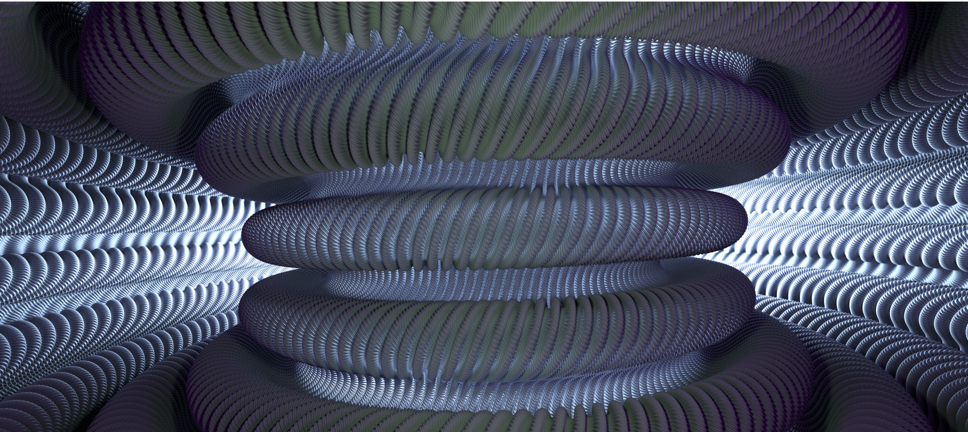
Embeddings of the square flat torus and smooth fractals

Boris Thibert
LJK / Université de Grenoble

Workshop on geometric structures with symmetry and periodicity   8 June, 2014



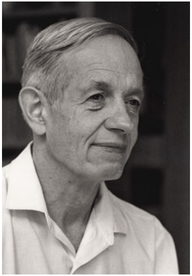
HEVEA TEAM

VINCENT BORRELLI
ICJ, Lyon

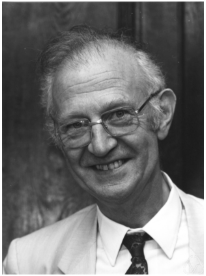
SAÏD JABRANE
ICJ, Lyon

FRANCIS LAZARUS
Gipsa-Lab, Grenoble

Isometric embeddings



John F. Nash



Nicolaas Kuiper

Nash-Kuiper Theorem (1954-55-86)

Let (M^n, g) be compact Riemannian manifold and

$$f_0 : (M^n, g) \to \mathbb{E}^{n+1}$$

be a strictly short embedding ($f_0^* \langle \cdot, \cdot \rangle_{\mathbb{E}^{n+1}} < g$).
Then for every $\varepsilon > 0$, there exists a C^1 isometric embedding

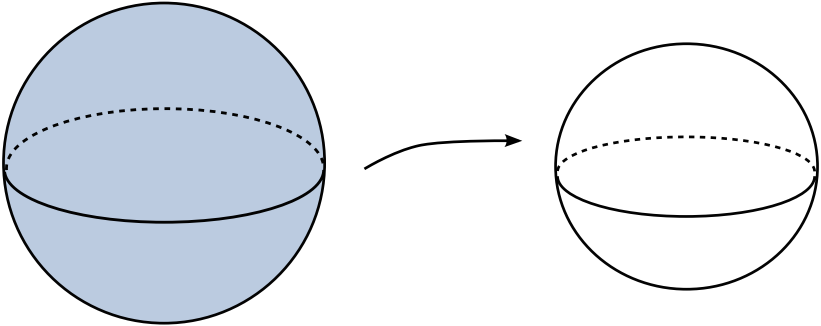
$$f : (M^n, g) \to \mathbb{E}^{n+1}$$

such that $\|f - f_0\|_{C^0} \leq \varepsilon$.

Nash-Kuiper sphere

Nash Kuiper theorem

The unit sphere $\mathbb{S}^2$ can be C^1 isometrically embedded in a ball of radius $r < 1$.



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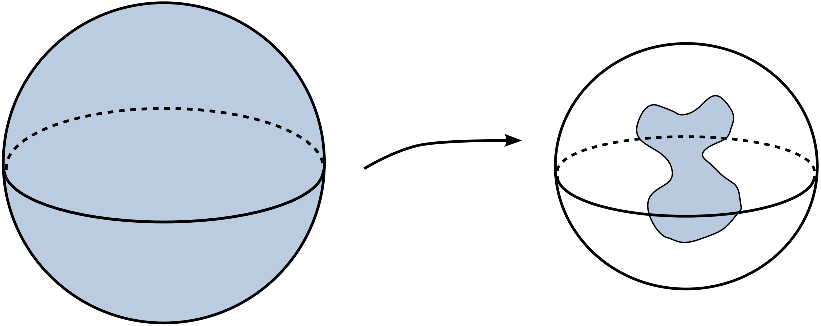
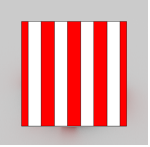


Flat torus

Definition

A quotient $\mathbb{E}^2/\Lambda$ of the two dimensional Euclidean space by a lattice $\Lambda \subset \mathbb{E}^2$ is called a *flat torus*.



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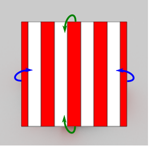

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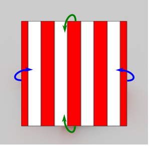

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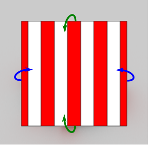

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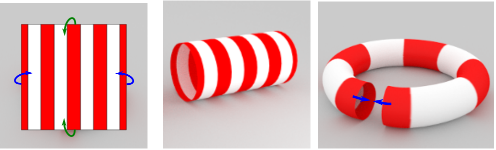

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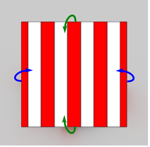   

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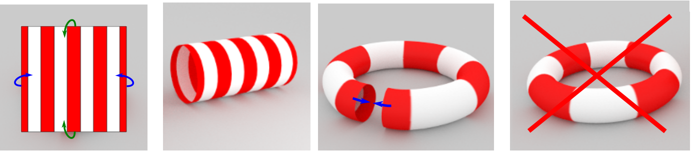

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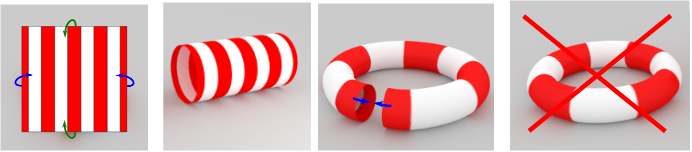
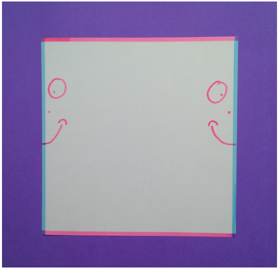


(C) Patrick Massot

Nash Kuiper theorem

The square flat torus admits a C^1 isometric embedding in $\mathbb{E}^3$.

