Asymptotic properties of some random polytopes

Pierre Calka

UNIVERSITÉ DE ROUEN

INRIA Sophia Antipolis, 9 December 2010

Outline

Random convex hulls in the unit-ball

Random tessellations in $\mathbb{R}^d$. Typical cell

Convergence of the boundary and consequences

Joint work with **Tomasz Schreiber** (Toruń University, Poland) and **Joseph Yukich** (Lehigh University, USA)

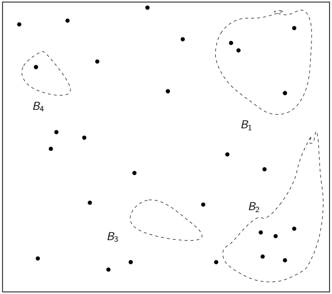
Outline

Random convex hulls in the unit-ball
- Poisson point process
- Model
- Historical background

Random tessellations in $\mathbb{R}^d$. Typical cell

Convergence of the boundary and consequences

Poisson point process



Poisson point process of intensity measure μ

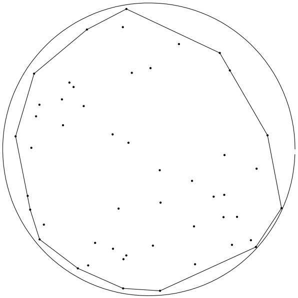
:=locally finite set $\mathbf{X}$ of $\mathbb{R}^d$ such that

- $\#(\mathbf{X} \cap B_1)$ Poisson r.v. of mean $\mu(B_1)$
- $\#(\mathbf{X} \cap B_1), \cdots, \#(\mathbf{X} \cap B_n)$ independent

$(B_1, \cdots, B_n \in \mathcal{B}(\mathbb{R}^d),\ B_i \cap B_j = \emptyset,\ i \neq j)$

If $\mu = \lambda \mathrm{d}x$, $\mathbf{X}$ homogeneous of intensity λ

Random convex hulls in the unit-ball



- $\mathbf{X}$ homogeneous Poisson point process in the unit-ball $\mathbb{B}^d$ of intensity $\lambda > 0$ (resp. set of n independent and uniform points)
- Convex hull K_λ (resp. K'_n) of $\mathbf{X}$

Historical background

Non-asymptotic results

- **Sylvester** (1864): four-point problem

$$\frac{2}{3} \le p_C \le 1 - \frac{35}{12\pi^2}$$

$p_C :=$ probability that 4 i.i.d. uniform points in a convex set $C \subset \mathbb{R}^2$ are extreme

- **Wendel** (1962): $\mathbb{P}\{0 \notin K_n'\} = 2^{-(n-1)} \sum_{k=0}^{d-1} \binom{n-1}{k}$ $(n \ge d)$

- **Efron** (1965): $f_0(\cdot) :=$ # vertices, $V_d(\cdot)$:=volume, $\kappa_d := V_d(\mathbb{B}^d)$

$$\kappa_d \mathbb{E} f_0(K_n') = n\left(\kappa_d - \mathbb{E} V_d(K_{n-1}')\right)$$

Buchta (2005): identities relating higher moments

Historical background

LLN

Rényi & Sulanke (1963), **Wieacker** (1978), **Schneider & Wieacker** (1980), **Schneider** (1988), **Bárány & Buchta** (1993)

$$\kappa_d - V_d(K_\lambda) \approx c_d \lambda^{-\frac{2}{d+1}}, \quad f_0(K_\lambda) \approx c_d' \lambda^{\frac{d-1}{d+1}}$$

CLT

Reitzner (2005), **Vu** (2006), **Schreiber & Yukich** (2008), **Bárány & Reitzner** (2009), **Bárány et al.** (2009)

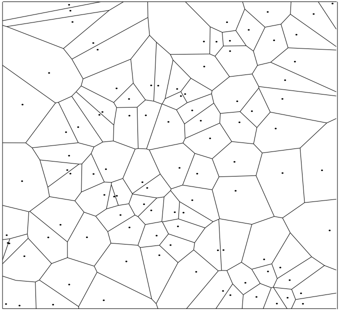
Aims

Description of the boundary of K_λ

CLT with variance asymptotics

Outline

Poisson-Voronoi tessellations



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^d$ of intensity measure $\mathrm{d}x$
- For a nucleus $x \in \mathbf{X}$, cell

