Each answer has to be justified. The points marked for each exercise give an indication of the relative importance.

Ex. 1 — (1 point) Consider a graph, where each link $l \in E$ is affected by a delay $D_l(y_l)$ depending on the amount of traffic on that link, y_l. The delay function is assumed to be convex, increasing and differentiable. Consider the following routing optimization problem:

$$\begin{aligned}
\underset{\mathbf{x}\in\mathbb{R}^{|R|},\mathbf{y}\in\mathbb{R}^{|E|}}{\text{minimize}} \quad & \sum_{l\in E} y_l D_l(y_l) \\
\text{subject to} \quad & f_s = \sum_{r|s(r)=s} x_r \ \ \forall s \in S \\
& y_l = \sum_{r|l\in r} x_r \ \ \forall l \in E \\
& x_r \geq 0 \ \ \forall r \in R
\end{aligned}$$

1. What is the system minimizing?
2. Write the Lagrangian function relative to the first two sets of constraints (i.e. ignoring $x_r \geq 0$).
3. Consider the particular case of a network made by two nodes u and v connected by $|E| = 3$ parallel edges. f_{uv} traffic has to be routed between u and v. The delays on the links are respectively:
   - $D_1(y_1) = 1 + y_1$
   - $D_2(y_2) = 2 + \frac{1}{2}y_2$
   - $D_3(y_3) = 4 + \frac{1}{100}y_3$

   Determine the optimal routing if $f_{uv} = 1$ and if $f_{uv} = 3$.

Ex. 2 — [2 points] Consider two flows sharing a common link l. The link communicates the current amount of traffic on it (y_l) to both sources. Each source adapts its rate according to the following equation:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \frac{1}{\sqrt{x_1}} - y_l^2, \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= 10\left(\frac{1}{\sqrt{x_2}} - y_l^2\right).
\end{aligned}$$

1. What optimization problem is implicitly maximized by the two sources?
2. What are the optimal solutions of this optimization problem?
3. Will the flow rates converge to an optimal solution? Why?

Ex. 3 — [3 points] Consider a single source transmitting through L parallel links. The rate on link l is x_l and causes a congestion cost equal to $M_l(x_l)$, where $M_l()$ is a convex increasing differentiable function. The utility for the source to transmit at a total rate $X = \sum_{l=1}^{L} x_l$ is $U(X)$, where $U()$ is a concave increasing differentiable function. The goal is to maximize the total social welfare defined as utility of the source minus congestion costs of the links.

1. Formulate mathematically the corresponding optimization problem.

2. Can you identify properties of the optimal rate allocation? Which links are used at the optimum? What is the corresponding rate?