The square flat torus



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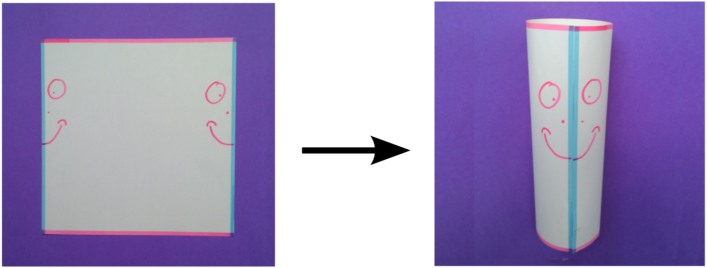

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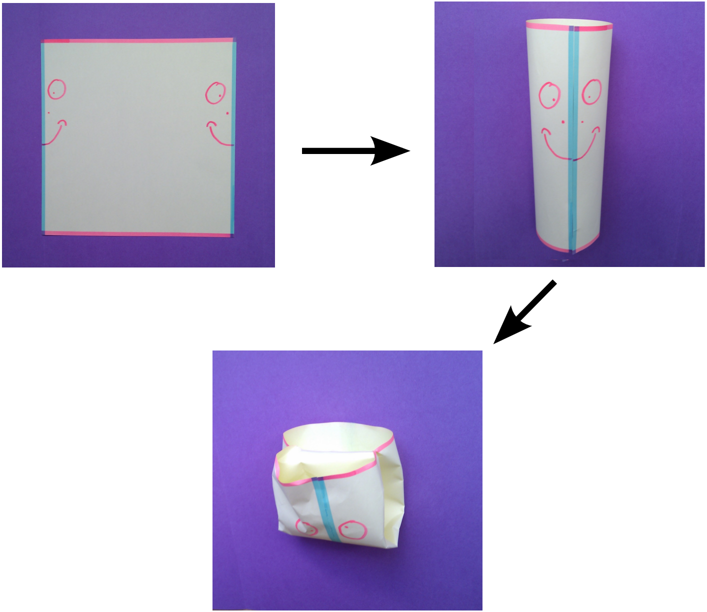

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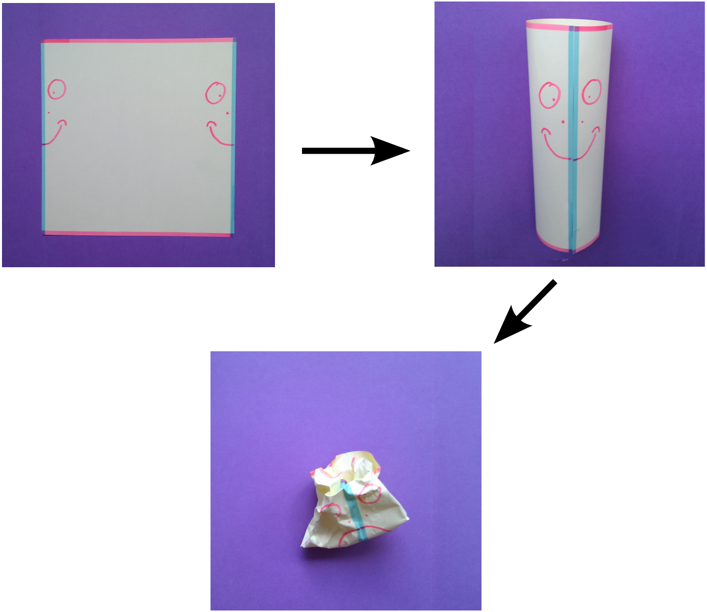
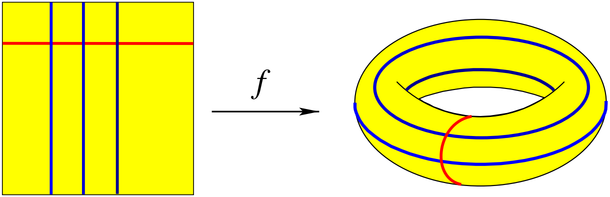


Brief history

- **Janet-Cartan (1926-27)** Local analytic isometric embedding of (M^n, g) into $\mathbb{E}^{\frac{n(n+1)}{2}}$
- **Nash (1954)** Global C^1-isometric embedding of (M^n, g) into $\mathbb{E}^{k \geq n+2}$ (if a C^∞ embedding into $\mathbb{E}^k$ exists)
- **Kuiper (1955)** Global C^1-isometric embedding of (M^n, g) into $\mathbb{E}^{k \geq n+1}$ (if a C^∞ embedding into $\mathbb{E}^k$ exists)
- **Nash (1956)** Global C^∞-isometric embedding of (M^n, g) into $\mathbb{E}^k$, $k = \frac{n}{2}(3n + 11)$.
- **Gromov (1973)** h-principle and convex integration theory.
- ...

Differential system

- Let $f : (\mathbb{T}^2, g) \to (\mathbb{R}^3, \langle \cdot, \cdot \rangle_{\mathbb{E}^3})$ be an embedding.



- $f : (\mathbb{T}^2, g) \to (\mathbb{R}^3, \langle \cdot, \cdot \rangle_{\mathbb{E}^3})$ is isometric if $\quad f^*\langle \cdot, \cdot \rangle_{\mathbb{E}^3} = g.$

$$< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} >= g_{1,1} \qquad < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >= g_{1,2} \qquad < \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y} >= g_{2,2}$$

- $f : (\mathbb{T}^2, g) \to (\mathbb{R}^3, \langle \cdot, \cdot \rangle_{\mathbb{E}^3})$ is strictly short if

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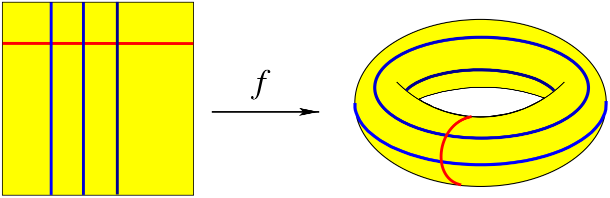

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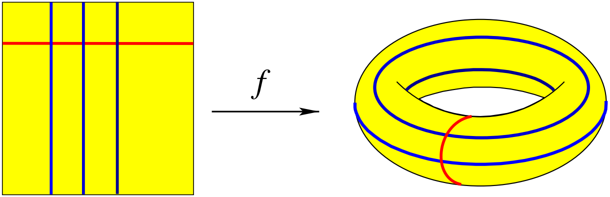

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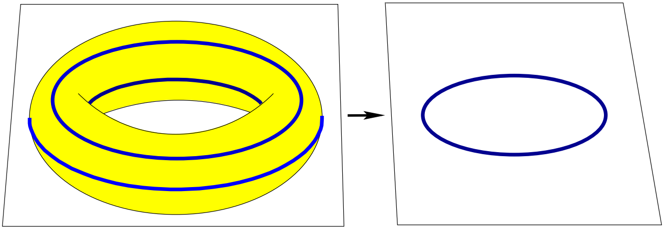
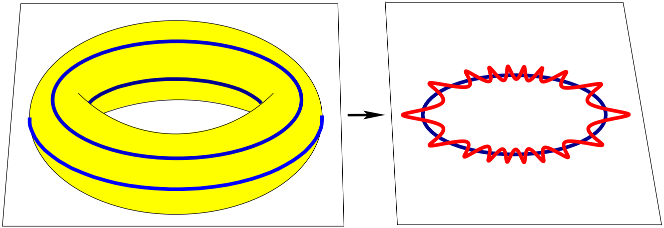
Idea of convex integration (isometric relation)

