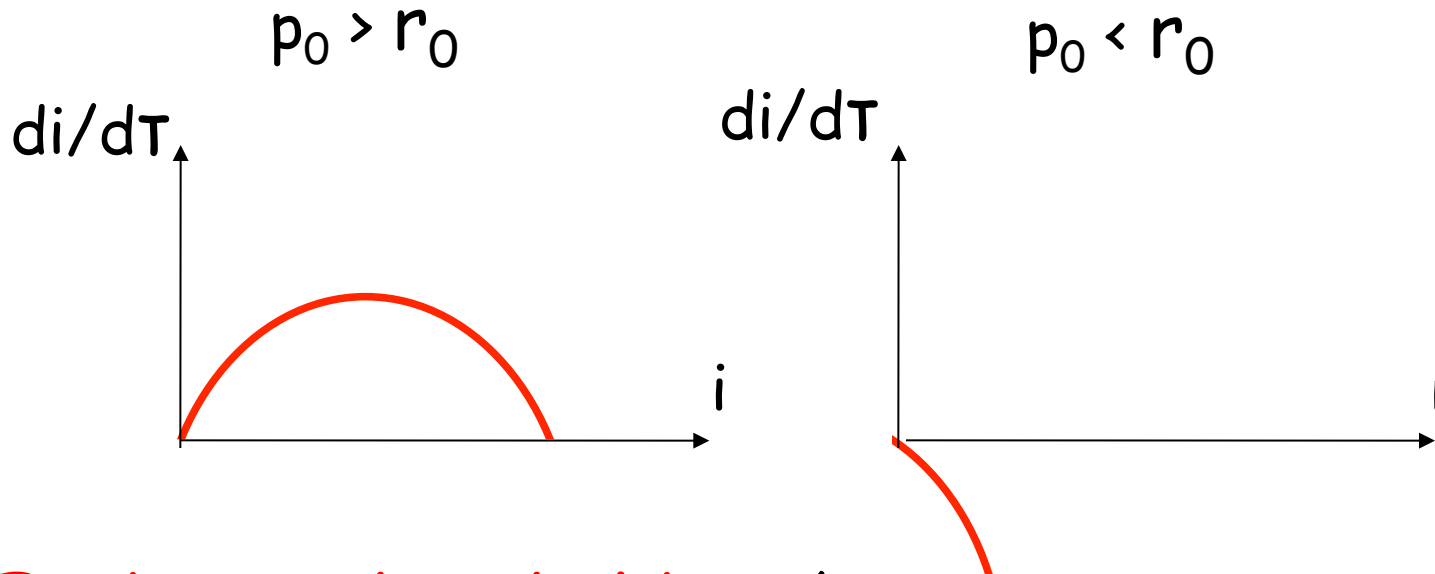
At each slot there is a probability p that two given nodes meet, a probability r that a node recovers.

Let's practise

- Can we propose a Markov Model for SIS?
 - No need to calculate the transition matrix
- If it is possible, derive a Mean Field model for SIS
 - Do we need some scaling?

Study of the SIS model

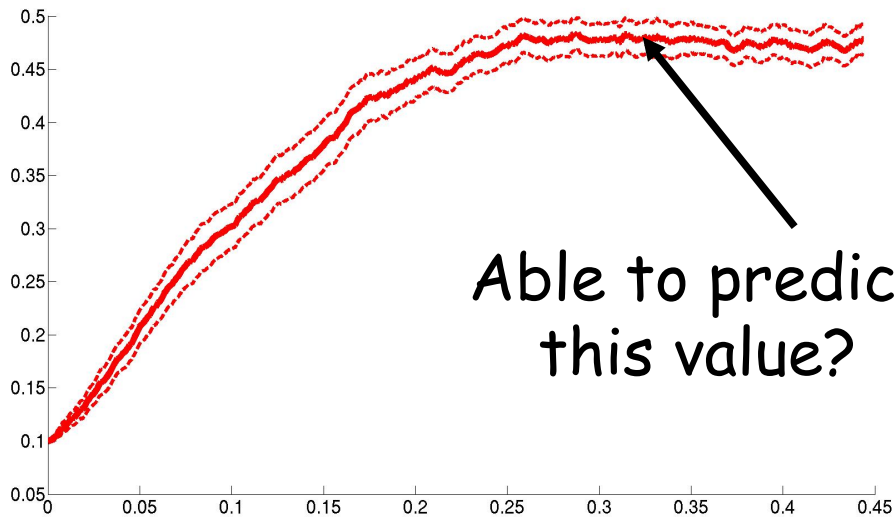
- We need $p^{(N)}=p_0/N^2$ and $r^{(N)}=r_0/N$
- If we choose $\varepsilon(N)=1/N$, we get
 - $di(t)/dt= p_0 i(t)(1-i(t)) - r_0 i(t)$



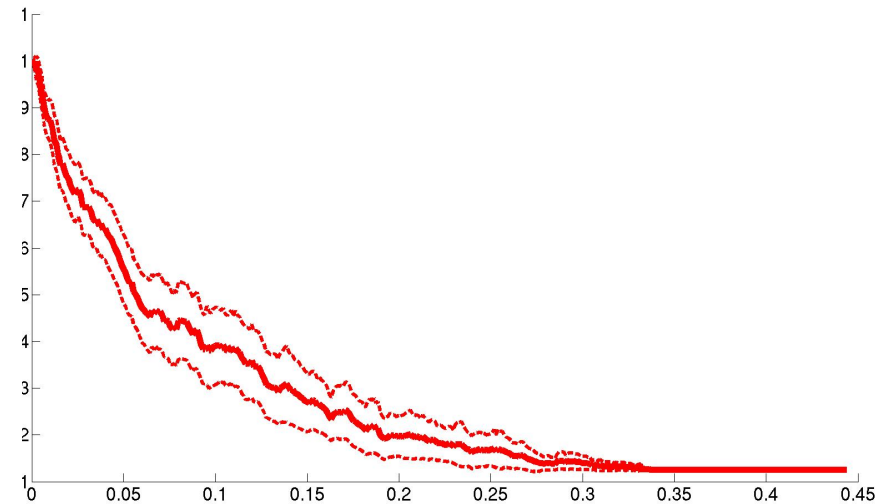
Epidemic Threshold: p_0/r_0

$N=80, p_0=0.1$

$r_0 = 0.05$



$r_0 = 0.125$



Study of the SIS model

- $\mu_2(t) = i(t)$
- $di(t)/dt = p_0 i(t)(1-i(t)) - r_0 i(t)$
- Equilibria, $di(t)/dt = 0$
 - $i(\infty) = 1 - r_0/p_0$ or $i(\infty) = 0$
 - If $i(0) > 0$ and $p_0 > r_0 \Rightarrow \mu_2(\infty) = 1 - r_0/p_0$

