Distributed Optimization and Games

Introduction to Game Theory

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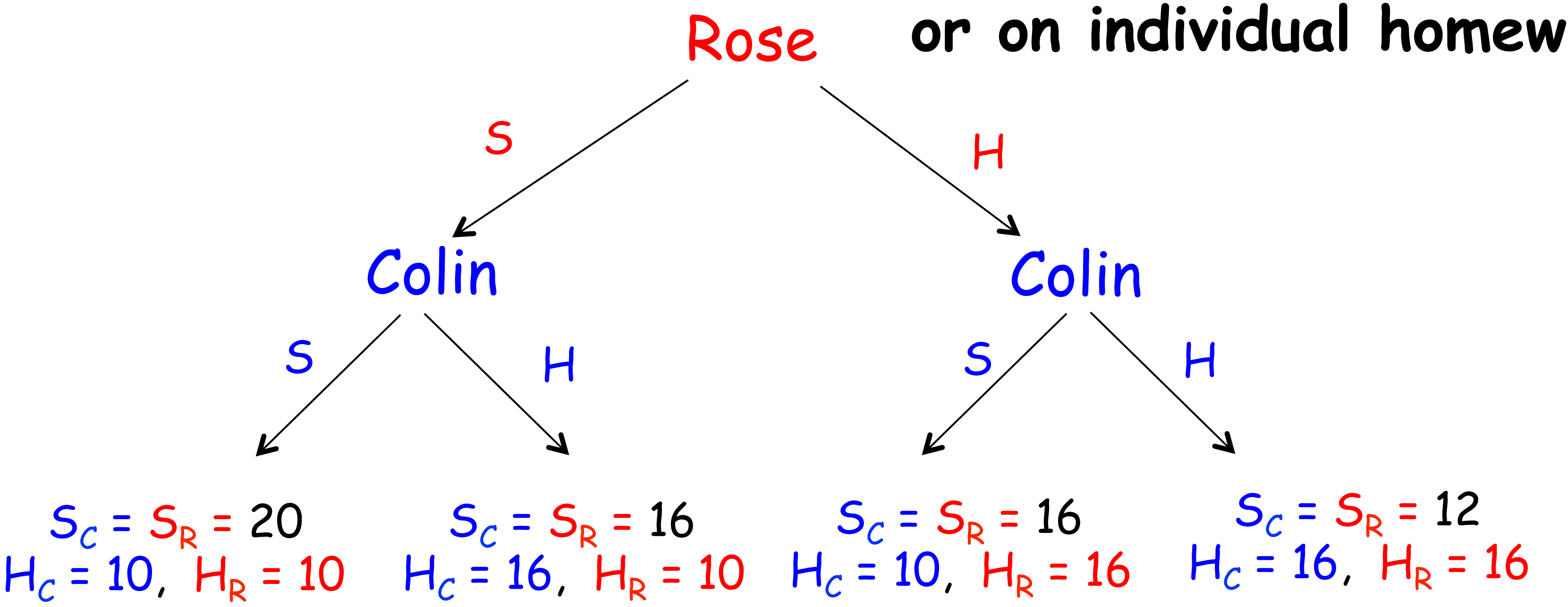
What is Game Theory About?

- ❒ Mathematical/Logical analysis of situations of conflict and cooperation

Invest on scribe notes or on individual homework?



Rose

S

H

Colin

Colin

S

H

S

H

$S_C = S_R = 20$
$H_C = 10, H_R = 10$

$S_C = S_R = 16$
$H_C = 16, H_R = 10$

$S_C = S_R = 16$
$H_C = 10, H_R = 16$

$S_C = S_R = 12$
$H_C = 16, H_R = 16$

- ❒ Goal: to prescribe how rational players should act

What is a Game?

- A Game consists of
  - at least two players
  - a set of strategies for each player
  - a preference relation over possible outcomes
- Player is general entity
  - individual, company, nation, protocol, animal, etc
- Strategies
  - actions which a player chooses to follow
- Outcome
  - determined by mutual choice of strategies
- Preference relation
  - modeled as utility (payoff) over set of outcomes