- Initial stricly short embedding f_0 : $f_0^*\langle\cdot,\cdot\rangle_{\mathbb{E}^3} < g$



Idea of convex integration (isometric relation)

- Initial stricly short embedding f_0 : $\quad f_0^*\langle\cdot,\cdot\rangle_{\mathbb{E}^3} < g$
- We add corrugations (oscillations)



Outline

1. One dimensional convex integration.
2. Convex integration on the torus.
3. Smooth fractals.

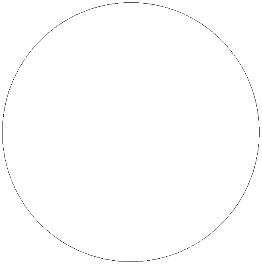
One dimensional convex integration

Input :

$$f_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2 \;\text{ such that }\; \text{length}(f_0) < r$$

Our goal is to to build

$$f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2 \text{ such that } \begin{cases} \\ \|f - f_0\|_{C^0} \text{ small} \end{cases}$$



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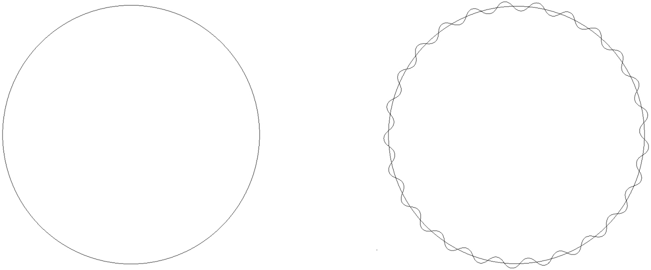


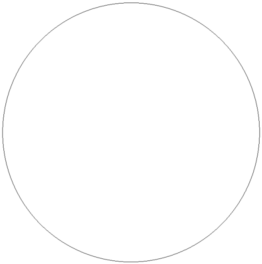
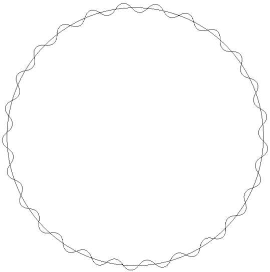
One dimensional convex integration

Input :

$$f_0 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2 \text{ such that } \|f_0'(u)\| < r$$

Our goal is to to build

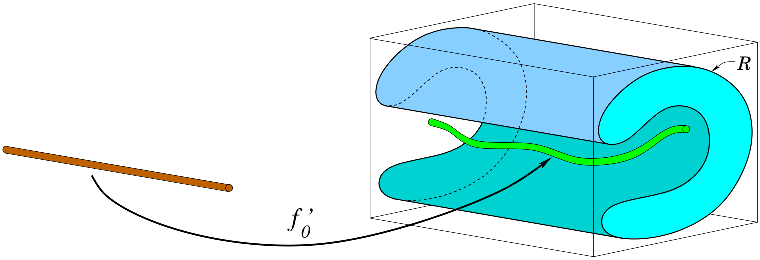
$$f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2 \text{ such that } \begin{cases} \|f'(u)\| = r \quad (f'(u) \in \mathcal{R}) \\ \|f - f_0\|_{C^0} \text{ small} \end{cases}$$

General principle of the construction

The differential relation $\mathcal{R}$ is the set of constraint :

- We have $f_0'(u) \notin \mathcal{R}$.
- We want $f'(u) \in \mathcal{R}$.



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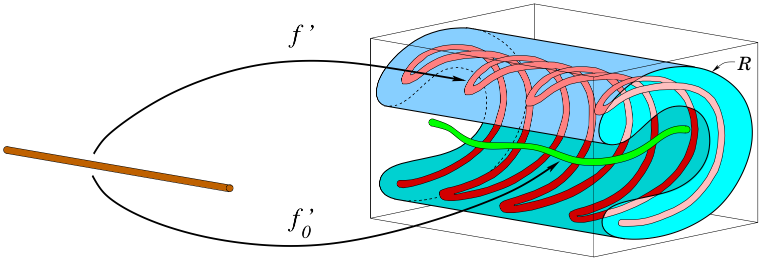
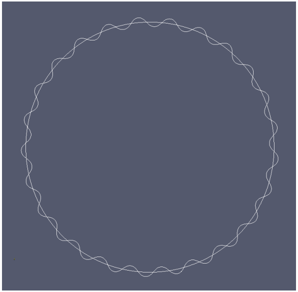


Finally

$$f(u) := f_0(0) + \int_0^u h_s(Ns)ds,$$

with

$$h_u(s) = r\Big(\cos(\alpha\cos(2\pi s))\mathbf{e_1}(u) + \sin(\alpha\cos(2\pi s))\mathbf{n}(u)\Big)$$



Lemma

- $\|f - f_0\|_{C^0} = O(\frac{1}{N})$
- $f'(u) \in \mathcal{R}$

Outline

1. One dimensional convex integration.
2. Convex integration on the torus.
3. Smooth fractals.

General problem

Input : A strictly short embedding $f_0 : (\mathbb{T}^2, g) \longrightarrow \mathbb{E}^3$

Goal : We look for an isometric embedding :

$$f : (\mathbb{T}^2, g) \longrightarrow \mathbb{E}^3.$$

Approach : one parameter version of convex integration

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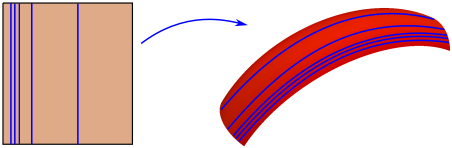

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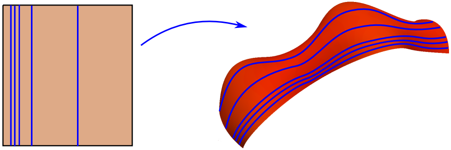

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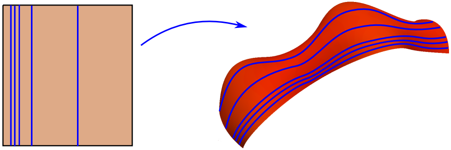

$\Rightarrow$ The isometric default is reduced in one direction.

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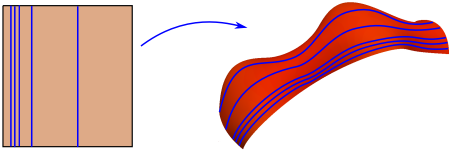

$\Rightarrow$ The isometric default is reduced in one direction.
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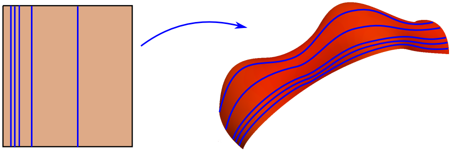

$\Rightarrow$ The isometric default is reduced in one direction.
$\Rightarrow$ We need to corrugate in a second direction...
$\Rightarrow$ Then in a third direction...