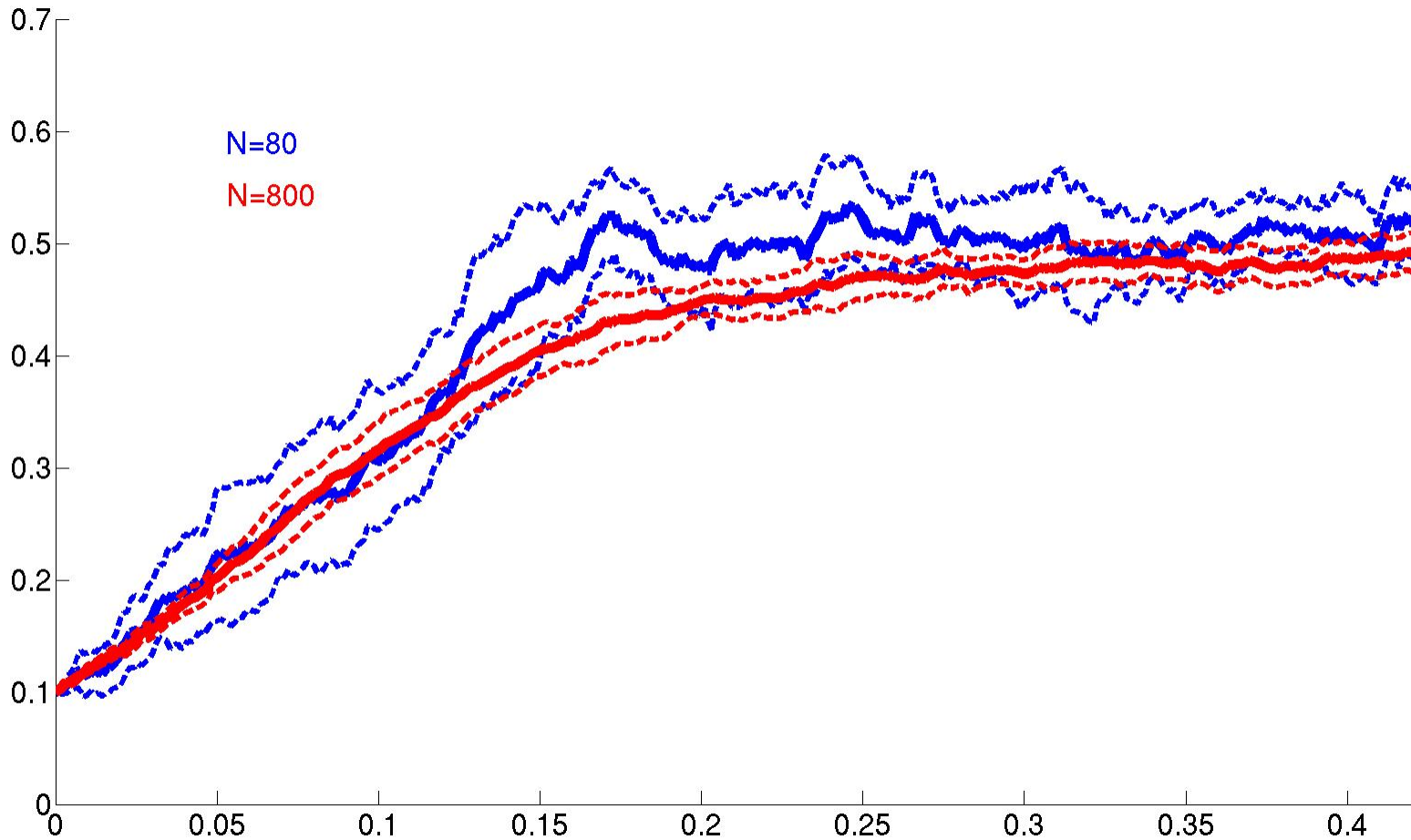
Study of the SIS model

- If $i(0) > 0$ $p_0 > r_0$, $\mu_2(\infty) = 1 - r_0/p_0$
- $\text{Prob}(X_1^{(N)}(k) = 1) \approx i(k\varepsilon(N))$
 - $\text{Prob}(X_1^{(N)}(\infty) = 1) \approx \mu_2(\infty) = i(\infty) = 1 - r_0/p_0$
- What is the steady state distribution of the MC?
 - $(0, 0, 0, \dots, 0)$ is the unique absorbing state and it is reachable from any other state
 - Who is lying here?

Back to the Convergence Result

- Define $\underline{\mathbf{M}}^{(N)}(t)$ with t real, such that
 - $\underline{\mathbf{M}}^{(N)}(k\varepsilon(N)) = \mathbf{M}^{(N)}(k)$ for k integer
 - $\underline{\mathbf{M}}^{(N)}(t)$ is affine on $[k\varepsilon(N), (k+1)\varepsilon(N)]$
- Consider the Differential Equation
 - $d\boldsymbol{\mu}(t)/dt = \mathbf{f}(\boldsymbol{\mu})$, with $\boldsymbol{\mu}(0) = \mathbf{m}_0$
- Theorem
 - For all $T > 0$, if $\mathbf{M}^{(N)}(0) \rightarrow \mathbf{m}_0$ in probability (/mean square) as $N \rightarrow \infty$, then
$$\sup_{0 \leq t \leq T} \|\underline{\mathbf{M}}^{(N)}(t) - \boldsymbol{\mu}(t)\| \rightarrow 0$$
in probability (/mean square)

Some examples



Nothing to do with $t=\infty$?

- Theorem 3: The limits when N diverges of the stationary distributions of $\mathbf{M}^{(N)}$ are included in the Birkhoff center of the ODE
 - Birkhoff center: the closure of all the recurrent points of the ODE (independently from the initial conditions)
 - What is the Birkhoff center of $di(t)/dt = p_0 i(t)(1-i(t)) - r_0 i(t)$?