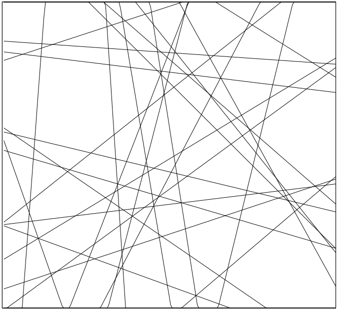
$$C(x|\mathbf{X}) := \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - x'\| \; \forall x' \in \mathbf{X}\}$$

- Tessellation: set of cells $C(x|\mathbf{X})$

Properties: isotropic, stationary, ergodic.

References: **Descartes** (1644), **Gilbert** (1961), **Okabe et al.** (1992)

Poisson hyperplane tessellations



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^d$ of intensity measure $\|x\|^{\alpha-d}\mathrm{d}x$ $_{(\alpha \geq 1)}$
- For $x \in \mathbf{X}$, polar hyperplane

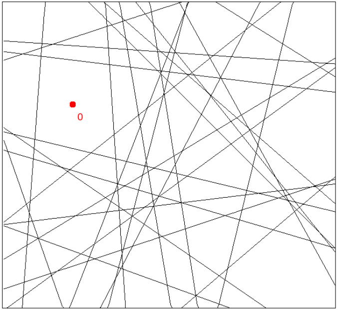
  $$H_x := \{y \in \mathbb{R}^d : \langle y - x, x \rangle = 0\}$$

- Tessellation: set of connected components of $\mathbb{R}^d \setminus \cup_{x \in \mathbf{X}} H_x$

Properties: isotropic, stationary iff $\alpha = 1$, ergodic.

References: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

Poisson hyperplane tessellations



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^d$ of intensity measure $\|x\|^{\alpha-d}\mathrm{d}x$ $(\alpha \geq 1)$
- For $x \in \mathbf{X}$, polar hyperplane

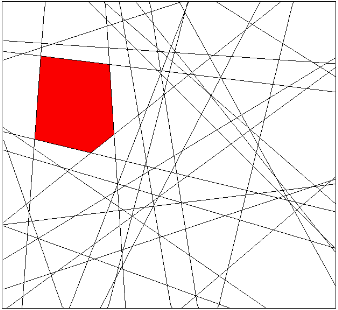
$$H_x := \{y \in \mathbb{R}^d : \langle y - x, x \rangle = 0\}$$

- Tessellation: set of connected components of $\mathbb{R}^d \setminus \cup_{x \in \mathbf{X}} H_x$

Properties: isotropic, stationary iff $\alpha = 1$, ergodic.

References: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

Poisson hyperplane tessellations



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^d$ of intensity measure $\|x\|^{\alpha-d}\mathrm{d}x$ $(\alpha \geq 1)$
- For $x \in \mathbf{X}$, polar hyperplane

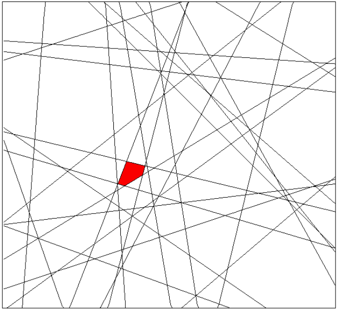
$$H_x := \{y \in \mathbb{R}^d : \langle y - x, x \rangle = 0\}$$

- Tessellation: set of connected components of $\mathbb{R}^d \setminus \cup_{x \in \mathbf{X}} H_x$

Properties: isotropic, stationary iff $\alpha = 1$, ergodic.