A stage

Stage theorem [Borrelli, Jabrane, Lazarus, ~]

$$\|g - F_{N_1,N_2,N_3}{}^*\langle\cdot,\cdot\rangle_{\mathbb{E}^3}\|_{C^0} = O\left(\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3}\right)$$

Problems

- $\lim_{N_i\to\infty} F_{N_1,N_2,N_3} = f_0$ (only C^0 convergence)
- The differential relation of the isometries is closed.

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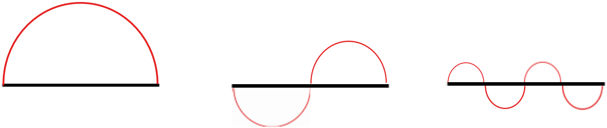

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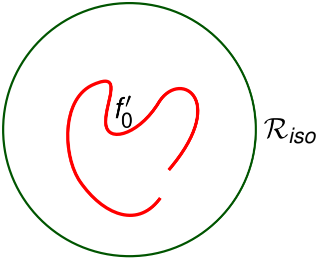

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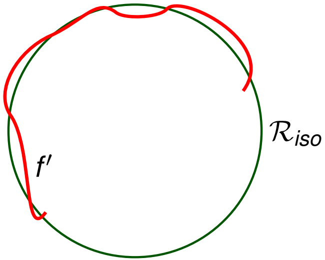

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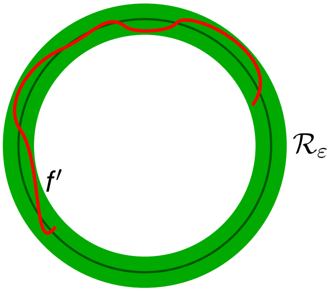

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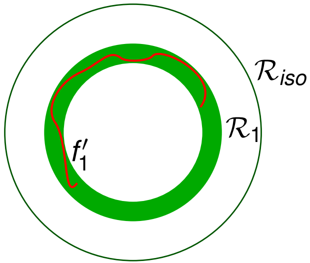

A stage

Stage theorem [Borrelli, Jabrane, Lazarus, ~]

$$\|g - F_{N_1,N_2,N_3}{}^*\langle\cdot,\cdot\rangle_{\mathbb{E}^3}\|_{C^0} = O(\frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3})$$

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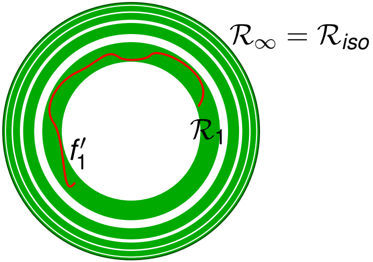

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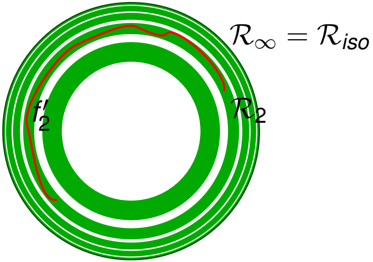

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Stage theorem [Borrelli, Jabrane, Lazarus, ~]

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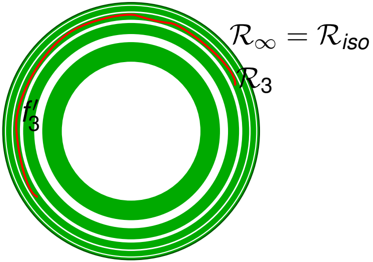


$\Rightarrow$ Build iteratively (F_k) where F_k depends on $N_{k,1}$, $N_{k,2}$, $N_{k,3}$ and F_{k-1}.

Implementation : choice of the N_k's

A sufficient condition

$$\|\mu_{k,j} - f^*_{k,j}\langle\cdot,\cdot\rangle_{\mathbb{R}^3}\|_{C^0} = 0\left(\frac{1}{N_{k,j}}\right) < \tau(\delta_{k+1} - \delta_k)\rho_{min}(D_k)$$

This imposes high frequences of corrugation :

611
69,311
20,914,595
6,572,411,478

With 10 points per oscillation, we need a mesh of $(10 \times 6572411478)^2 \approx 4.3\,10^{19}$ vertices !

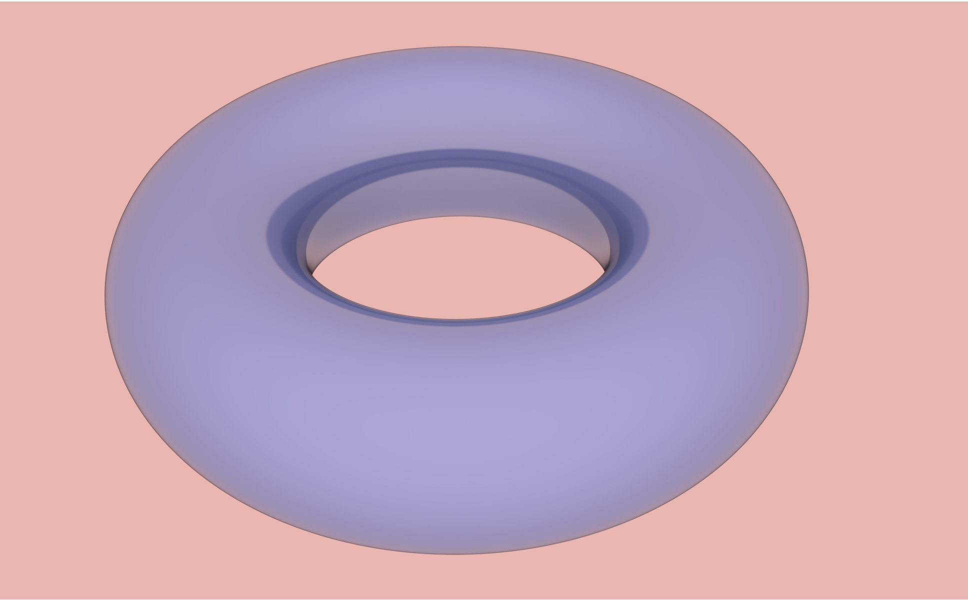
Choice of the N_k

We reduce these numbers to :

$$\begin{matrix} 12 \\ 80 \\ 500 \\ 9000 \end{matrix}$$

We already need $100,000 * 20,000 = 2$ milliards of vertices !