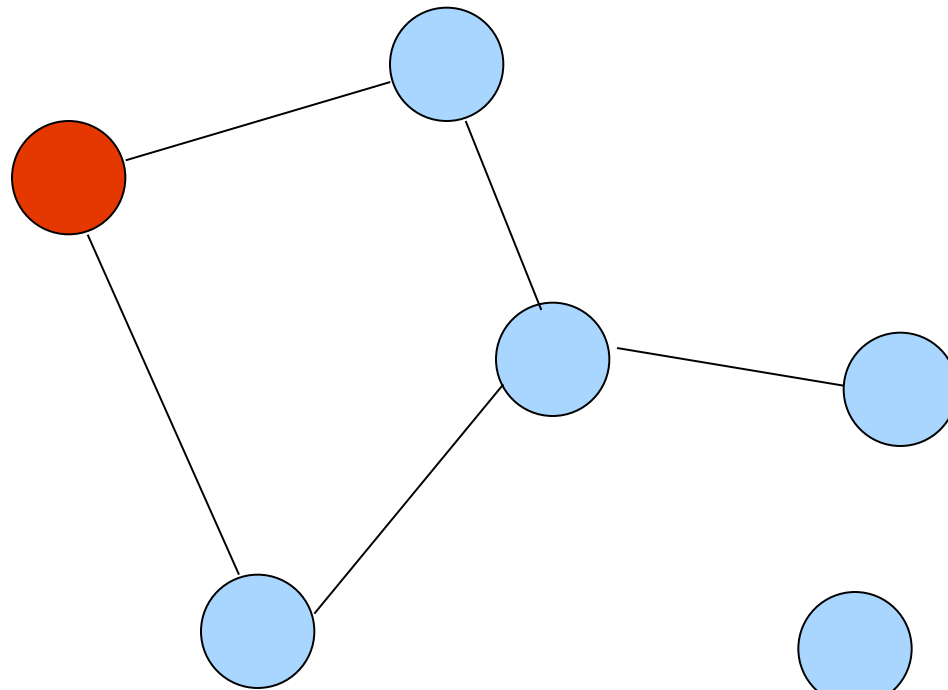
Nothing to do with $t=\infty$?

- Theorem 3: The limits when N diverges of the stationary distributions of $\mathbf{M}^{(N)}$ are included in the Birkhoff center of the ODE
- Corollary: If the ODE has a unique stationary point \mathbf{m}^* , the sequence of stationary distributions $\mathbf{M}^{(N)}$ converges to \mathbf{m}^*

Outline

- Limit of Markovian models
- Mean Field (or Fluid) models
 - exact results
 - Extensions
 - Epidemics on graphs
 - Reference: ch. 9 of Barrat, Barthélemy, Vespignani "Dynamical Processes on Complex Networks", Cambridge press
 - Applications to networks

SI on a graph



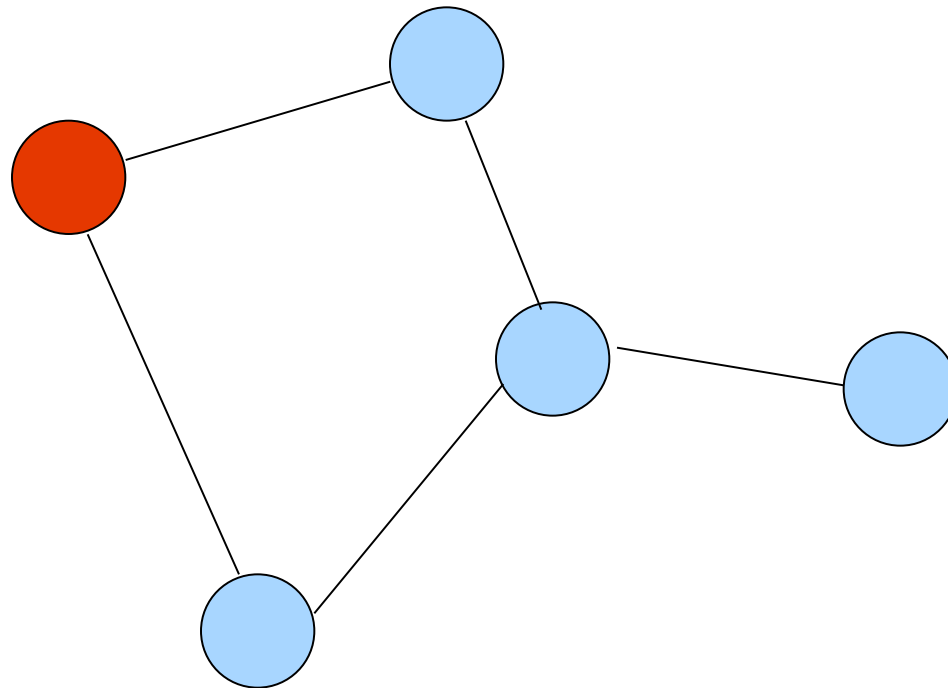
 Susceptible

 Infected

At each time slot, each link outgoing from an infected node spreads the disease with probability p_g

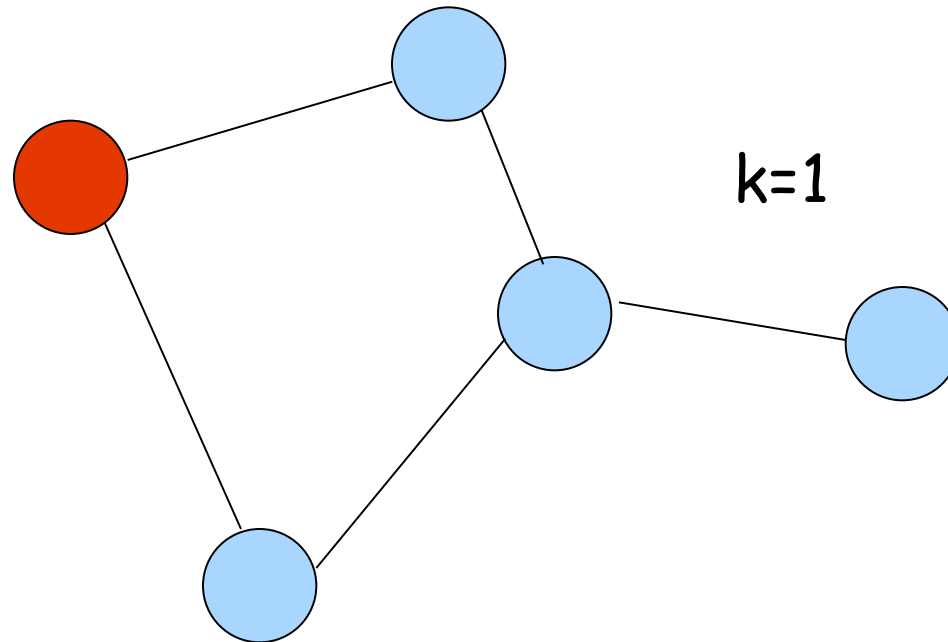
Can we apply Mean Field theory?