References: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

Poisson hyperplane tessellations



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^d$ of intensity measure $\|x\|^{\alpha-d}\mathrm{d}x$ $(\alpha \geq 1)$
- For $x \in \mathbf{X}$, polar hyperplane

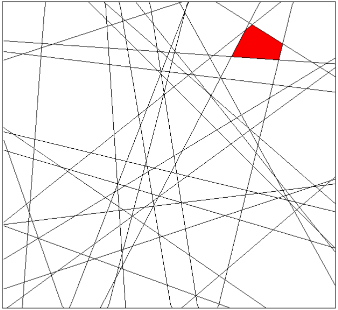
$$H_x := \{y \in \mathbb{R}^d : \langle y - x, x\rangle = 0\}$$

- Tessellation: set of connected components of $\mathbb{R}^d \setminus \cup_{x\in\mathbf{X}} H_x$

Properties: isotropic, stationary iff $\alpha = 1$, ergodic.

References: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

Poisson hyperplane tessellations



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^d$ of intensity measure $\|x\|^{\alpha-d}\mathrm{d}x$ $(\alpha \geq 1)$
- For $x \in \mathbf{X}$, polar hyperplane

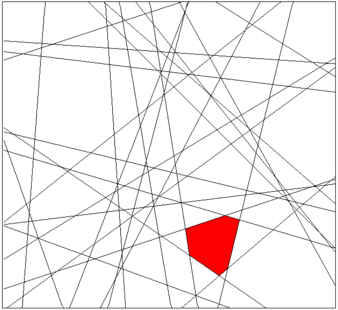
$$H_x := \{y \in \mathbb{R}^d : \langle y - x, x \rangle = 0\}$$

- Tessellation: set of connected components of $\mathbb{R}^d \setminus \cup_{x \in \mathbf{X}} H_x$

Properties: isotropic, stationary iff $\alpha = 1$, ergodic.

References: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

Poisson hyperplane tessellations



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^d$ of intensity measure $\|x\|^{\alpha-d}\mathrm{d}x$ $(\alpha \geq 1)$
- For $x \in \mathbf{X}$, polar hyperplane

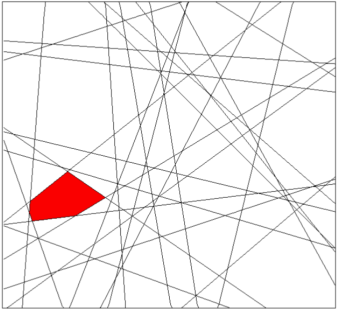
$$H_x := \{y \in \mathbb{R}^d : \langle y - x, x \rangle = 0\}$$

- Tessellation: set of connected components of $\mathbb{R}^d \setminus \cup_{x\in\mathbf{X}} H_x$

Properties: isotropic, stationary iff $\alpha = 1$, ergodic.

References: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

Poisson hyperplane tessellations



- $\mathbf{X}$ Poisson point process in $\mathbb{R}^d$ of intensity measure $\|x\|^{\alpha-d}\mathrm{d}x$ $(\alpha \geq 1)$
- For $x \in \mathbf{X}$, polar hyperplane

$$H_x := \{y \in \mathbb{R}^d : \langle y - x, x \rangle = 0\}$$

- Tessellation: set of connected components of $\mathbb{R}^d \setminus \cup_{x \in \mathbf{X}} H_x$

Properties: isotropic, stationary iff $\alpha = 1$, ergodic.

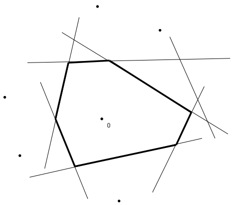
References: **Meijering** (1953), **Miles** (1964), **Stoyan et al.** (1987)

Zero-cell and typical cell

- Zero-cell: unique cell containing the origin

  (Crofton cell for a stationary Poisson hyperplane tessellation)

- Typical cell: cell $\mathcal{C}$ 'chosen uniformly' among all cells
  Three equivalent ways to define its distribution:
  - limits of ergodic means (*Reference*: **Cowan** (1977))
  - use of a Palm measure (*Reference*: **Mecke** (1967))
  - explicit realization:



Poisson-Voronoi tessellation:

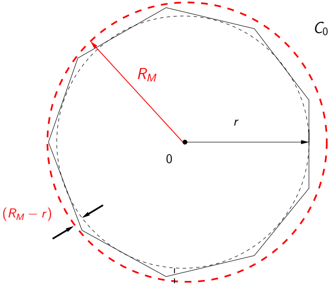
$$\mathcal{C} \overset{D}{=} C(0|\mathbf{X} \cup \{0\})$$

zero-cell of a hyperplane tessellation ($\alpha = d$)

