Ridges of Polynomial Parametric Surfaces and Algorithms for the Analysis of an Algebraic Curve

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Journées Informatique et Géométrie JIG07

Outline

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1. Umbilics and Ridges on smooth generic surfaces
2. Implicit structure of ridges of a smooth parametric surface
3. Topological approximation on polynomial parametric surfaces
4. Generalization to an Arbitrary Curve

Monge patch

- Locally in the Monge coordinate system :

$$z = \frac{1}{2}(k_1 x^2 + k_2 y^2) + \frac{1}{6}(b_0 x^3 + 3b_1 x^2 y + 3b_2 x y^2 + b_3 y^3) + \frac{1}{24}(c_0 x^4 + 4c_1 x^3 y + 6c_2 x^2 y^2 + 4c_3 x y^3 + c_4 y^4) + \dots$$

- k_1 is the maximal principal curvature : blue curvature.
  k_2 is the minimal principal curvature : red curvature.
- **Umbilics** are characterized by $k_1 = k_2$
- Taylor expansion of the blue curvature along the blue curvature line going through the origin :

$$k_1(x) = k_1 + b_0 x + \dots$$

- Rk : switching the orientation of the principal directions reverts the sign of odd degree coefficients.

Blue (red) ridges

Expansion of k_1 along the blue line d_1:

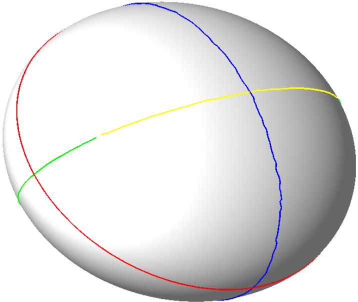
$$k_1(x) = k_1 + b_0 x + \frac{P_1}{2(k_1 - k_2)} x^2 + \dots \qquad P_1 = 3b_1^2 + (k_1 - k_2)(c_0 - 3k_1^3).$$

A blue ridge point is characterized by $b_0 =< \nabla k_1, d_1 >= 0$.

- elliptic if $P_1 < 0$ then the blue curvature is maximal along its line;
- hyperbolic if $P_1 > 0$ then the blue curvature is minimal along its line.

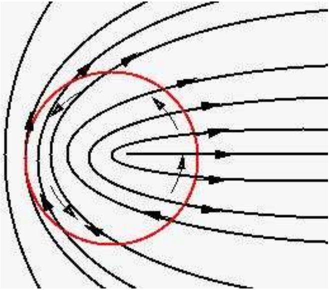
Remark : Two types of Red ridges

- Red curves (minimum).
- Yellow curves (maximum).



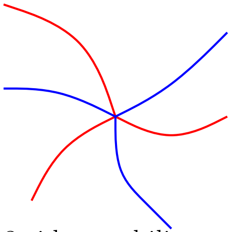
Orientation of principal directions

- The principal direction d_1 is not globally orientable.
- The sign of $b_0 =< \nabla k_1, d_1 >$ is not well defined.

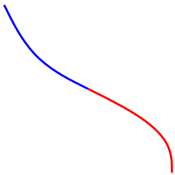


The principal field is not orientable around an umbilic.

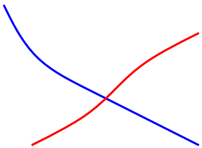
Special points of the ridge curve



3-ridge umbilic



1-ridge umbilic



Purple point

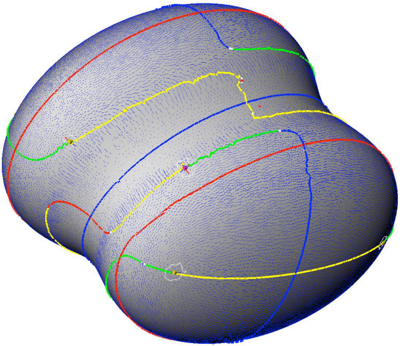
Difficulties of ridge extraction

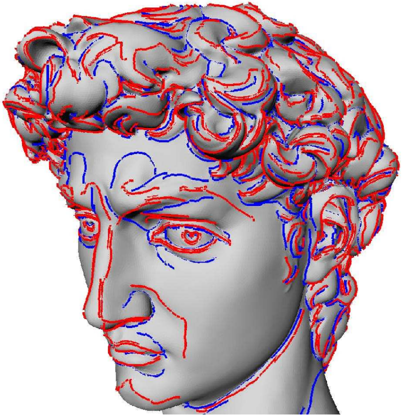
- Need derivatives up to the fourth order.
- Identify Umbilics and Purple points.
- Orientation problem.