- ❑ Formally not, because in a graph the different nodes are not equivalent...
- ❑ ...but we are stubborn



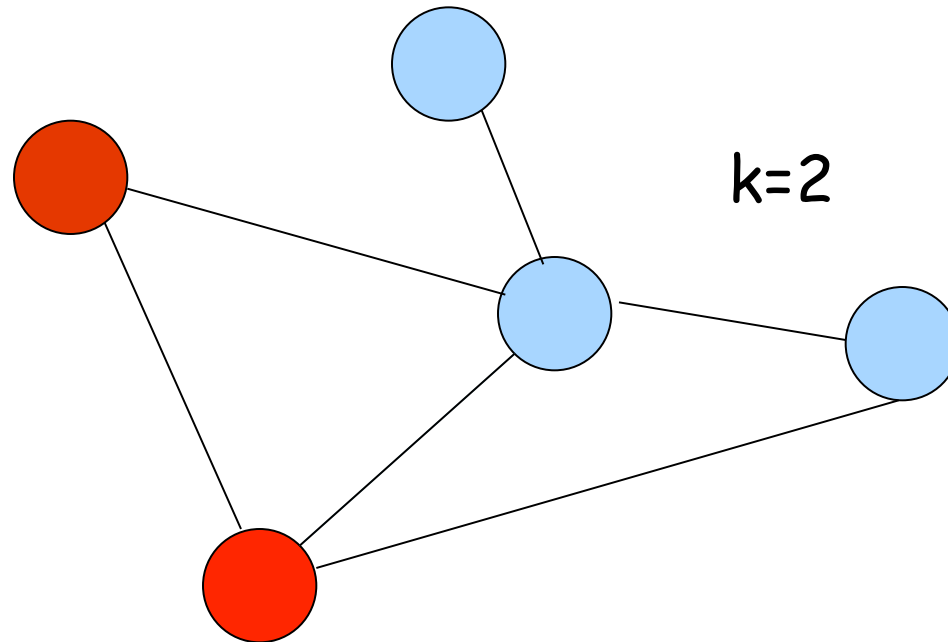
Derive a Mean Field model

- Consider all the nodes equivalent
- e.g. assume that at each slot the graph changes, while keeping the average degree $\langle d \rangle$
 - Starting from an empty network we add a link with probability $\langle d \rangle / (N-1)$



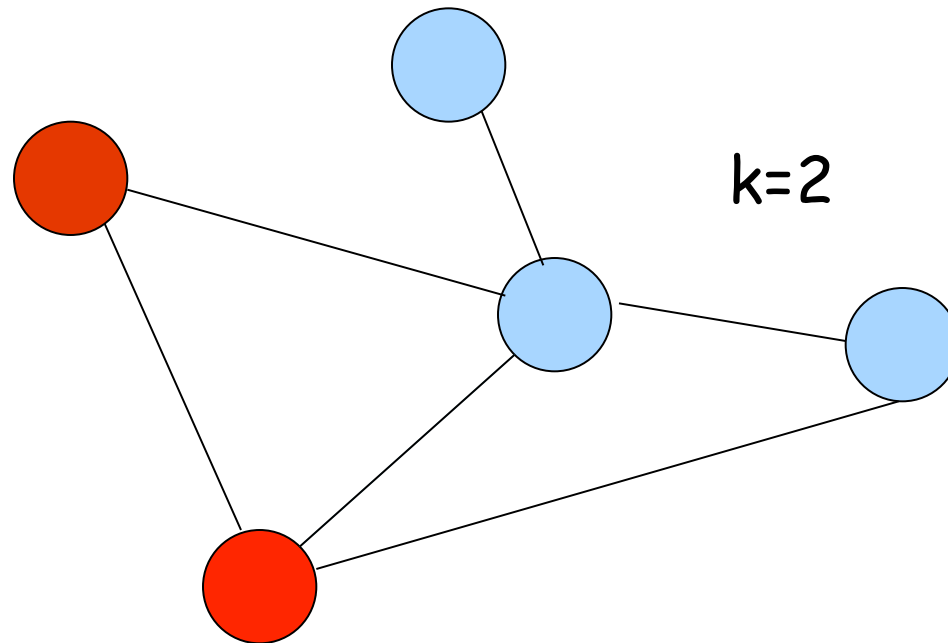
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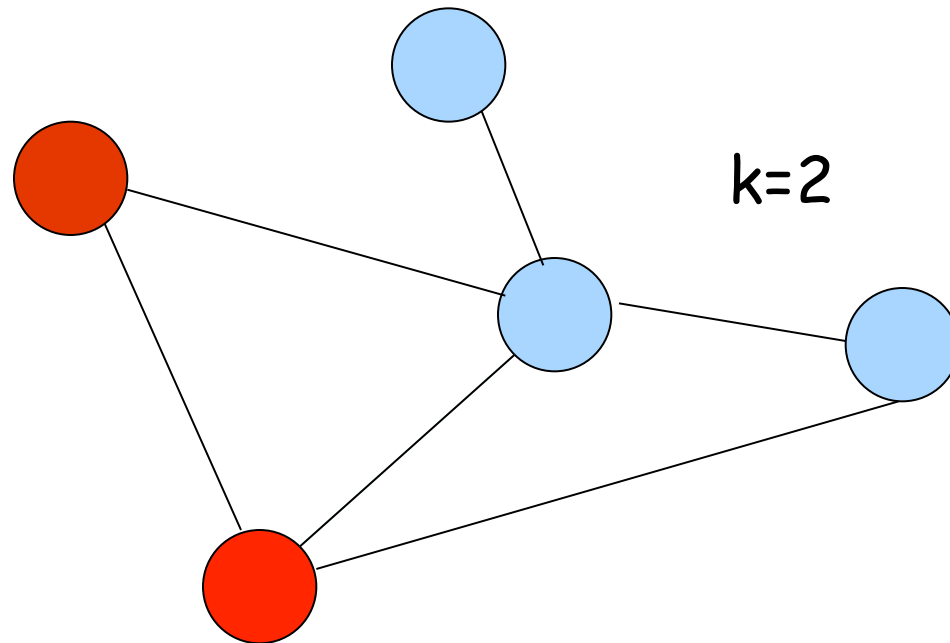
Derive a Mean Field model

- i.e. at every slot we consider a sample of an ER graph with N nodes and probability $\langle d \rangle / (N-1)$
 - Starting from an empty network we add a link with probability $\langle d \rangle / (N-1)$



Derive a Mean Field model

- If $I(k)=I$, the prob. that a given susceptible node is infected is $q_I=1-(1-\langle d \rangle/(N-1) p_g)^I$
- and $(I(k+1)-I(k)|I(k)=I) =_d \text{Bin}(N-I, q_I)$