Images



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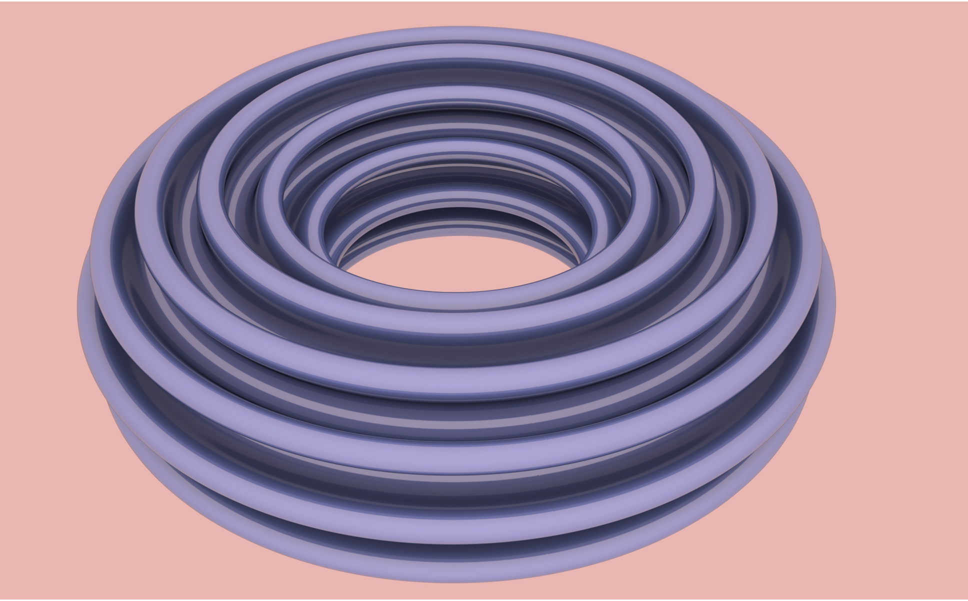

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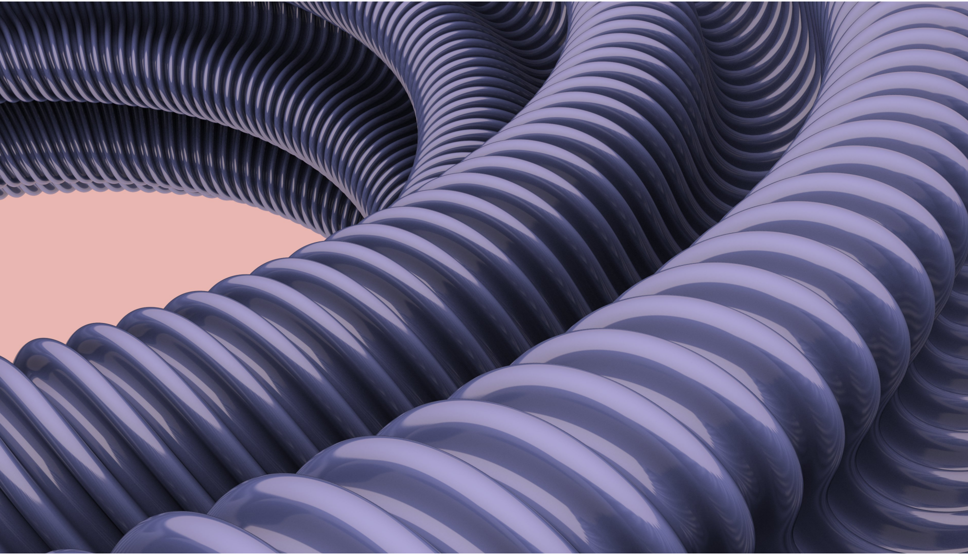

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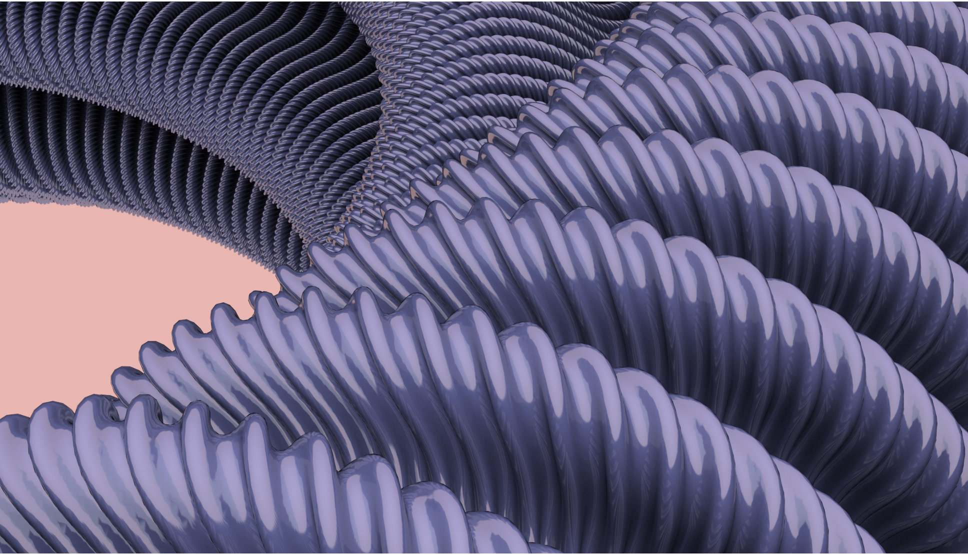

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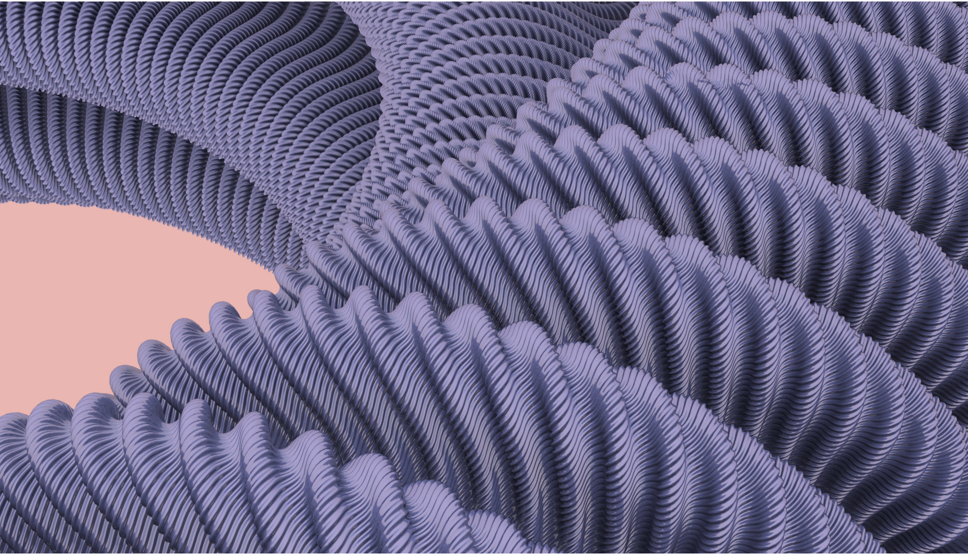

