Asymptotic properties of some random polytopes

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Random convex hulls in the unit-ball

Random tessellations in $\mathbb{R}^d$. Typical cell

Convergence of the boundary and consequences

Joint work with Tomasz Schreiber (Toruń University, Poland) and Joseph Yukich (Lehigh University, USA)
Outline

Random convex hulls in the unit-ball
    Poisson point process
    Model
    Historical background

Random tessellations in $\mathbb{R}^d$. Typical cell

Convergence of the boundary and consequences
Poisson point process

Poisson point process of intensity measure $\mu$

:= locally finite set $X$ of $\mathbb{R}^d$ such that

- $(X \cap B_1)$ Poisson r.v. of mean $\mu(B_1)$
- $(X \cap B_1), \cdots, (X \cap B_n)$ independent

$(B_1, \cdots, B_n \in B(\mathbb{R}^d), B_i \cap B_j = \emptyset, i \neq j)$

If $\mu = \lambda dx$, $X$ homogeneous of intensity $\lambda$
Random convex hulls in the unit-ball

- \( \mathbf{X} \) homogeneous Poisson point process in the unit-ball \( \mathbb{B}^d \) of intensity \( \lambda > 0 \) (resp. set of \( n \) independent and uniform points)

- Convex hull \( K_\lambda \) (resp. \( K'_n \)) of \( \mathbf{X} \)
Historical background

**Non-asymptotic results**

- **Sylvester** (1864): four-point problem

  \[
  \frac{2}{3} \leq p_C \leq 1 - \frac{35}{12\pi^2}
  \]

  \(p_C\) := probability that 4 i.i.d. uniform points in a convex set \(C \subset \mathbb{R}^2\) are extreme

- **Wendel** (1962): \(\mathbb{P}\{0 \notin K'_n\} = 2^{-(n-1)} \sum_{k=0}^{d-1} \binom{n-1}{k}\) \((n \geq d)\)

- **Efron** (1965): \(f_0(\cdot) := \# \) vertices, \(V_d(\cdot) := \)volume, \(\kappa_d := V_d(\mathbb{B}^d)\)

  \[
  \kappa_d \mathbb{E} f_0(K'_n) = n \left( \kappa_d - \mathbb{E} V_d(K'_{n-1}) \right)
  \]

- **Buchta** (2005): identities relating higher moments
Historical background

**LLN**


\[ \kappa_d - V_d(K_\lambda) \approx c_d \lambda^{-\frac{2}{d+1}}, \quad f_0(K_\lambda) \approx c'_d \lambda^{\frac{d-1}{d+1}} \]

**CLT**

Reitzner (2005), Vu (2006), Schreiber & Yukich (2008), Bárány & Reitzner (2009), Bárány et al. (2009)

**Aims**

Description of the boundary of \( K_\lambda \)
CLT with variance asymptotics
Outline

Random convex hulls in the unit-ball

Random tessellations in $\mathbb{R}^d$. Typical cell
- Poisson-Voronoi tessellations
- Poisson hyperplane tessellations
- Zero-cell and typical cell
- Distributional and asymptotic results
- Connection with random convex hulls

Convergence of the boundary and consequences
Poisson-Voronoi tessellations

- **X** Poisson point process in $\mathbb{R}^d$ of intensity measure $d\lambda$

- For a nucleus $x \in X$, cell

$$C(x|X) := \{ y \in \mathbb{R}^d : \|y - x\| \leq \|y - x'\| \ \forall x' \in X\}$$

- **Tessellation**: set of cells $C(x|X)$

**Properties**: isotropic, stationary, ergodic.

**References**: Descartes (1644), Gilbert (1961), Okabe et al. (1992)
Poisson hyperplane tessellations

- **X**: Poisson point process in $\mathbb{R}^d$ of intensity measure $||x||^{\alpha-d}dx$ ($\alpha \geq 1$)

- For $x \in X$, polar hyperplane

  \[ H_x := \{ y \in \mathbb{R}^d : \langle y - x, x \rangle = 0 \} \]

- **Tessellation**: set of connected components of $\mathbb{R}^d \setminus \bigcup_{x \in X} H_x$

*Properties*: isotropic, stationary iff $\alpha = 1$, ergodic.