Derive a Mean Field model

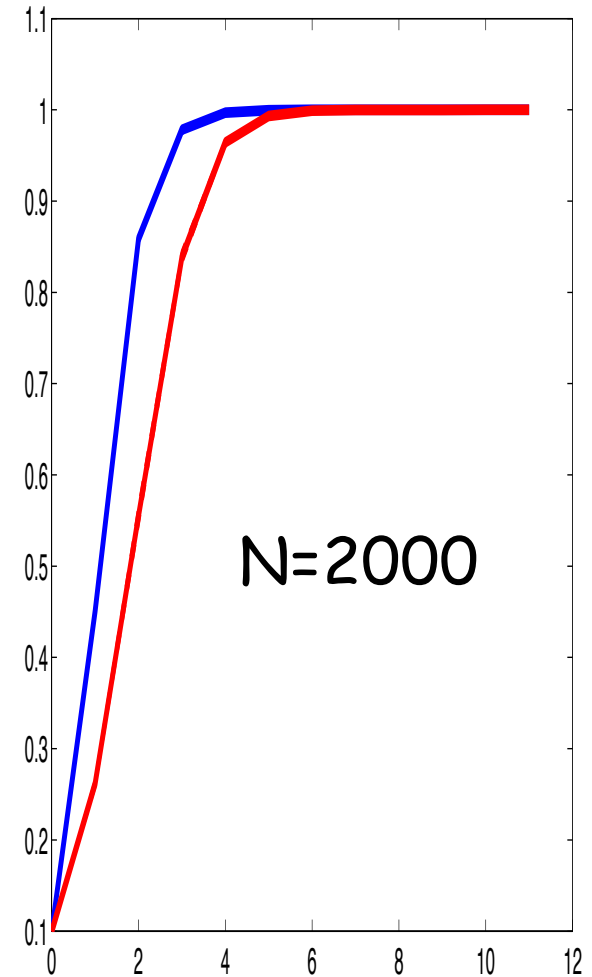
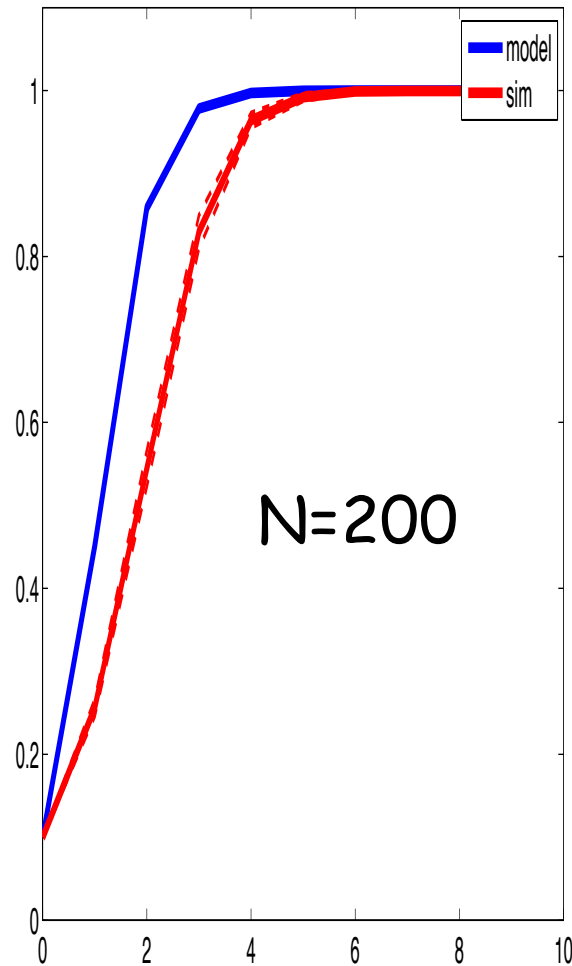
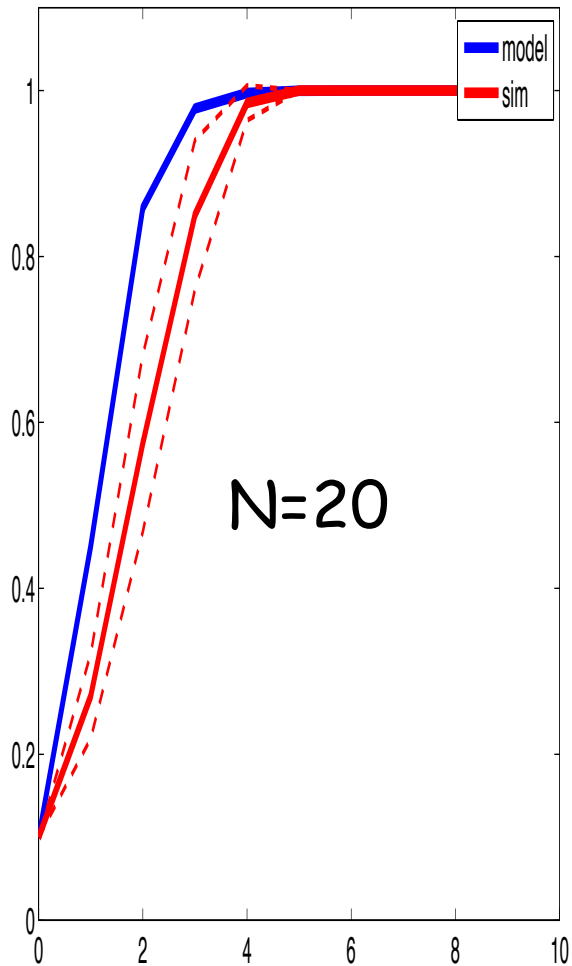
- If $I(k)=I$, the prob. that a given susceptible node is infected is $q_I=1-(1-\langle d \rangle/(N-1) p_g)^I$
- and $(I(k+1)-I(k)|I(k)=I) =_d \text{Bin}(N-I, q_I)$
 - Equivalent to first SI model where $p=\langle d \rangle/(N-1) p_g$
 - We know that we need $p^{(N)}=p_0/N^2$
- $i^{(N)}(k) \approx \mu_2(k \varepsilon(N))=1/((1/i_0-1) \exp(-k p_0/N)+1)=$
 $= 1/((1/i_0-1) \exp(-k \langle d \rangle p_g)+1)$
 - The percentage of infected nodes becomes significant after the **outbreak time** $1/(\langle d \rangle p_g)$
- How good is the approximation practically?
 - It depends on the graph!