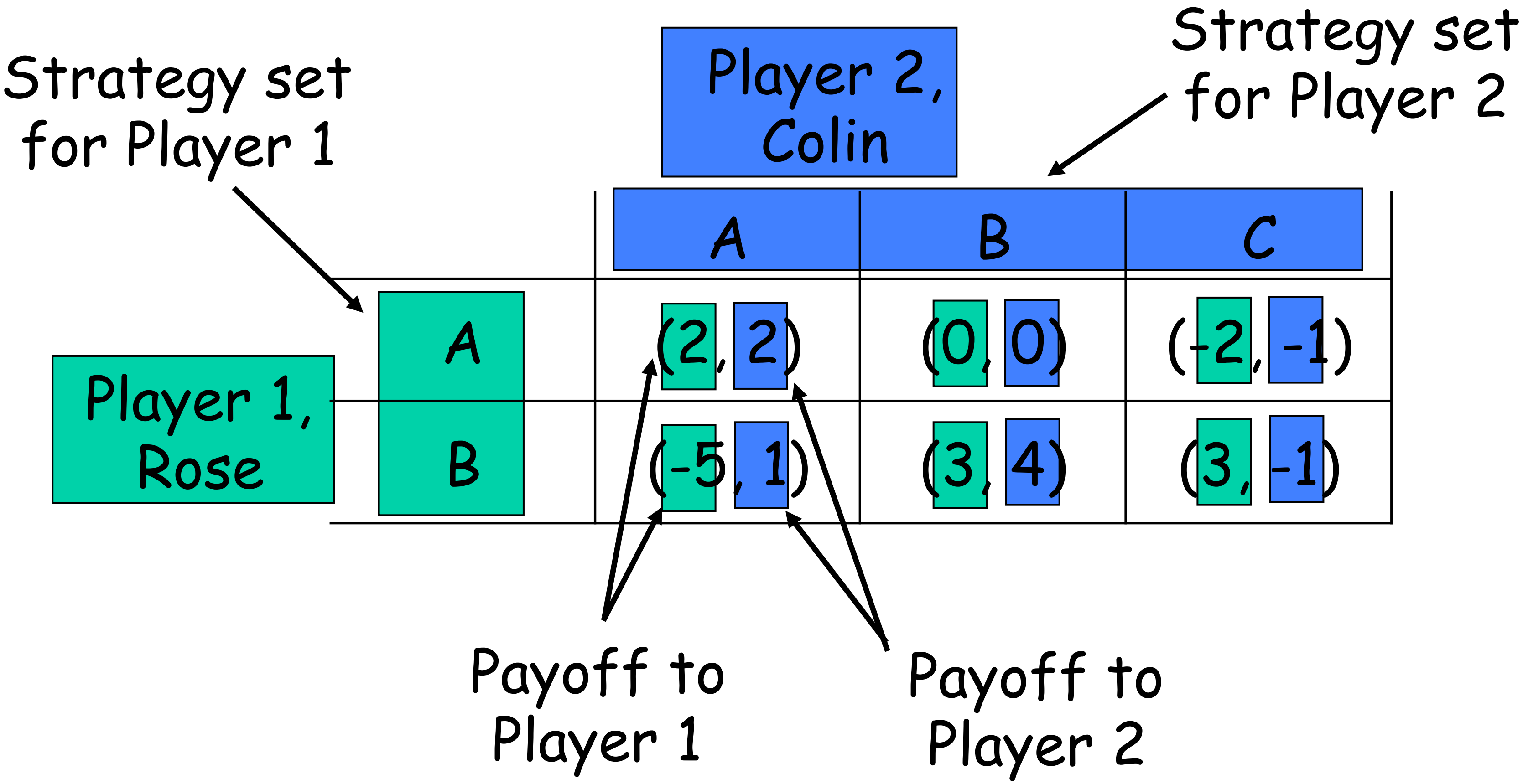
Short history of GT

- Forerunners:
  - Waldegrave's first minimax mixed strategy solution to a 2-person game (1713), Cournot's duopoly (1838), Zermelo's theorem on chess (1913), Borel's minimax solution for 2-person games with 3 or 5 strategies (20s)
- 1928: von Neumann's theorem on two-person zero-sum games
- 1944: von Neumann and Morgenstern, Theory of Games and Economic Behaviour
- 1950-53: Nash's contributions (Nash equilibrium, bargaining theory)
- 1952-53: Shapley and Gillies' core (basic concept in cooperative GT)
- 60s: Aumann's extends cooperative GT to non-transferable utility games
- 1967-68: Harsanyi's theory of games of incomplete information
- 1972: Maynard Smith's concept of an Evolutionarily Stable Strategy
- Nobel prizes in economics
  - 1994 to Nash, Harsanyi and Selten for "their pioneering analysis of equilibria in the theory of non-cooperative games"
  - 2005 to Aumann and Schelling "for having enhanced our understanding of conflict and cooperation through game-theory analysis"
  - 2012 to Roth and Shapley "for the theory of stable allocations and the practice of market design"
- Movies:
  - 2001 "A beautiful mind" on John Nash's life
- See also:
  - www.econ.canterbury.ac.nz/personal_pages/paul_walker/gt/hist.htm