Advertisement: ridges on meshes with CGAL!

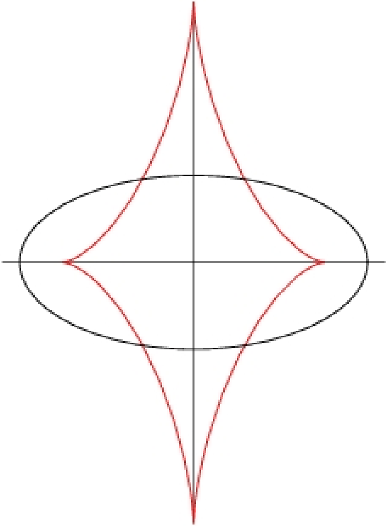
Brand new CGAL 3.3 Packages for

- Estimation of Local Differential Quantities from point clouds
- Extraction of Ridges and Umbilics on meshes





Alternative characterization of ridges



Focal curve (red) of an ellipse

- The blue focal surface is the locus of centers of blue osculating spheres.
- Blue ridges are in correspondence with singularities of the blue focal surface : they are the contact points of spheres centered on these singularities.

Previous work and contributions

- Ridges and umbilics as **singularities of focal surfaces** : 1-submanifold in $\mathbb{R}^7$ (Porteous).
- Ridges as **extrema of principal curvatures** : need different analysis **away from**, and **at** umbilics (Giblin).
- **Extraction** in practice (Morris, Thirion, Ohtake, Polthier) : problem of orientations, focus on a subset avoiding umbilics.

Our contributions : **Focus on topology**

- Meshes
  - **Sufficient conditions** to report the topology.
  - Application of the **jet fitting** for extraction.
- Parametric surfaces (joint work with J.-C. Faugère and F. Rouillier)
  - **Global** implicit structure in the parametric domain.
  - Computation of topology for **polynomial** surfaces.

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   -
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   -
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   -
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   -

Problem statement

- The surface is parameterized:

$$\Phi : (u, v) \in \mathbb{R}^2 \longrightarrow \Phi(u, v) \in \mathbb{R}^3$$

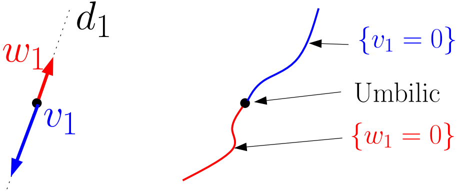
- Find a well defined function

$$P : (u, v) \in \mathbb{R}^2 \longrightarrow P(u, v) \in \mathbb{R}$$

such that $P = 0$ is **the ridge curve in the parametric domain.**

Solving the orientation problem

- Consider blue and red ridges together $< \nabla k_1, d_1 > \times < \nabla k_2, d_2 >$ is **orientation independent** (Gaussian Extremality, J-P. Thirion).
- Use **two vector fields** v_1 and w_1 orienting d_1 such that: $v_1 = w_1 = 0$ characterizes umbilics. (R. Morris)



- v_1 and w_1 are computed from the two dependent equations of the eigenvector system for d_1.

Some technicalities

- $p_2 = (k_1 - k_2)^2 = 0$ characterizes **umbilics.**
  It is a smooth function of the second derivatives of Φ.
- Define a, a', b, b' such that:

$$\sqrt{p_2}\langle \nabla k_1, v_1 \rangle = a\sqrt{p_2} + b \quad (1)$$

$$\sqrt{p_2}\langle \nabla k_1, w_1 \rangle = a'\sqrt{p_2} + b' \quad (2)$$

  These are **smooth functions** of the derivatives of Φ up to the third order.
- Note that the multiplication by $\sqrt{p_2}$ avoids the problem of non differentiability of k_1 at umbilics.

Main result

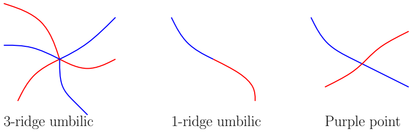
Theorem

The ridge curve has equation $P = ab' - a'b = 0$.