Let's try on Erdős-Rényi graph

- Remark: in the calculations above we had a different sample of an ER graph at each slot, in what follows we consider a single sample

ER $\langle d \rangle = 20$, $p_g = 0.1$, 10 runs

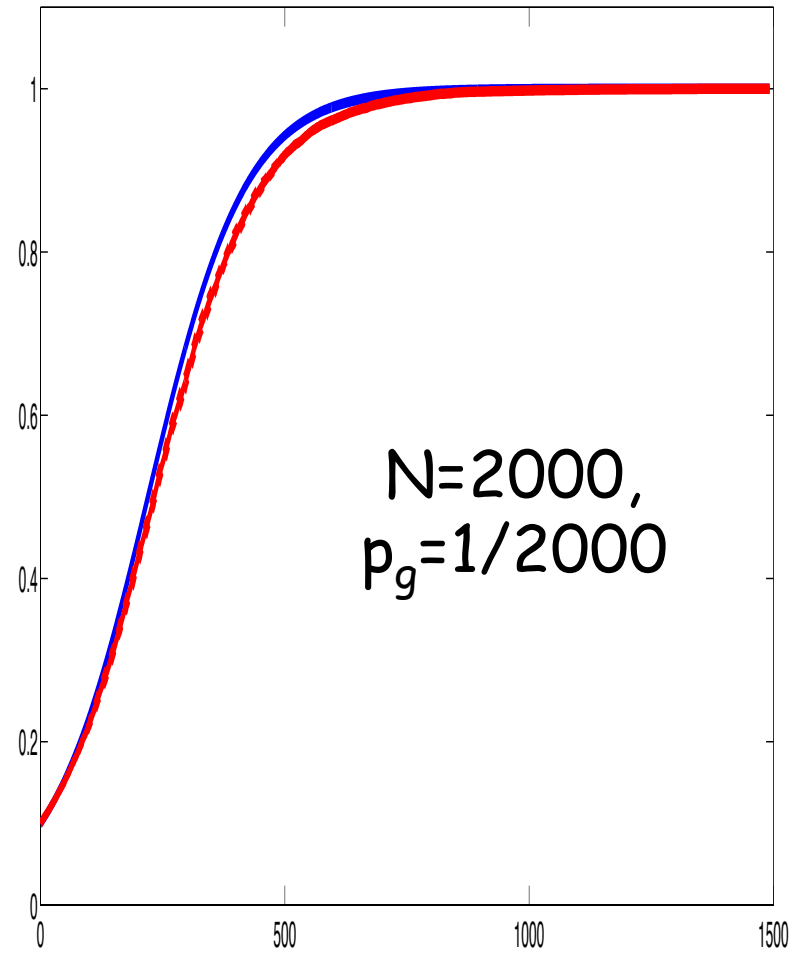
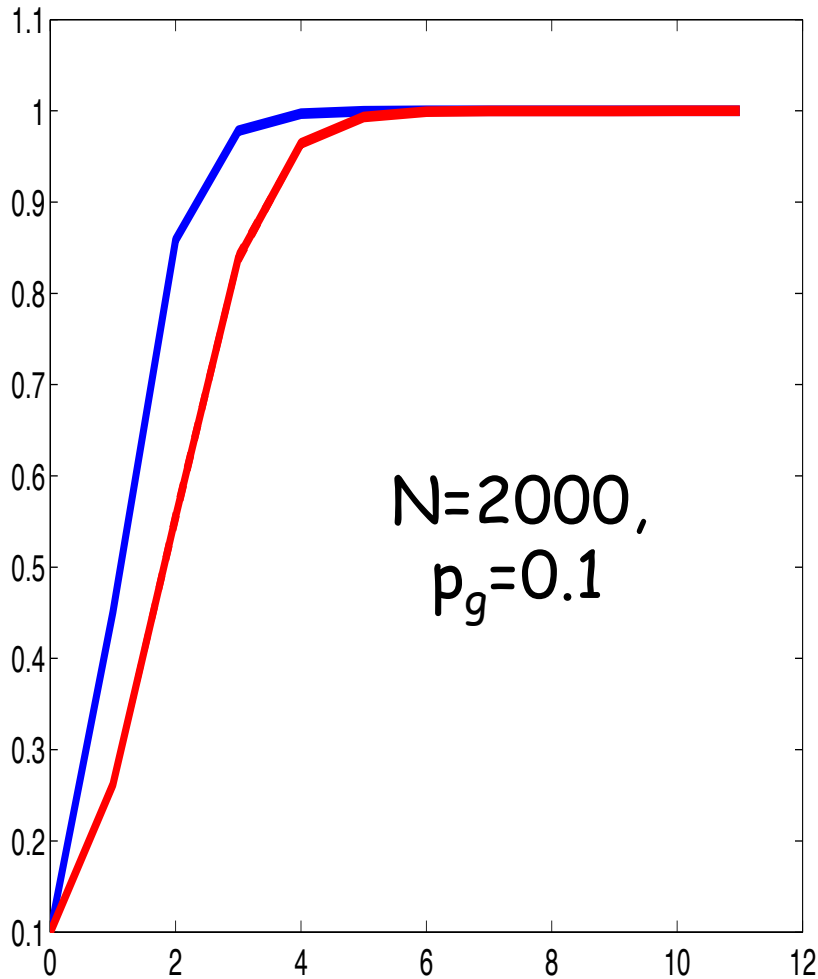
$$i^{(N)}(k) \approx 1 / ((1/i_0 - 1) \exp(-k \langle d \rangle p_g) + 1)$$



Lesson 1

- System dynamics is more *deterministic* the larger the network is
- For given $\langle d \rangle$ and p_g , the MF solution shows the same relative error