Distributional and asymptotic results

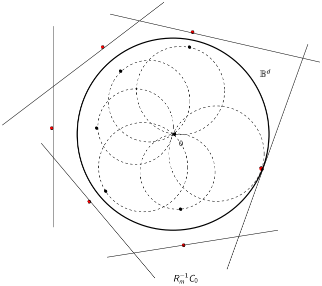
- Explicit integral formula for the distribution of the number of hyperfaces and conditional distribution of the cell
- **D. G. Kendall**'*s conjecture (40s):*

  'when it is large, the zero-cell is close to the spherical shape'.
  (**Hug**, **Reitzner** & **Schneider**, 2004)



Particular case:
zero-cell conditioned on having large inradius R_m

Connection with random convex hulls



R_m:= inradius of the zero-cell C_0 of a hyperplane tessellation
Effect of the inversion $I(x) := x/\|x\|^2$, $x \neq 0$ on $R_m^{-1}C_0$:

- Point process outside $\mathbb{B}^d \xrightarrow{I}$ point process in $\mathbb{B}^d$
- Line process $\xrightarrow{I}$ Germ-grain process in $\mathbb{B}^d$
- Number of hyperfaces of C_0 = number of extreme points in $\mathbb{B}^d$

Outline

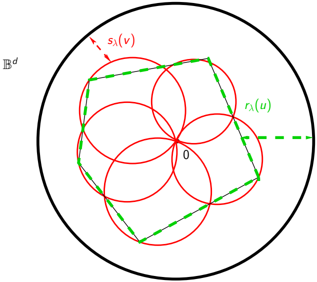
Random convex hulls in the unit-ball

Random tessellations in $\mathbb{R}^d$. Typical cell

Convergence of the boundary and consequences

- Radius-vector function and support function
- Scaling limits
- Paraboloid growth process
- Paraboloid hull process
- Central limit theorems and variance asymptotics
- Brownian limits
- Extreme values
- Dual results for zero-cells of hyperplane tessellations

Radius-vector function and support function



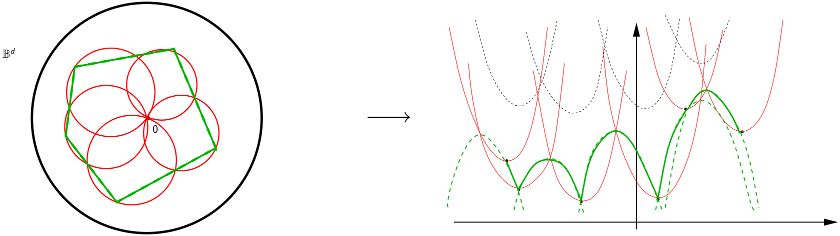
Defect radius-vector function

$$r_\lambda(u) := 1 - \sup\{r > 0 : ru \in K_\lambda\}, \ \ u \in \mathbb{S}^{d-1}$$

Defect support function

$$s_\lambda(u) := 1 - \sup\{\langle x, u\rangle : x \in K_\lambda\} = 1 - \sup\{r > 0 : ru \in \bigcup_{x \in K_\lambda} B(x/2, \|x\|/2)\}$$

Scaling limits



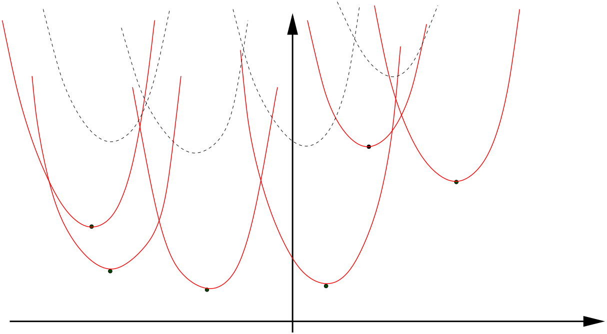
$$\hat r_\lambda(v) := \lambda^{\frac{2}{d+1}} r_\lambda(\exp_{d-1}(\lambda^{-\frac{1}{d+1}} v)),\ \hat s_\lambda(v) := \lambda^{\frac{2}{d+1}} r_\lambda(\exp_{d-1}(\lambda^{-\frac{1}{d+1}} v))$$