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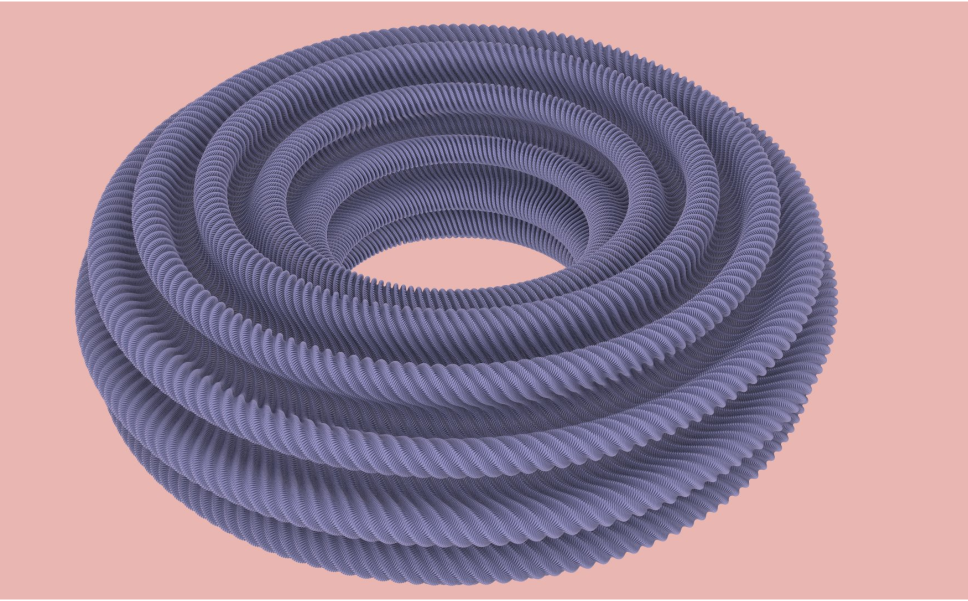
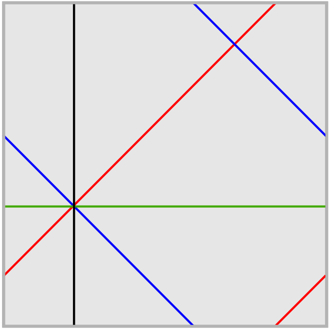
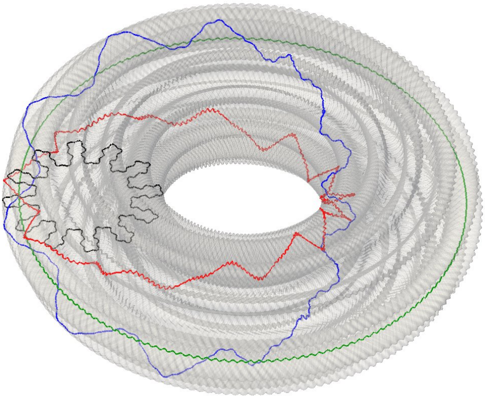


Outline

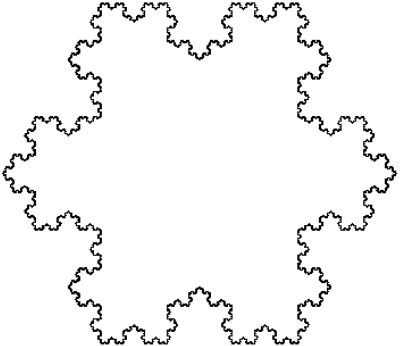
1. One dimensional convex integration.
2. Convex integration on the torus.
3. Smooth fractals.

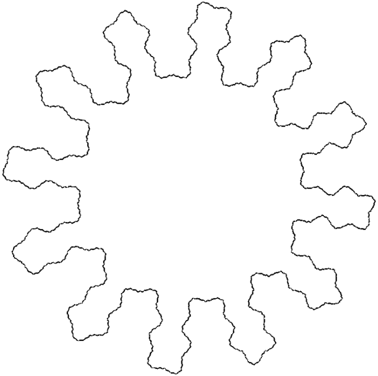
Curves





More images





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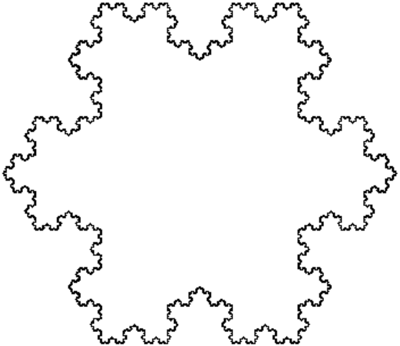

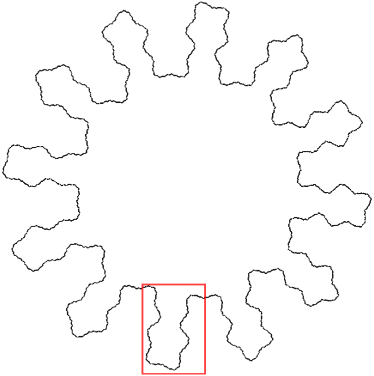

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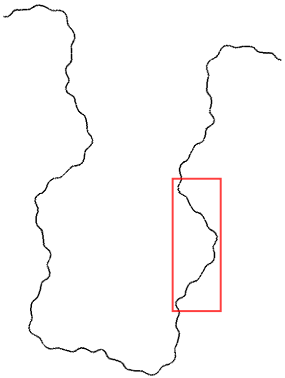

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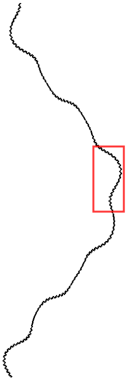

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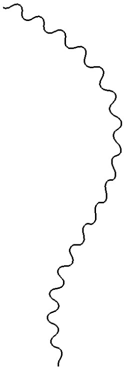
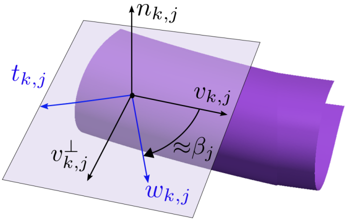
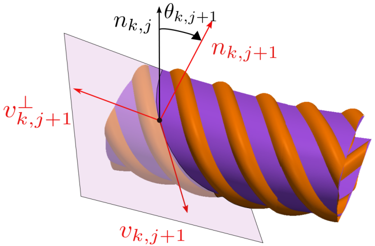


Structure of the Gauss map





The Gauss map is given by

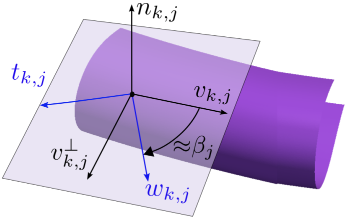
$$\begin{pmatrix} v^{\perp}_{\infty} \\ v_{\infty} \\ \mathbf{n}_{\infty} \end{pmatrix} = \prod_{k=0}^{\infty} \left(\prod_{j=1}^{3} \mathcal{C}_{k,j} \right) \begin{pmatrix} v^{\perp}_{0} \\ v_{0} \\ \mathbf{n}_{0} \end{pmatrix}$$