ER $\langle d \rangle = 20$, 10 runs

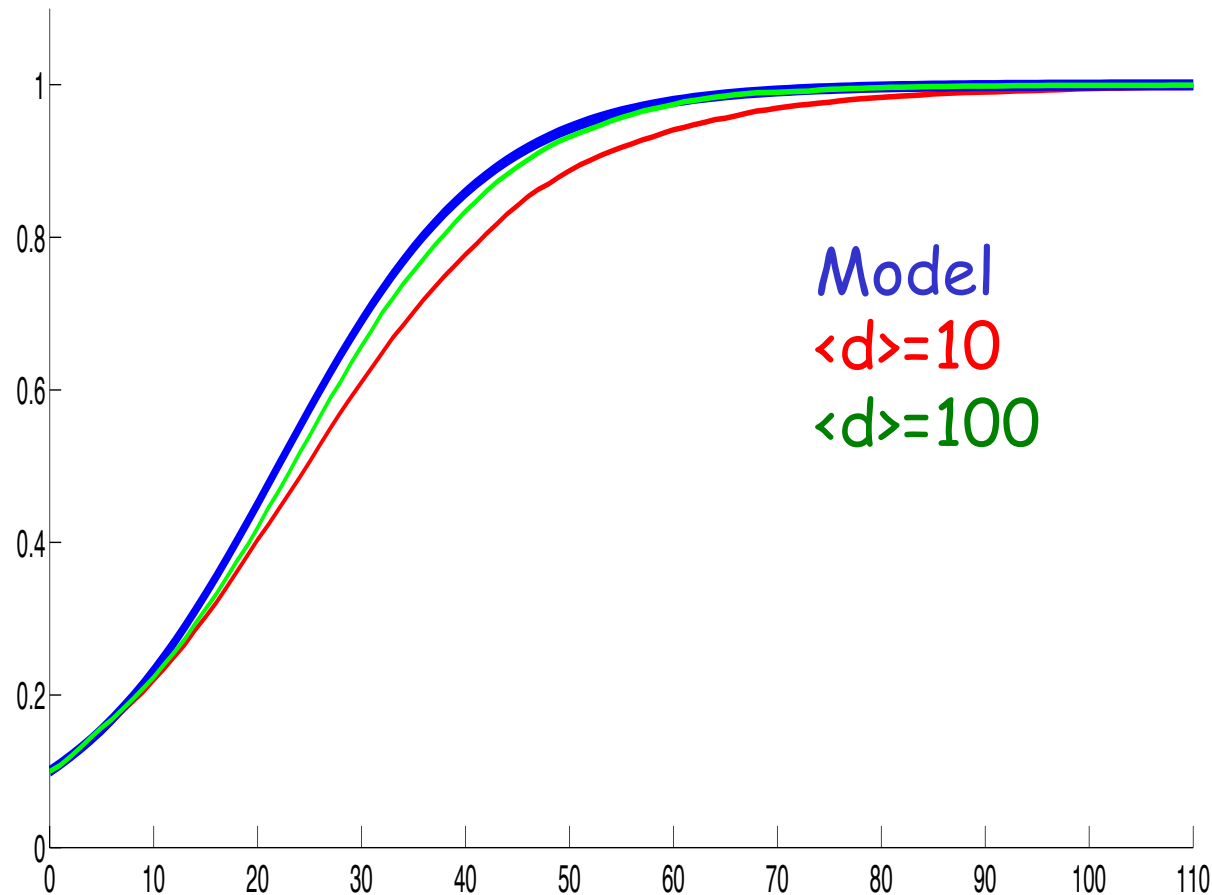


Lesson 2

- For given $\langle d \rangle$, the smaller the infection probability p_g the better the MF approximation
 - Why?

Changing the degree

ER $N=1000$, $\langle d \rangle p_g = 0.1$, 10 runs

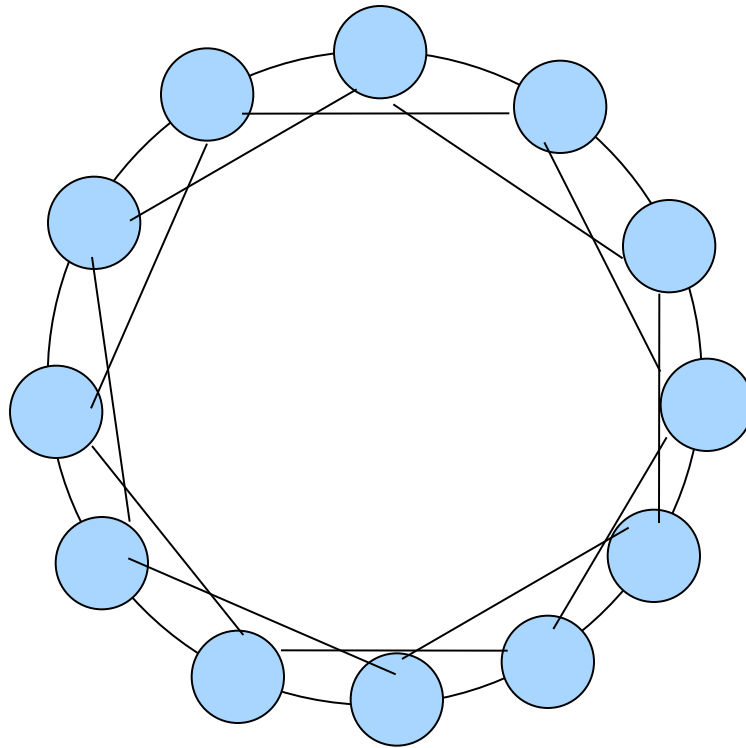


$$i^{(N)}(k) \approx 1 / ((1/i_0 - 1) \exp(-k \langle d \rangle p_g) + 1)$$

Lesson 3

- Given $\langle d \rangle p_g$, the more the graph is connected, the better the MF approximation
 - Why?

A different graph Ring(N,k)



Lesson 4

- The smaller the clustering coefficient, the better the MF approximation
 - Why?

Heterogeneous Networks

- Denote $P(d)$ the probability that a node has degree d
- If the degree does not change much, we can replace d with $\langle d \rangle$
 - what we have done for ER graphs (N,p)
 - Binomial with parameters $(N-1,p)$
- How should we proceed (more) correctly?
 - Split the nodes in degree classes
 - Write an equation for each class
- Remark: following derivation will not be as rigorous as previous ones

Heterogeneous Networks

- N_d number of nodes with degree d ($=N \cdot P(d)$)
- I_d : number of infected nodes with degree d
- Given node i with degree d and a link e_{ij} , what is the prob. that j has degree d' ?
 - $P(d')$? NO
- and if degrees are uncorrelated? i.e. Prob(neighbour has degree d' | node has a degree d) independent from d ,
 - $P(d')$? NO
 - Is equal to $d' / \langle d \rangle P(d')$

Heterogeneous Networks

- Given n (susc.) with degree d and a link e_{nj}
- Prob. that j has degree d' is
 - $d' / \langle d \rangle P(d')$
- Prob. that j has degree d' and is infected
 - $d' / \langle d \rangle P(d') I_{d'} / N_{d'}$
 - more correct $(d'-1) / \langle d \rangle P(d') I_{d'} / N_{d'}$
- Prob. that n is infected through link e_{nj} is
 - $p = p_g \sum_{d'} (d'-1) / \langle d \rangle P(d') I_{d'} / N_{d'}$
- Prob. that n is infected through one link
 - $1 - (1-p)^d$