Applications of Game Theory

- Economy
- Politics (vote, coalitions)
- Biology (Darwin's principle, evolutionary GT)
- Anthropology
- War
- Management-labor arbitration
- Philosophy (morality and free will)
- National Football league draft
- "Recently" applied to computer networks
  - Nagle, RFC 970, 1985: "datagram networks as a multi-player game"
  - wider interest starting around 2000

Matrix Game (Normal form)



Strategy set for Player 1

Strategy set for Player 2

		Player 2, Colin		
		A	B	C
Player 1, Rose	A	(2, 2)	(0, 0)	(-2, -1)
	B	(-5, 1)	(3, 4)	(3, -1)

Payoff to Player 1

Payoff to Player 2

- Simultaneous play
  - players analyze the game and then write their strategy on a piece of paper

Students' game

		Colin	
		S	H
Rose	S	15, 15	13, 16
	H	16, 13	14, 14

More Formal Game Definition

- Normal form (strategic) game
  - a finite set N of players
  - a set strategies S_i for each player $i \in N$
  - payoff function $u_i(s)$ for each player $i \in N$
    - where $s \in S = \times_{j \in N} S_j$ is an outcome
    - sometimes also $u_i(A,B,\ldots) \quad A \in S_1, B \in S_2, \ldots$
    - $u_i : S \rightarrow \Re$

Two-person Zero-sum Games

- One of the first games studied
  - most well understood type of game
- Players interest are strictly opposed
  - what one player gains the other loses
  - game matrix has single entry (gain to player 1)
- A “strong” solution concept

Dominance

- Strategy S *(weakly) dominates* a strategy T if every possible outcome when S is chosen is at least as good as corresponding outcome in T, and one is strictly better
  - S *strictly dominates* T if every possible outcome when S is chosen is strictly better than corresponding outcome in T
- Dominance Principle
  - rational players never choose dominated strategies
- Higher Order Dominance Principle
  - iteratively remove dominated strategies

Higher order dominance may be enough

Colin

		S	H
Rose	S	15, 15	13, 16
	H	16, 13	14, 14

Rose's
S strategy
dominated
By H

GT prescribes:

Rose H - Colin H

Higher order dominance may be enough

GT prescribes:

