Each answer has to be justified. The points marked for each exercise give an indication of the relative importance.

Ex. 1 — (3 points) Consider a graph, where each link $l \in E$ is affected by a delay $D_l(y_l)$ depending on the amount of traffic on that link, y_l. The delay function is assumed to be convex, increasing and differentiable. Each source s must transmit a total flow with rate f_s across a set of routes $R(s)$. Let R denote the set of all routes, i.e. $R = \cup_s R(s)$.

$$
\begin{aligned}
\underset{\mathbf{x}\in\mathbb{R}^{|R|},\mathbf{y}\in\mathbb{R}^{|E|}}{\text{minimize}} \quad & \sum_{l\in E} \int_0^{y_l} D_l(t)\mathrm{d}t \\
\text{subject to} \quad & f_s = \sum_{r\in R(s)} x_r \;\; \forall s \in S \\
& y_l = \sum_{r|l\in r} x_r \;\; \forall l \in E \\
& x_r \geq 0 \;\; \forall r \in R
\end{aligned}
$$

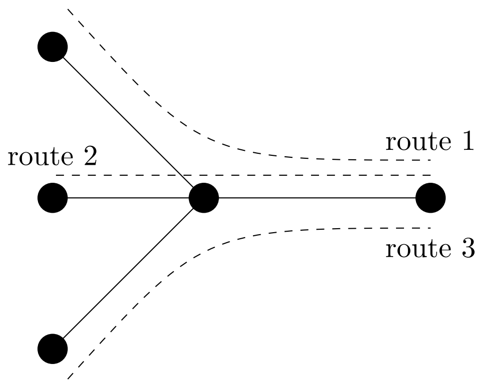
where the functions $D_l(t)$ are increasing and differentiable.

1. Explain the constraints.
2. Use the method of Lagrangian multipliers to characterize the global minimum.

Ex. 2 — (2 points) User r can transmit at any rate x_r on route r, obtaining a utility $U_r(x_r)$, where the function $U_r(.)$ is increasing, concave and differentiable over $\mathbb{R}$. Consider the following network utility maximization problem.

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^{|R|}}{\text{maximize}} \quad & \sum_{r\in R} U_r(x_r) \\ \text{subject to} \quad & \sum_{r|l\in r} x_r \leq C_l \ \ \forall l \in E \\ & x_r \geq 0 \ \ \forall r \in R \end{aligned}$$

1. Consider the particular case of the network represented below, where every edge has capacity $C = 1$. The utilities on the first two routes are equal: $U_1(x) = U_2(x) = 1 - e^{-x}$. Determine the optimal allocations in the two following cases:
   a) $U_3(x) = 1 - e^{-x}$
   b) $U_3(x) = 1 - e^{-2x}$



Ex. 3 — (4 points) Under the same notation of the first exercise, consider the following problem:

$$
\begin{aligned}
\underset{\mathbf{x}\in\mathbb{R}^{|R|},\mathbf{y}\in\mathbb{R}^{|E|}}{\text{minimize}} \quad & \sum_{l\in E} y_l D_l(y_l) \\
\text{subject to} \quad & f_s = \sum_{r\in R(s)} x_r \ \ \forall s \in S \\
& y_l = \sum_{r|l\in r} x_r \ \ \forall l \in E \\
& x_r \geq 0 \ \ \forall r \in R
\end{aligned}
$$

Each source s updates the transmission rates on its available routes $R(s)$ according to the following differential equation:

$$
\frac{dx_r}{dt} = \left[\frac{1}{|R(s)|}\sum_{u\in R(s)}\sum_{l\in u} D_l(y_l) + y_l D_l'(y_l)\right] - \left[\sum_{l\in r} D_l(y_l) + y_l D_l'(y_l)\right] \quad \forall r \in R(s),
$$

where $|R(s)|$ is the size of the set $R(s)$.

1. Shows that the total rate transmission rate from source s does not change over time. If source s starts transmitting a total rate equal to f_s, what will happen later?
2. What represents the first term in the right hand side of the differential equation? What the second term? Explain qualitatively the transmission rate dynamics.
3. Show that the transmission rates will converge to an optimal solution of the minimization problem. Hints: define a suited Lyapunov function, determine its sign by using Jensen's inequality.