Heterogeneous Networks

$$\square E[(I_d(k+1) - I_d(k) | \mathbf{I}(k) = \mathbf{I})] = (N_d - I_d)(1 - (1-p)^d)$$

$$- p = p_g \sum_{d'} (d'-1) / \langle d \rangle P(d') I_{d'} / N_{d'}$$

$$\square f_d^{(N)}(i) = (1 - i_d)(1 - (1-p)^d)$$

$$- i_d = I_d / N_d$$

$$- \text{if we choose } p_g = p_{g0} / N$$

$$- f_d(i) = p_{g0} (1 - i_d) d \underbrace{\sum_{d'} (d'-1) / \langle d \rangle P(d') i_{d'}}_{\Theta}$$

$$\square di_d(t)/dt = f_d(i(t)) = p_{g0} (1 - i_d(t)) d \Theta(t)$$

Heterogeneous Networks

- $di_d(t)/dt = f_d(i(t)) = p_{g0} (1 - i_d(t)) d \Theta(t)$,
 - for $d=1,2,\dots$
 - $\Theta(t) = \sum_{d'} (d'-1) / \langle d \rangle P(d') i_{d'}(t)$
 - $i_d(0) = i_{d0}$, for $d=1,2,\dots$
- If $i_d(0) \ll 1$, for *small* t
 - $di_d(t)/dt \approx p_{g0} d \Theta(t)$
 - $d\Theta(t)/dt = \sum_{d'} (d'-1) / \langle d \rangle P(d') di_{d'}(t)/dt$
 $\approx p_{g0} \sum_{d'} (d'-1) / \langle d \rangle P(d') d' \Theta(t) =$
 $= p_{g0} (\langle d^2 \rangle - \langle d \rangle) / \langle d \rangle \Theta(t)$

Heterogeneous Networks

- $d\Theta(t)/dt \approx p_{g_0}(\langle d^2 \rangle - \langle d \rangle) / \langle d \rangle \Theta(t)$
 - Outbreak time: $\langle d \rangle / ((\langle d^2 \rangle - \langle d \rangle) p_{g_0})$
 - For ER $\langle d^2 \rangle = \langle d \rangle (\langle d \rangle + 1)$, we find the previous result, $1 / (\langle d \rangle p_{g_0})$
 - What about for Power-law graphs, $P(d) \sim d^{-\gamma}$?
- For the SIS model:
 - $d\Theta(t)/d \approx p_{g_0}(\langle d^2 \rangle - \langle d \rangle) / \langle d \rangle \Theta(t) - r_0 \Theta(t)$
 - Epidemic threshold: $p_{g_0} (\langle d^2 \rangle - \langle d \rangle) / (\langle d \rangle r_0)$

Outline

- Limit of Markovian models
- Mean Field (or Fluid) models
 - exact results
 - extensions
 - **Applications**
 - Bianchi's model
 - Epidemic routing

Decoupling assumption in Bianchi's model

- Assuming that retransmission processes at different nodes are independent
 - Not true: if node i has a large backoff window, it is likely that also other nodes have large backoff windows
- We will provide hints about why it is possible to derive a Mean Field model...
- then the decoupling assumption is guaranteed asymptotically

References

- Benaim, Le Boudec, "A Class of Mean Field Interaction Models for Computer and Communication Systems", LCA-Report-2008-010
- Sharma, Ganesh, Key, "Performance Analysis of Contention Based Medium Access Control Protocols", IEEE Trans. Info. Theory, 2009
- Bordenave, McDonarl, Proutière, "Performance of random medium access control, an asymptotic approach", Proc. ACM Sigmetrics 2008, 1-12, 2008

Bianchi's model

- N nodes,
- K possible stages for each node, in stage i ($i=1, \dots, V$) the node transmits with probability $q^{(N)}_i$ (e.g. $q^{(N)}_i = 1/W^{(N)}_i$)
- If a node in stage i experiences a collision, it moves to stage $i+1$
- If a node transmits successfully, it moves to stage 1

Mean Field model

- We need to scale the transmission probability: $q^{(N)}_i = q_i/N$
- $f^{(N)}(\mathbf{m}) = E[\mathbf{M}^{(N)}(k+1) - \mathbf{M}^{(N)}(k) | \mathbf{M}^{(N)}(k) = \mathbf{m}]$
- $f_1^{(N)}(\mathbf{m}) = E[M_1^{(N)}(k+1) - M_1^{(N)}(k) | M_1^{(N)}(k) = \mathbf{m}]$
- $P_{\text{idle}} = \prod_{i=1, \dots, V} (1 - q_i^{(N)})^{m_i N}$
- The number of nodes in stage 1
 - increases by one if there is one successful transmission by a node in stage $i > 1$
 - Decreases if a node in stage 1 experiences a collision

Mean field model

□ $P_{\text{idle}} = \prod_{i=1, \dots, V} (1 - q_i^{(N)})^{m_i N} \rightarrow \exp(-\sum_i q_i m_i)$

- Define $\tau(m) = \sum_i q_i m_i$

□ The number of nodes in stage 1

- increases by one if there is one successful transmission by a node in stage $i > 1$
 - with prob. $\sum_{i > 1} m_i N q_i^{(N)} P_{\text{idle}} / (1 - q_i^{(N)})$
- Decreases if a node in stage 1 experiences a collision
 - with prob. $m_1 N q_1^{(N)} (1 - P_{\text{idle}}) / (1 - q_1^{(N)})$

□ $f_1^{(N)}(\mathbf{m}) = E[M_1^{(N)}(k+1) - M_1^{(N)}(k) | M_1^{(N)}(k) = \mathbf{m}] =$

$= \sum_{i > 1} m_i q_i^{(N)} P_{\text{idle}} / (1 - q_i^{(N)})$

$- m_1 q_1^{(N)} (1 - P_{\text{idle}}) / (1 - q_1^{(N)})$

Mean field model

- $P_{\text{idle}} = \prod_{i=1, \dots, V} (1 - q_i^{(N)})^{m_i N} \rightarrow \exp(-\sum_i q_i m_i)$
 - Define $\tau(\mathbf{m}) = \sum_i q_i m_i$
 - $f_1^{(N)}(\mathbf{m}) = \sum_{i>1} m_i q_i^{(N)} P_{\text{idle}} / (1 - q_i^{(N)}) - m_1 q_1^{(N)} (1 - P_{\text{idle}} / (1 - q_1^{(N)}))$
 - $f_1^{(N)}(\mathbf{m}) \sim 1/N \left(\sum_{i>1} m_i q_i e^{-\tau(\mathbf{m})} - m_1 q_1 (1 - e^{-\tau(\mathbf{m})}) \right)$
 - $f_1^{(N)}(\mathbf{m})$ vanishes and $\varepsilon(N) = 1/N$, continuously differentiable in \mathbf{m} and in $1/N$
 - This holds also for the other components
 - Number of transitions bounded
- \Rightarrow We can apply the Theorem