Rose C - Colin B

Colin

		A	B	C	D
Rose	A	12	-1	1	0
	B	3	1	4	-18
	C	5	2	4	3
	D	-16	0	5	-1

(Weakly) Dominated by C

A priori D is **not** dominated by C

Strictly dominated by B

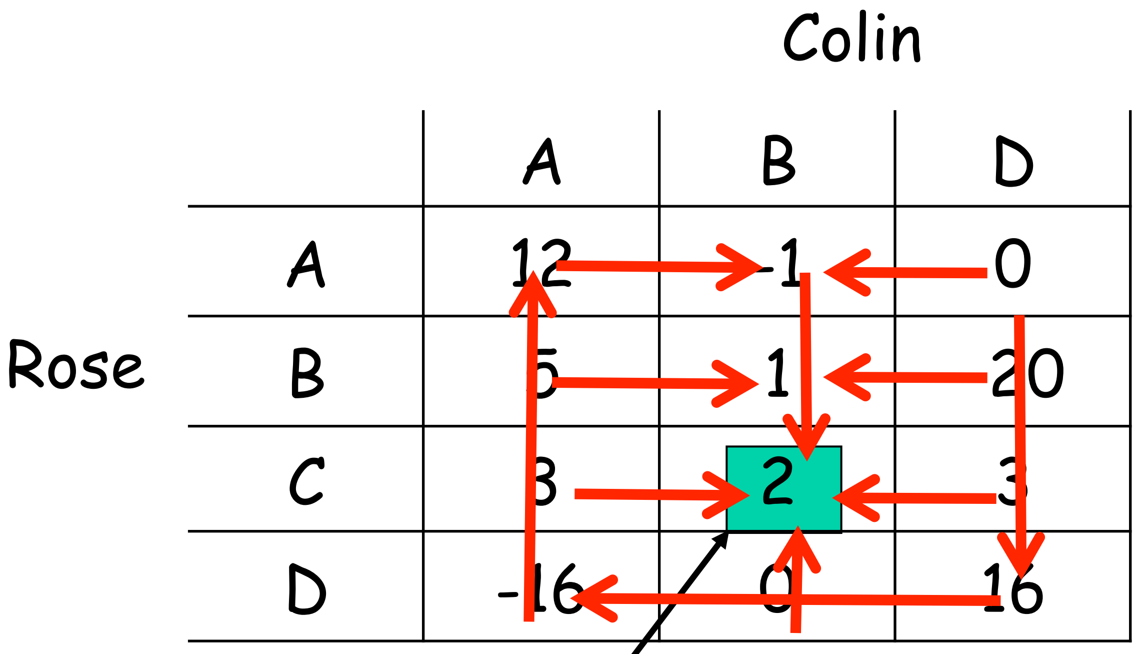
… but not in general

Colin

		A	B	C	D
	A	12	-1	1	0
Rose	B	5	1	7	-20
	C	3	2	4	3
	D	-16	0	0	16

dominated
strategy
(dominated by B)

Analyzing the Reduced Game: Movement Diagram



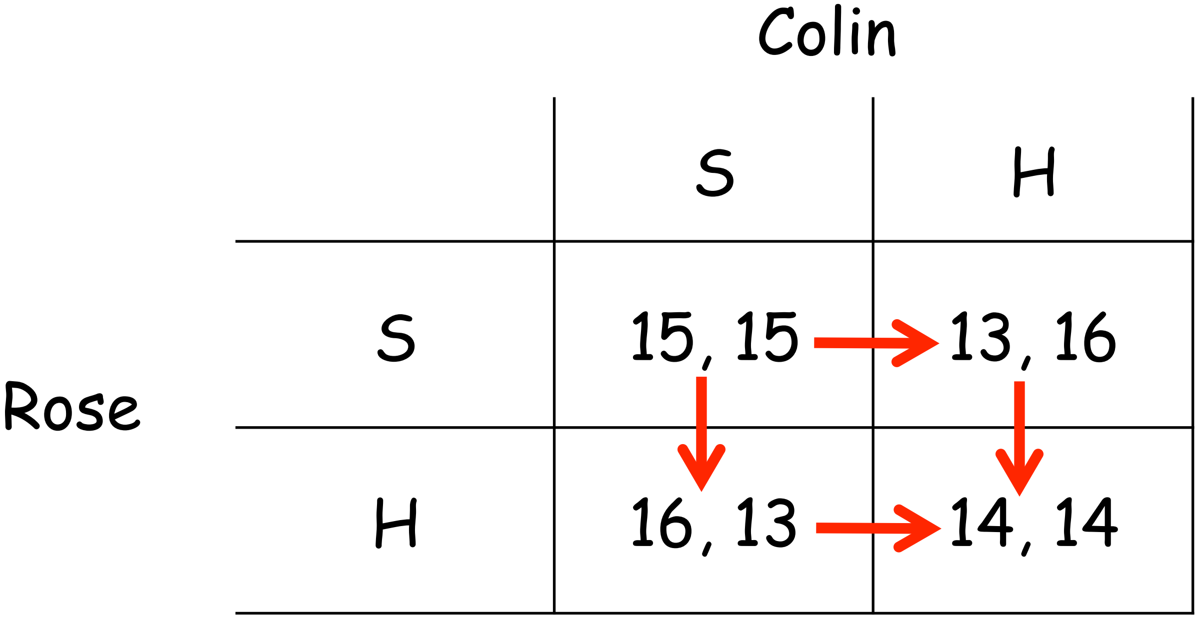
		Colin		
		A	B	D
Rose	A	12	-1	0
	B	5	1	20
	C	3	2	3
	D	-16	0	16

If Rose plays D,
A is Colin's
best response

Outcome (C, B) is “stable”

- ***Pure strategy Nash Equilibrium***
- mutual best responses

Students' game



		Colin	
		S	H
Rose	S	15, 15	13, 16
	H	16, 13	14, 14

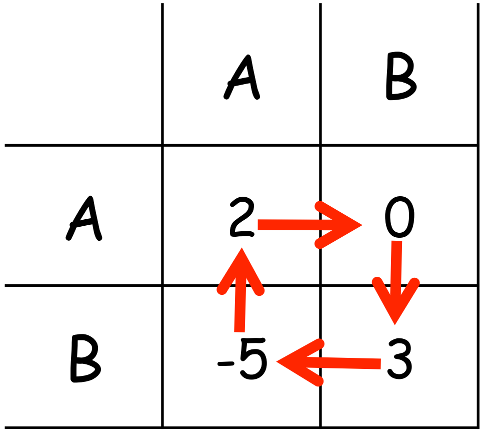
Games without pure strategy NE

- An example?