$(v \in \mathbb{R}^{d-1})$

$\exp_{d-1} : \mathbb{R}^{d-1} \simeq T_{u_0}\mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ exponential map at some fixed $u_0 \in \mathbb{S}^{d-1}$

$\hat r_\lambda$ and $\hat s_\lambda$ converge in distribution in $(\mathcal{C}(B_{d-1}(0,R)), \|\cdot\|_\infty)$, $R > 0$.

Paraboloid growth process



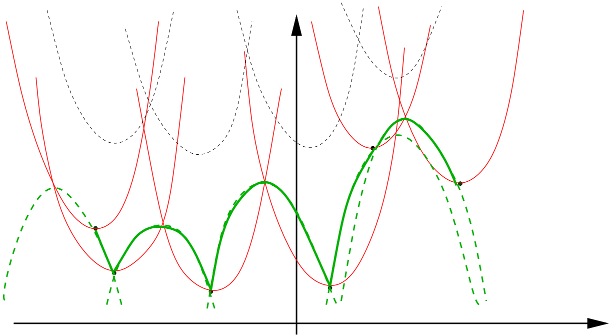
$\mathcal{P} :=$ homogeneous Poisson point process on $\mathbb{R}^{d-1} \times \mathbb{R}_+$

$\Pi^{\uparrow} := \{(v,h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h \geq \|v\|^2/2\}$

$\Psi := \bigcup_{x \in \mathcal{P}} (x + \Pi^{\uparrow})$

The boundary $\partial\Psi$ is the limit of $\hat{s}_\lambda$.

Paraboloid hull process



$\mathcal{P}$:= homogeneous Poisson point process on $\mathbb{R} \times \mathbb{R}_+$
$\Pi^{\downarrow} := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h \leq -\|v\|^2/2\}$
$\Phi := \bigcup_{x \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : x + \Pi^{\downarrow} \cap \mathcal{P} = \emptyset} (x + \Pi^{\downarrow})$

The boundary $\partial\Phi$ is the limit of $\hat{r}_\lambda$.

Central limit theorems and variance asymptotics

$f_k(K_\lambda) :=$ number of k-faces of K_λ

$V_k(K_\lambda):=$ k-th intrinsic volume of K_λ $\qquad (0 \le k \le d)$

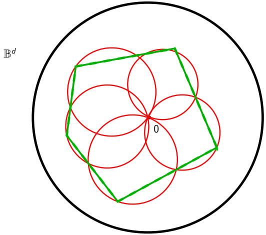
$$\lambda^{-\frac{d-1}{d+1}} \mathrm{Var}[f_k(K_\lambda)] \to \sigma_{f_k}^2 \text{ and } \lambda^{\frac{d+3}{d+1}} \mathrm{Var}[V_k(K_\lambda)] \to \sigma_{V_k}^2.$$

($\sigma_{f_k}^2$, $\sigma_{V_k}^2$ explicit in terms of Φ and Ψ)

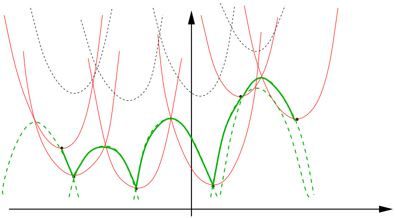
$f_k(K_\lambda)$ and $V_k(K_\lambda)$ satisfy CLTs with convergence rates.

Idea: each quantity can be written $\sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda)$ where $\xi(x, \mathcal{P}_\lambda) = 0$ if x not extreme and where $\xi(x, \mathcal{P}_\lambda)$ depends on $\mathcal{P}_\lambda$ only in a vicinity of x.