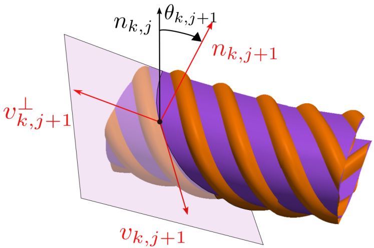
Corrugation theorem [Borrelli, Jabrane, Lazarus, ~, PNAS '12]

We have $\mathcal{C}_{k,j+1} = \mathcal{L}_{k,j+1} \cdot \mathcal{R}_{k,j}$, where

$$\mathcal{L}_{k,j+1} = \begin{pmatrix} \cos\theta_{k,j+1} & 0 & \sin\theta_{k,j+1} \\ 0 & 1 & 0 \\ -\sin\theta_{k,j+1} & 0 & \cos\theta_{k,j+1} \end{pmatrix} + O\left(\frac{1}{N_{k,j+1}}\right)$$

Structure of the Gauss map





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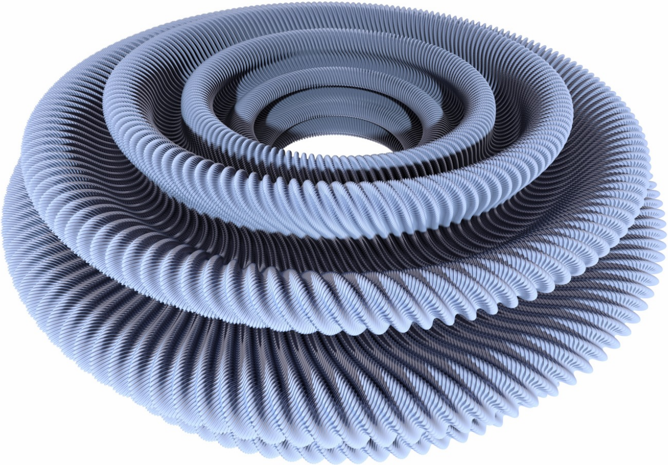
$$\mathcal{R}_{k,j} = \begin{pmatrix} \cos\beta_j & \sin\beta_j & 0 \\ -\sin\beta_j & \cos\beta_j & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(\varepsilon_{k,j})$$

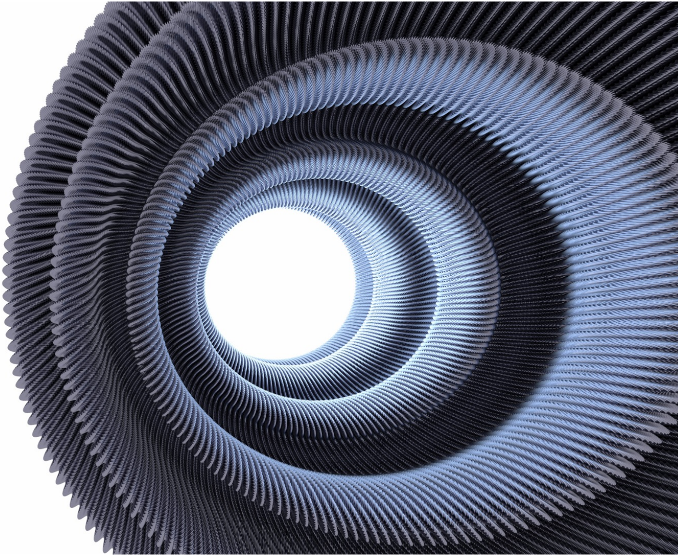
where $\theta_{k,j}(s,u) = \alpha_{k,j}(s,u)\cos(2\pi N_{k,j}u)$, $\varepsilon_{k,j} := \|\langle\cdot,\cdot\rangle_{\mathbb{E}^2} - f^*_{k,j}\langle\cdot,\cdot\rangle_{\mathbb{E}^3}\|$ and $\beta_j := \angle(\ker \ell_j, \ker \ell_j)$.

Images



(Rendered by D. Rohmer)

Images



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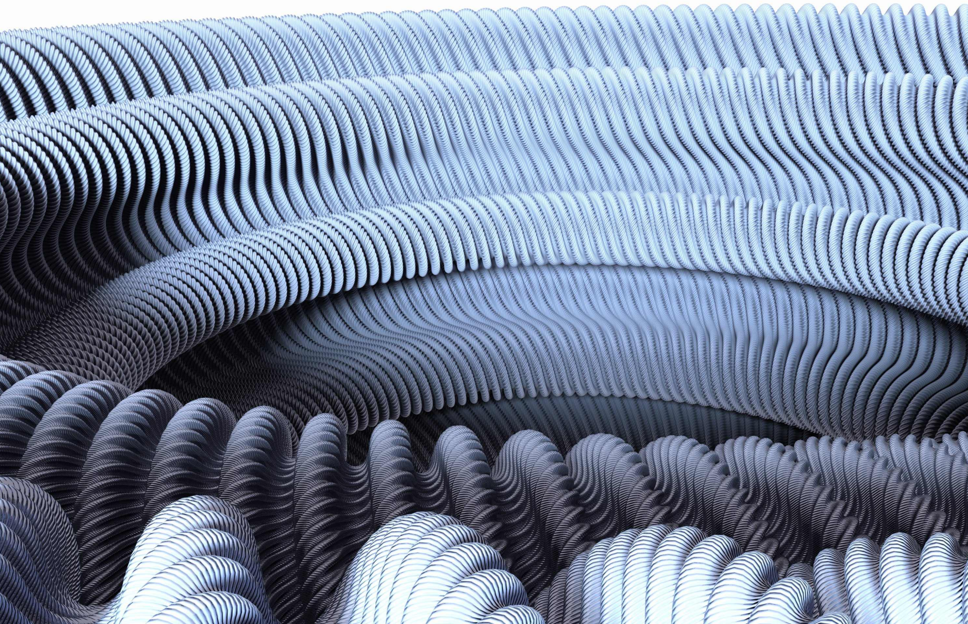

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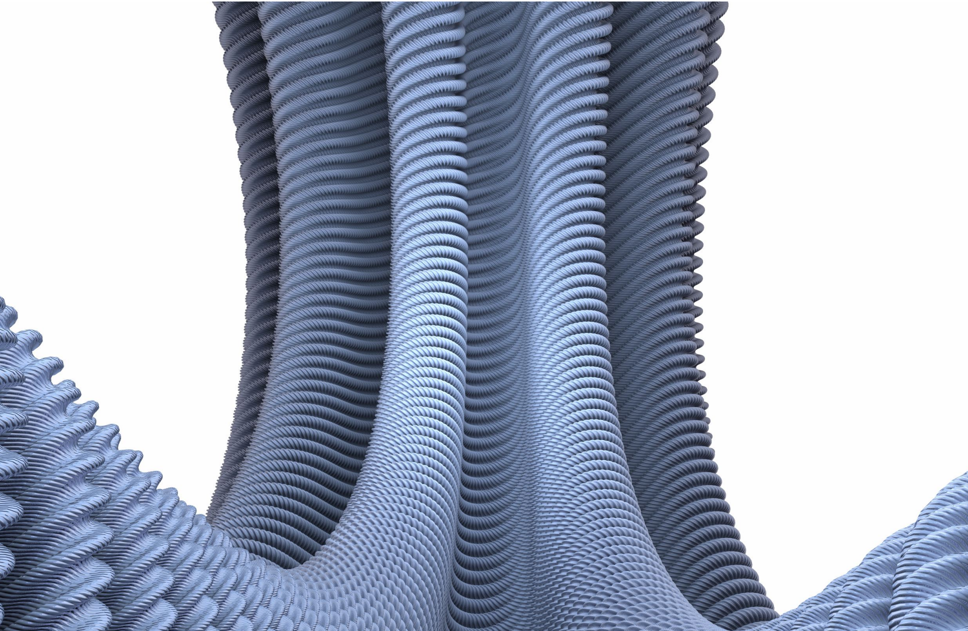