	R	P	S
R	0	-1	1
P	1	0	-1
S	-1	1	0

Games without pure strategy NE

- An example? An even simpler one



	A	B
A	2	0
B	-5	3

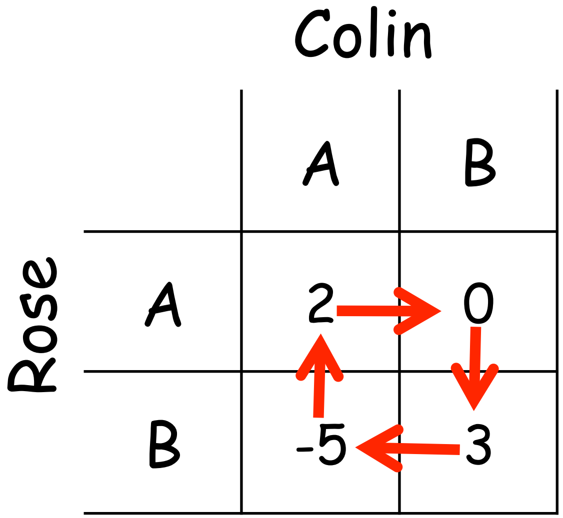
Some practice: find all the pure strategy NE

	A	B	C	D
A	3	2	4	2
B	2	1	3	0
C	2	2	2	2

	A	B	C
A	-2	0	4
B	2	1	3
C	3	-1	-2

	A	B	C
A	4	3	8
B	9	5	1
C	2	7	6

Games with no pure strategy NE



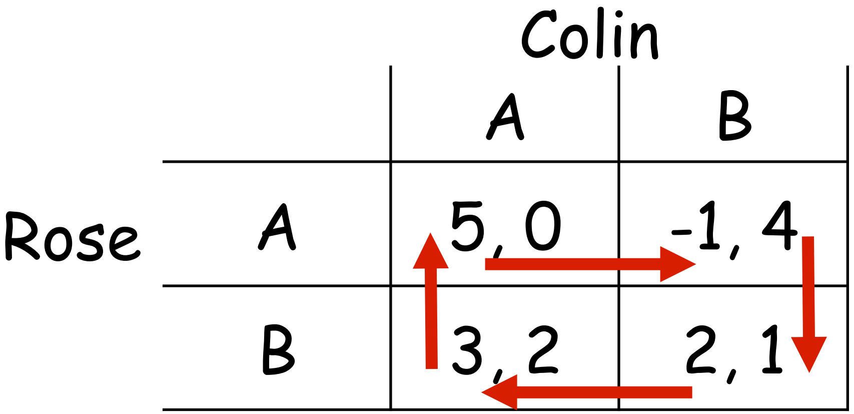
Colin

Rose

	A	B
A	2	0
B	-5	3

- What should players do?
  - resort to randomness to select strategies

Games with no pure strategy NE



		Colin	
		A	B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1

- …but we can find mixed strategies equilibria

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy (*equalizing strategy*), that equalizes the opponent payoffs
  - how to calculate it?

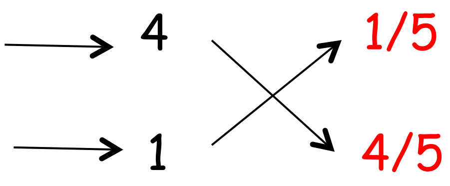
		Colin	
		A	B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

		Colin	
		A	B
Rose	A	-0	-4
	B	-2	-1

Rose considers
Colin's game



4 1/5

1 4/5

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

Colin considers *Rose’s game*

		Colin	
		A	B
Rose	A	5	-1
	B	3	2
		3/5	2/5

Mixed strategies equilibria

- Same idea of equilibrium
  - each player plays a mixed strategy, that equalizes the opponent payoffs
  - how to calculate it?