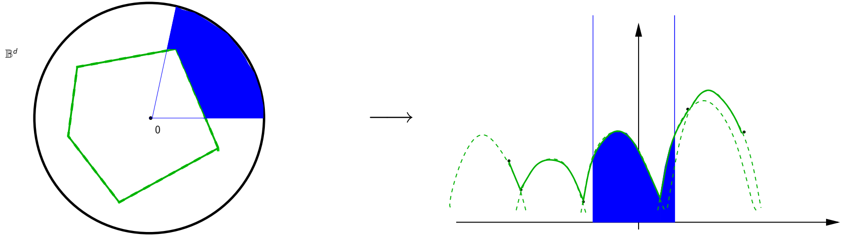
Brownian limits



$\longrightarrow$



Brownian limits



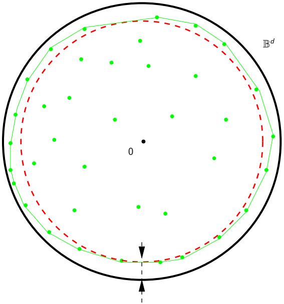
$$V_\lambda(v) := \int_{\exp([0,v])} r_\lambda(w) \mathrm{d}\sigma_{d-1}(w), \; W_\lambda(v) := \int_{\exp([0,v])} s_\lambda(w) \mathrm{d}\sigma_{d-1}(w)$$

(σ_{d-1} := uniform measure on $\mathbb{S}^{d-1}$)

$$\lambda^{\frac{d+3}{2d+2}}[V_\lambda(\cdot) - \mathbb{E}V_\lambda(\cdot)] \quad (\text{resp. } \lambda^{\frac{d+3}{2d+2}}[W_\lambda(\cdot) - \mathbb{E}W_\lambda(\cdot)])$$

converges in $\mathcal{C}(\mathbb{R}^{d-1})$ to a Brownian sheet of explicit variance.

Extreme values

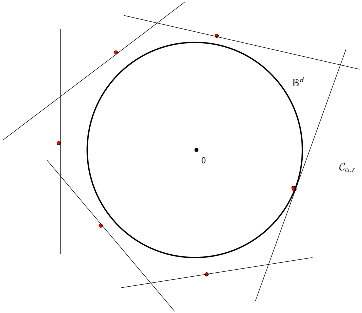


$S_\lambda := \sup_{\mathbf{u}\in\mathbb{S}^{d-1}} s_\lambda(\mathbf{u}) = \sup_{\mathbf{u}\in\mathbb{S}^{d-1}} r_\lambda(\mathbf{u})$

$$S_\lambda = \lambda^{-\frac{2}{d+1}}[C_d^{(1)}\log(\lambda) + C_d^{(2)}\log(\log(\lambda)) + C_d^{(3)}(K_d + \xi_\lambda)]^{\frac{2}{d+1}}$$

where ξ_λ converges in distribution to the Gumbel law.

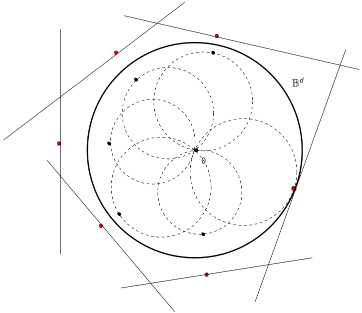
Dual results for zero-cells of hyperplane tessellations



C_0^α := zero cell from a hyperplane process of intensity $\|x\|^{\alpha-d}\mathrm{d}x$, $\alpha \geq 1$
$R_{int}^\alpha := \sup\{r > 0 : B(0,r) \subset C_0^\alpha\}$, $C_r^\alpha := C_0^\alpha$ conditioned on $\{R_{int}^\alpha \geq r\}$

$\mathcal{C}_{\alpha,r} := r^{-1} C_r^\alpha$
Examples: Poisson-Voronoi typical cell ($\alpha = d$), Crofton cell ($\alpha = 1$)

Dual results for zero-cells of hyperplane tessellations



Dual results

- Convergence to the paraboloid growth process
- Extreme results for the circumscribed radius
- CLT for the number of k-faces
- Brownian limit for the 'volume process'

Prospects

- Point process of extremal points
- Depoissonization
- Multivariate CLT
- Other convex sets

Thank you for your attention!