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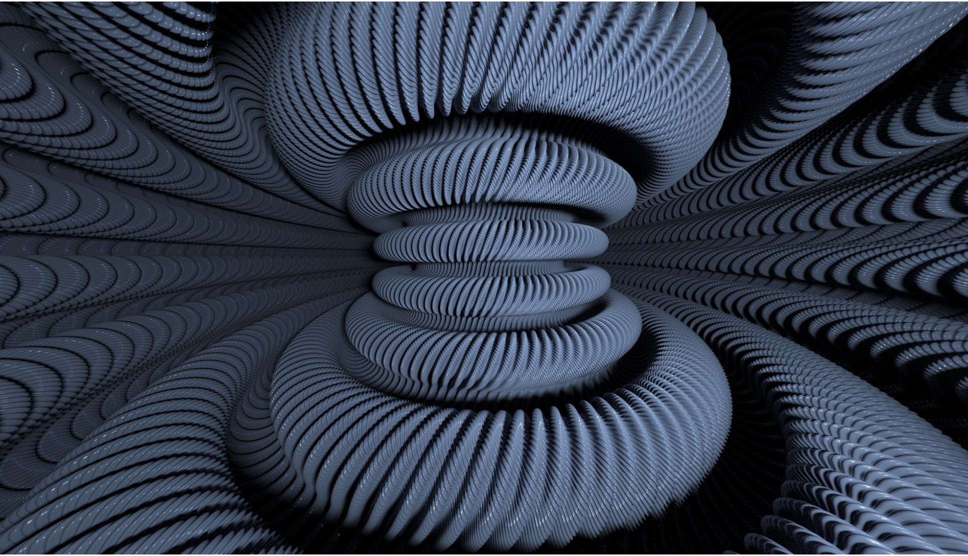

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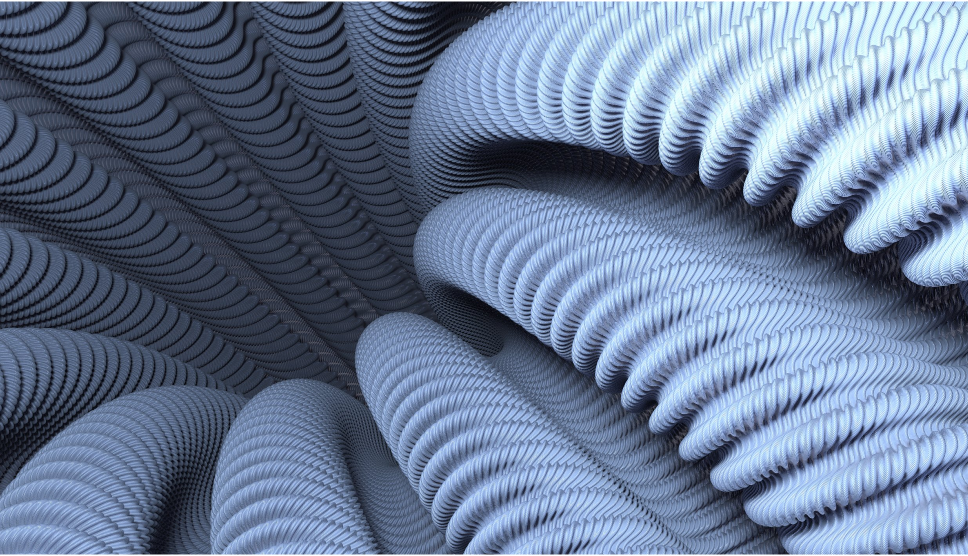

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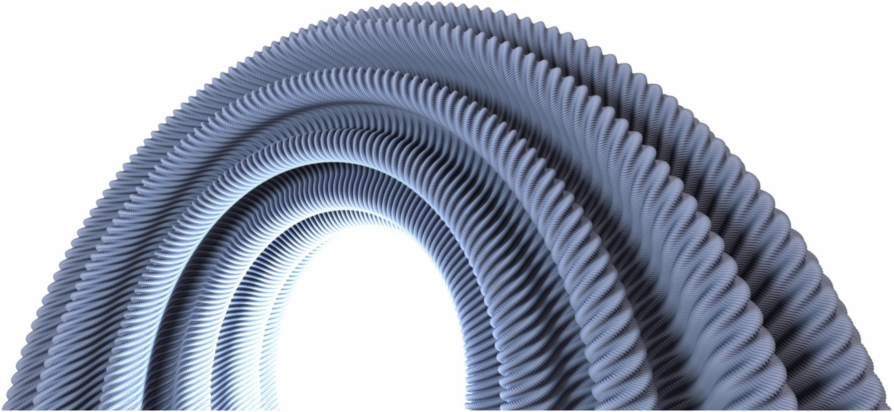

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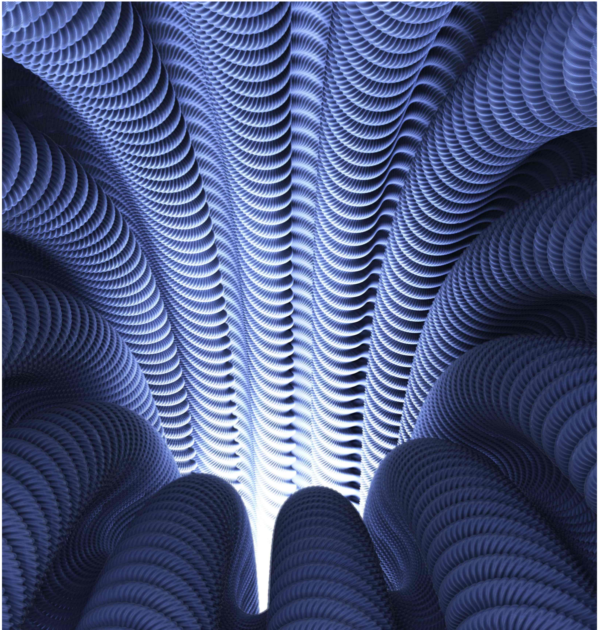

Thank you

More details :

- The HEVEA project : http://hevea.imag.fr
- A movie :
  http://www.youtube.com/watch?v=RYH_KXhF1SY
- Image des mathématiques : http://images.math.cnrs.fr/Gnash-un-tore-plat.html