		Colin	
		A	B
Rose	A	5, 0	-1, 4
	B	3, 2	2, 1

Rose playing (1/5,4/5)
Colin playing (3/5,2/5)
is an equilibrium

Rose gains 13/5
Colin gains 8/5

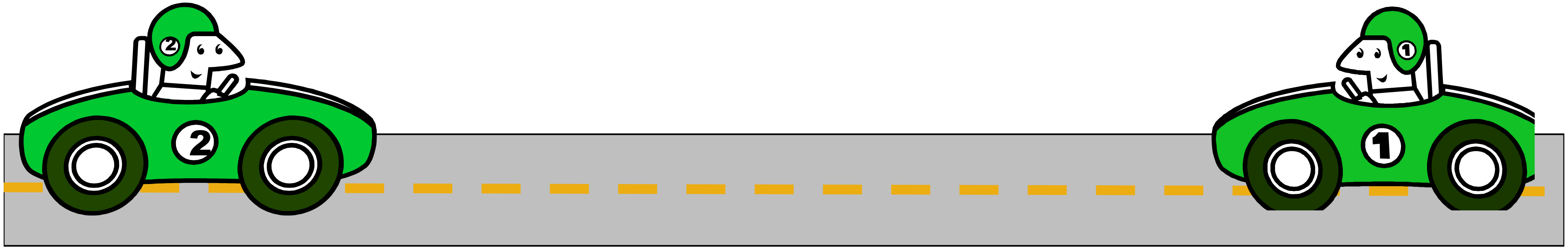
Good news:
Nash's theorem [1950]

- Every two-person games has at least one equilibrium either in pure strategies or in mixed strategies
  - Proved using fixed point theorem
  - generalized to N person game
- This equilibrium concept called Nash equilibrium in his honor
  - A vector of strategies (a profile) is a Nash Equilibrium (NE) if no player can unilaterally change its strategy and increase its payoff

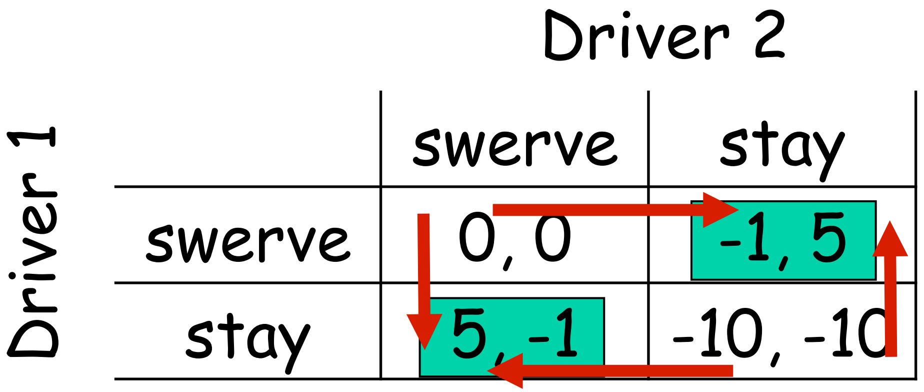
A useful property

- Given a finite game, a profile is a mixed NE of the game if and only if for every player i, every pure strategy used by i with non-null probability is a best response to other players mixed strategies in the profile
  - see Osborne and Rubinstein, *A course in game theory*, Lemma 33.2