For a point of this set one has:

- *If $p_2 = 0$, the point is an* **umbilic**.
- *If $p_2 \neq 0$ then*
  - *If $ab \neq 0$ or $a'b' \neq 0$ then the sign of one these non-vanishing products gives the* **color of the ridge point**.
  - *Otherwise, $a = b = a' = b' = 0$ and the point is a* **purple** *point.*

Singularities of the ridge curve via zero dimensional systems



- 3-ridge umbilics

$$S_{3R} = \{p_2 = P = P_u = P_v = 0, \delta(P_3) > 0\}$$

- 1-ridge umbilics

$$S_{1R} = \{p_2 = P = P_u = P_v = 0, \delta(P_3) < 0\}$$

- Purple points

$$S_p = \{a = b = a' = b' = 0,\ \delta(P_2) > 0,\ p_2 \neq 0\}$$

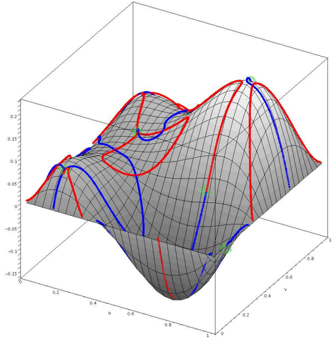
$\delta(P_2)$ ($\delta(P_3)$) is the discriminant of the quadratic (cubic) form of the 2^{nd} (3^{rd}) derivatives of P.

Example

For the degree 4 Bézier surface $\Phi(u,v) = (u, v, h(u,v))$ with

$$h(u,v) = 116u^4v^4 - 200u^4v^3 + 108u^4v^2 - 24u^4v - 312u^3v^4 + 592u^3v^3 - 360u^3v^2 + 80u^3v + 252u^2v^4 - 504u^2v^3 + 324u^2v^2 - 72u^2v - 56uv^4 + 112uv^3 - 72uv^2 + 16uv.$$

For a function h of total degree d, P has total degree at most $15d - 22$.



P is a bivariate polynomial

- total degree 84,
- degree 43 in u and v,
- 1907 terms,
- coefficients with up to 53 digits.

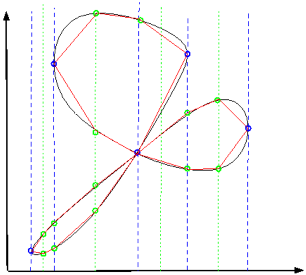
Outline

Topology of an algebraic curve

Polynomial parameterization $\Longrightarrow$ the ridge curve is **algebraic.**

Classical method (Cylindrical Algebraic Decomposition)

1. Compute x-coordinates of singular and critical points: α_i
   Assume generic position.
2. Compute intersection points between the curve and the fiber $x = \alpha_i$.
   Compute with polynomial with **algebraic** coefficients.
3. Connect points from fibers.



Our solution

1. Locate singular and critical points in 2D.
   No generic position assumption.
2. Compute regular intersection points between the curve and the fiber of singular and critical points.
   Compute with polynomial with **rational** coefficients.
3. Use the specific geometry of the ridge curve.
   We need to know how many branches of the curve pass through each singular point.
4. Connect points from fibers.

Algebraic tools

1. **Univariate** root isolation for polynomial with **rational** coefficients.
2. Solve zero dimensional systems I with **Rational Univariate Representation (RUR).**
   - Recast the problem to a univariate one with rational functions.
   - Let t be a separating polynomial and f_t the characteristic polynomial of the multiplication by t in the algebra $\mathbb{Q}[X_1,\ldots,X_n]/I$

$$\begin{array}{ccc} V(I)(\cap\mathbb{R}^n) & \approx & V(f_t)(\cap\mathbb{R}) \\ \alpha=(\alpha_1,\ldots,\alpha_n) & \rightarrow & t(\alpha) \\ \left(\frac{g_{t,X_1}(t(\alpha))}{g_{t,1}(t(\alpha))},\ldots,\frac{g_{t,X_n}(t(\alpha))}{g_{t,1}(t(\alpha))}\right) & \leftarrow & t(\alpha) \end{array}$$

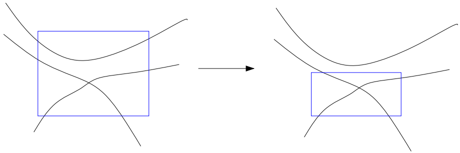
3. Evaluate the **sign of a polynomial** at the roots of another polynomial.

Step 1. Isolating study points

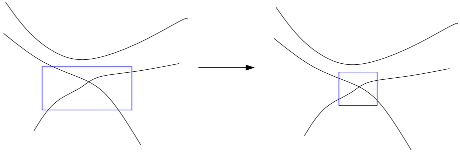
- Compute RUR of study points: 1-ridge umbilics, 3-ridge umbilics, purple points and critical points.
- Isolate study points in **boxes** $[x_i^1; x_i^2] \times [y_i^1; y_i^2]$, as small as desired.
- Identify study points with the same x-coordinate :
  - compute the characteristic polynomial of the multiplication by x for each system,
  - match the x-intervals with those of all the study points togather (given by the discriminant).