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Zero-cell and typical cell

- **Zero-cell**: unique cell containing the origin
  
  (Crofton cell for a stationary Poisson hyperplane tessellation)

- **Typical cell**: cell $C$ 'chosen uniformly' among all cells
  
  Three equivalent ways to define its distribution:
  
  - limits of ergodic means *(Reference: Cowan (1977))*
  
  - use of a Palm measure *(Reference: Mecke (1967))*
  
  - explicit realization:

  Poisson-Voronoi tessellation:

  $$ C \overset{D}{=} C(0|\mathbf{X} \cup \{0\}) $$

  zero-cell of a hyperplane tessellation ($\alpha = d$)
Explicit integral formula for the distribution of the number of hyperfaces and conditional distribution of the cell

**D. G. Kendall’s conjecture (40s):**

'when it is large, the zero-cell is close to the spherical shape'.

(Hug, Reitzner & Schneider, 2004)

Particular case:
zero-cell conditioned on having large inradius $R_m$
\( R_m := \text{inradius} \) of the zero-cell \( C_0 \) of a hyperplane tessellation

Effect of the inversion \( I(x) := x/\|x\|^2, \; x \neq 0 \) on \( R_m^{-1} C_0 \):

- Point process outside \( B^d \) \( \xrightarrow{I} \) point process in \( B^d \)
- Line process \( \xrightarrow{I} \) Germ-grain process in \( B^d \)
- Number of hyperfaces of \( C_0 \) = number of extreme points in \( B^d \)
Outline

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Random tessellations in $\mathbb{R}^d$. Typical cell

Convergence of the boundary and consequences
  Radius-vector function and support function
  Scaling limits
  Paraboloid growth process
  Paraboloid hull process
  Central limit theorems and variance asymptotics
  Brownian limits
  Extreme values
  Dual results for zero-cells of hyperplane tessellations
Radius-vector function and support function

Defect radius-vector function

$$r_\lambda(u) := 1 - \sup\{ r > 0 : ru \in K_\lambda \}, \quad u \in S^{d-1}$$

Defect support function

$$s_\lambda(u) := 1 - \sup\{ \langle x, u \rangle : x \in K_\lambda \} = 1 - \sup\{ r > 0 : ru \in \bigcup_{x \in K_\lambda} B(x/2, \|x\|/2) \}$$
Scaling limits

\[ \hat{r}_\lambda(v) := \lambda^{\frac{2}{d+1}} r_\lambda(\exp_{d-1}(\lambda^{-\frac{1}{d+1}} v)), \quad \hat{s}_\lambda(v) := \lambda^{\frac{2}{d+1}} r_\lambda(\exp_{d-1}(\lambda^{-\frac{1}{d+1}} v)) \]

\[ \exp_{d-1} : \mathbb{R}^{d-1} \simeq T_{u_0} \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1} \text{ exponential map at some fixed } u_0 \in \mathbb{S}^{d-1} \]

\[ \hat{r}_\lambda \text{ and } \hat{s}_\lambda \text{ converge in distribution in } (C(B_{d-1}(0, R)), \| \cdot \|_\infty), \ R > 0. \]
Paraboloid growth process

\[ \mathcal{P} := \text{homogeneous Poisson point process on } \mathbb{R}^{d-1} \times \mathbb{R}_+ \]

\[ \Pi^\uparrow := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+: h \geq \|v\|^2 / 2\} \]

\[ \Psi := \bigcup_{x \in \mathcal{P}} (x + \Pi^\uparrow) \]