Game of Chicken



- Game of Chicken (aka. Hawk-Dove Game)
  - driver who swerves looses



Driver 2

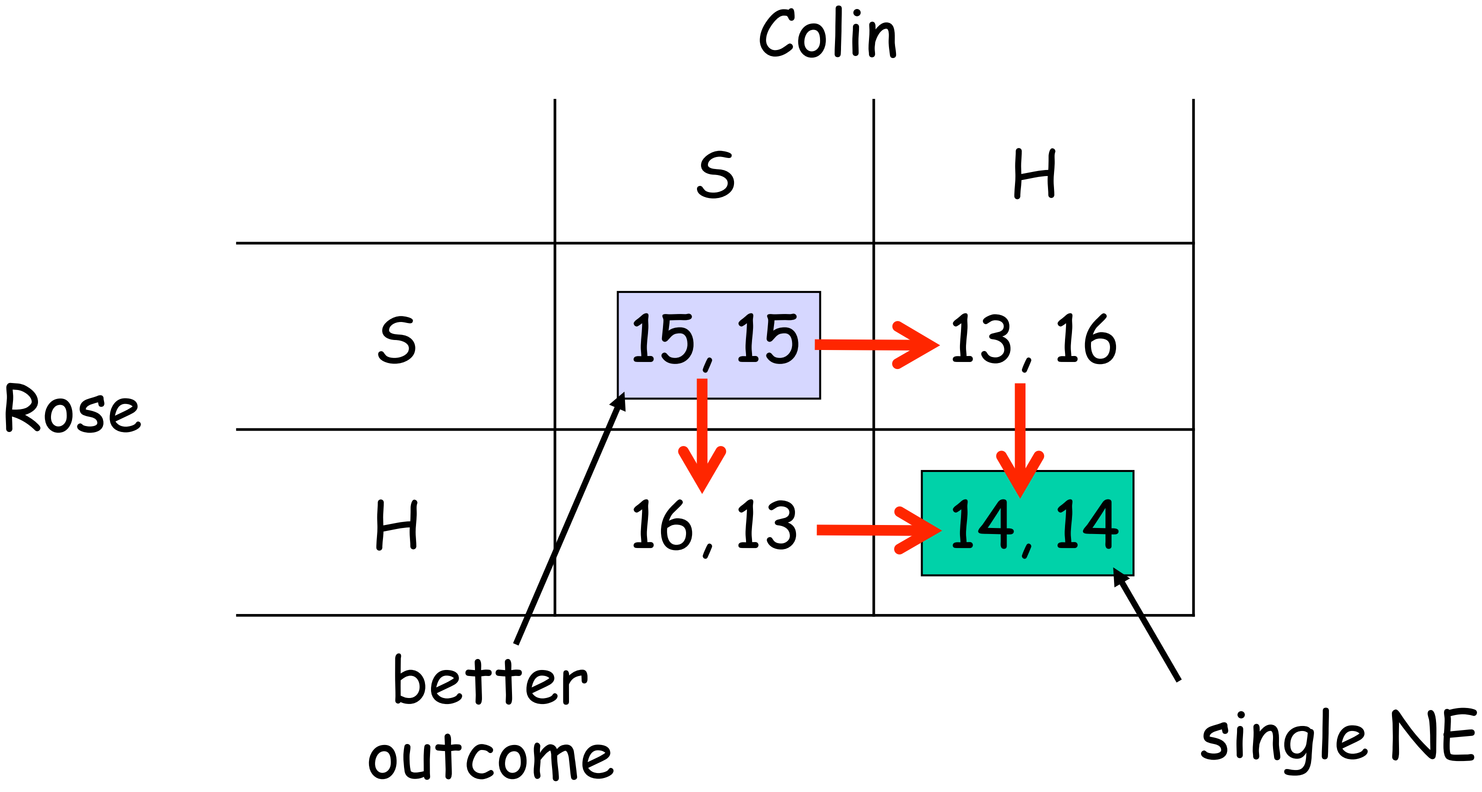
Driver 1		swerve	stay
	swerve	0, 0	-1, 5
	stay	5, -1	-10, -10

Drivers want to do opposite of one another

Two equilibria:
not equivalent
not interchangeable!