Step 2. Regularization of the study boxes

- Reduce a box until the right number of intersection points is reached wrt the study point type.

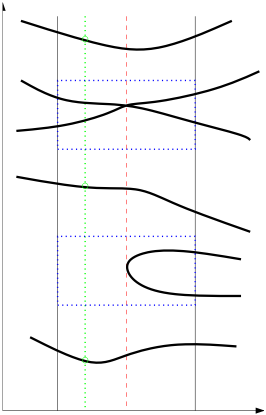


- Reduce to compute the number of branches connected to the right and left.



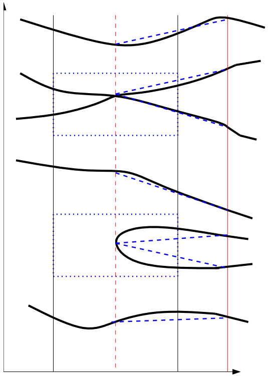
Step 3. Compute regular points in fibers

Outside study boxes, intersection between the curve and fibers of study points are **regular** points.
$\Longrightarrow$ Simple roots of the polynomial with rational coefficients $P(q, y)$ for any $q \in [x_i^1; x_i^2] \cap \mathbb{Q}$.



Step 4. Perform connections

- Add **intermediate** fibers.
- Compute the **color** of a ridge segment with a **sign** evaluation at the regular points of the intermediate fiber.
- One-to-one connection of points with multiplicity of branches.



Main result

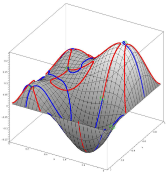
Theorem

The algorithm reports a graph isotopic to the ridge curve in the parametric domain.

In addition :

- The **position of the singularities** is as precise as the size of their boxes.
- A **more accurate geometrical representation** can be obtained with more intermediate fibers.

Example: degree 4 Bézier surface



- Computation with FGB (Groebner basis) and RS (RUR and isolation) softwares.
- Complexity measured by the number of **complex** roots.
- Domain of study $\mathcal{D} = [0,1] \times [0,1]$.

System	# of sol $\in \mathbb{C}$	$\in \mathbb{R}$	$\in \mathcal{D}$	FGb s.	RS s.
S_u	160	16	8	3	3
S_p	749	47	17	112	41
S_c	1432	44	19	270	166

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Local Geometry of Singularities

Main point:
Need to know the number of real branches connected to the right and to the left of each singular point.

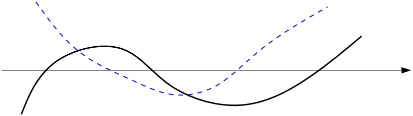
Solution:

- Rolle's Theorem: isolate roots of P by those of P'
- Teissier's Formula: make the connection between multiplicities of a singular point in a fiber and in some systems.

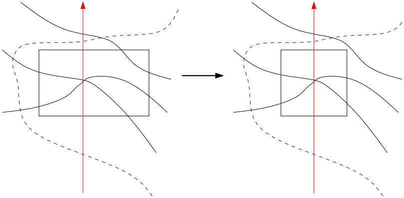
Application of Rolle's Theorem

Theorem

If β_i is a root of $P(y)$ of multiplicity k, then $P^{(k)}(y)$ vanishes between β_i and the other roots of P.



Apply to $P(y) = \partial^k f(\alpha_i, y)/\partial y^k$ for the singular point (α_i, β_j).



Teissier's Formula

Definition

$Int(f, g; p = (\alpha, \beta)) =$
multiplicity of intersection of f and g at p
multiplicity of p as zero of the ideal $< f, g >$

The RUR maps zeros of a system to roots of a univariate polynomial with the **same multiplicity.**

Theorem

$mult(f(\alpha, y); \beta)) = Int(f, f_y; p) - Int(f_x, f_y; p) - 1$

Note: $Int(f_x, f_y; p)$ = Milnor number of the singular point.

General Algorithm

1. Compute RURs for the systems $< f, f_y >$ and $< f_x, f_y >$
2. Processing singular points :
   - Compute $k = Int(f, f_y; p) - Int(f_x, f_y; p) - 1$
   - Refine boxes to avoid the curve $f_{y^k}(x, y)$
3. Count branches to the left and right of all critical points, refine the x-interval if needed.
4. Connect fibers.