The boundary \( \partial \Psi \) is the limit of \( \hat{s}_\lambda \).
Paraboloid hull process

\[ \mathcal{P} := \text{homogeneous Poisson point process on } \mathbb{R} \times \mathbb{R}_+ \]
\[ \Pi^\downarrow := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h \leq -\|v\|^2/2\} \]
\[ \Phi := \bigcup_{x \in \mathbb{R}^{d-1} \times \mathbb{R}_+: x + \Pi^\downarrow \cap \mathcal{P} = \emptyset} (x + \Pi^\downarrow) \]

The boundary \( \partial \Phi \) is the limit of \( \hat{r}_\lambda \).
Central limit theorems and variance asymptotics

\[ f_k(K_\lambda) := \text{number of } k\text{-faces of } K_\lambda \]

\[ V_k(K_\lambda) := k\text{-th intrinsic volume of } K_\lambda \quad (0 \leq k \leq d) \]

\[ \lambda^{-\frac{d-1}{d+1}} \text{Var}[f_k(K_\lambda)] \to \sigma_{f_k}^2 \quad \text{and} \quad \lambda^{\frac{d+3}{d+1}} \text{Var}[V_k(K_\lambda)] \to \sigma_{V_k}^2. \]

\[ (\sigma_{f_k}^2, \sigma_{V_k}^2 \text{ explicit in terms of } \Phi \text{ and } \Psi) \]

\[ f_k(K_\lambda) \text{ and } V_k(K_\lambda) \text{ satisfy CLTs with convergence rates.} \]

**Idea:** each quantity can be written \( \sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda) \) where \( \xi(x, \mathcal{P}_\lambda) = 0 \) if \( x \) not extreme and where \( \xi(x, \mathcal{P}_\lambda) \) depends on \( \mathcal{P}_\lambda \) only in a vicinity of \( x \).
Brownian limits
Brownian limits

\[ V_\lambda(v) := \int_{\exp([0,v])} r_\lambda(w) d\sigma_{d-1}(w), \quad W_\lambda(v) := \int_{\exp([0,v])} s_\lambda(w) d\sigma_{d-1}(w) \]

\((\sigma_{d-1} := \text{uniform measure on } \mathbb{S}^{d-1})\)

\[ \lambda \frac{d+3}{2d+2} [V_\lambda(\cdot) - \mathbb{E} V_\lambda(\cdot)] \quad (\text{resp. } \lambda \frac{d+3}{2d+2} [W_\lambda(\cdot) - \mathbb{E} W_\lambda(\cdot)]) \]

converges in \(\mathcal{C}(\mathbb{R}^{d-1})\) to a Brownian sheet of explicit variance.
Extreme values

\[ S_\lambda := \sup_{u \in S^{d-1}} s_\lambda(u) = \sup_{u \in S^{d-1}} r_\lambda(u) \]

\[ S_\lambda = \lambda^{-\frac{2}{d+1}} \left[ C_d^{(1)} \log(\lambda) + C_d^{(2)} \log(\log(\lambda)) + C_d^{(3)} (K_d + \xi_\lambda) \right]^{\frac{2}{d+1}} \]

where \( \xi_\lambda \) converges in distribution to the Gumbel law.
Dual results for zero-cells of hyperplane tessellations

\[ C_0^\alpha := \text{zero cell from a hyperplane process of intensity } \|x\|^{\alpha-d} \, dx, \quad \alpha \geq 1 \]

\[ R_{\text{int}}^\alpha := \sup \{ r > 0 : B(0, r) \subset C_0^\alpha \}, \quad C_r^\alpha := C_0^\alpha \text{ conditioned on } \{ R_{\text{int}}^\alpha \geq r \} \]

\[ C_{\alpha,r} := r^{-1} C_r^\alpha \]

*Examples:* Poisson-Voronoi typical cell \((\alpha = d)\), Crofton cell \((\alpha = 1)\)
Dual results for zero-cells of hyperplane tessellations

Dual results

- Convergence to the paraboloid growth process
- Extreme results for the circumscribed radius
- CLT for the number of $k$-faces
- Brownian limit for the 'volume process'
Prospects

- Point process of extremal points
- Depoissonization
- Multivariate CLT
- Other convex sets
Thank you for your attention!