- playing an equilibrium strategy does not lead to equilibrium

Students' game

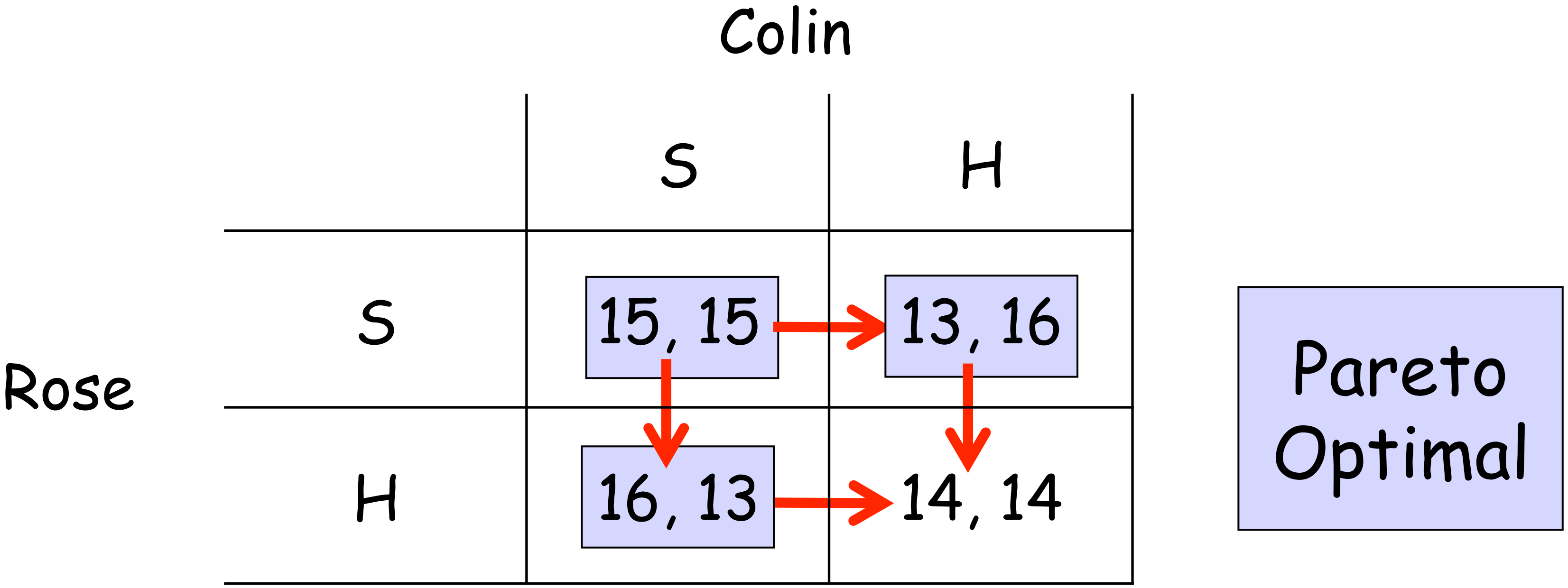


		Colin	
		S	H
Rose	S	15, 15	13, 16
	H	16, 13	14, 14

better outcome

single NE

Students' game



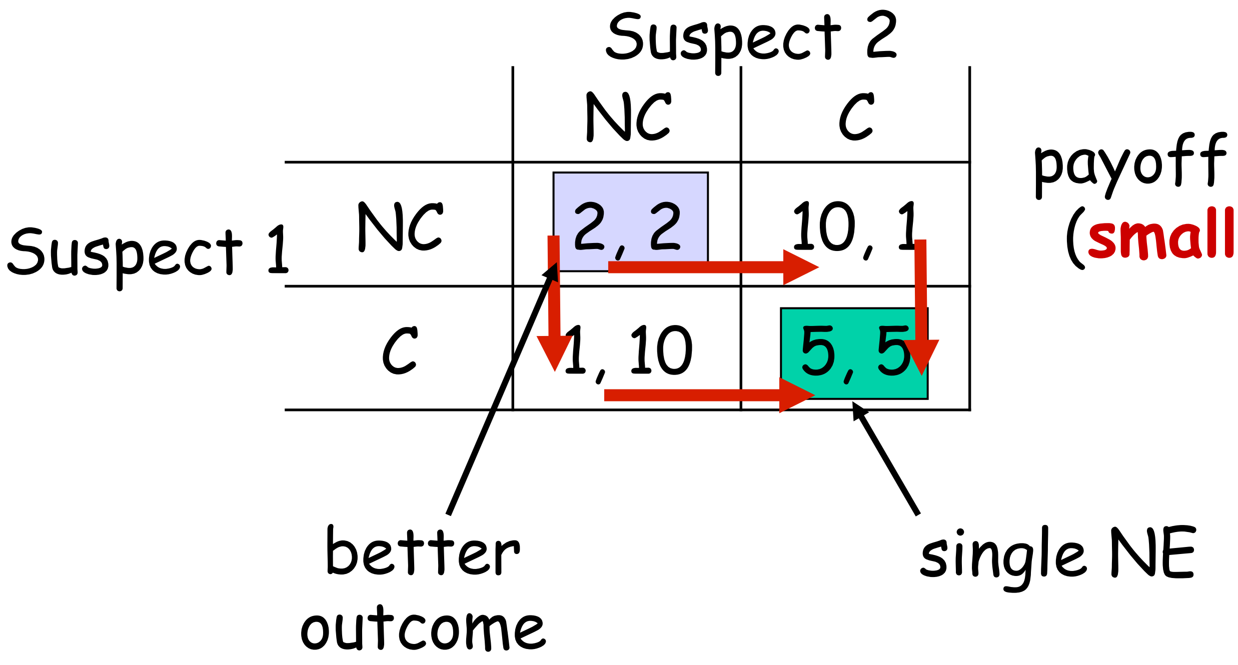
		Colin	
		S	H
Rose	S	15, 15	13, 16
	H	16, 13	14, 14

Pareto Optimal

- Def: outcome o* is Pareto Optimal if no other outcome would give to all the players a payoff not smaller and a payoff higher to at least one of them
- Conflict between group rationality (Pareto principle) and individual rationality (dominance principle)

Students' game = Prisoner's Dilemma

- One of the most studied and used games
  - proposed in 1950
- Two suspects arrested for joint crime
  - each suspect when interrogated separately, has option to confess



		Suspect 2	
		NC	C
Suspect 1	NC	2, 2	10, 1
	C	1, 10	5, 5

better outcome

single NE

payoff is years in jail
(**smaller is better**)