Don't worry, be noisy

The Effect of Noise on the Number of Extreme Points

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Motivation

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Worst case analysis

vs practical behavior

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Worst case analysis

Delaunay

$$\Omega\left(n^{\left\lceil \frac{d}{2} \right\rceil}\right)$$

vs practical behavior

almost linear ?

Motivation

Worst case analysis

Delaunay	Convex hull
$\Omega\left(n^{\left\lceil\frac{d}{2}\right\rceil}\right)$	$\Omega\left(n^{\left\lfloor\frac{d}{2}\right\rfloor}\right)$

vs practical behavior

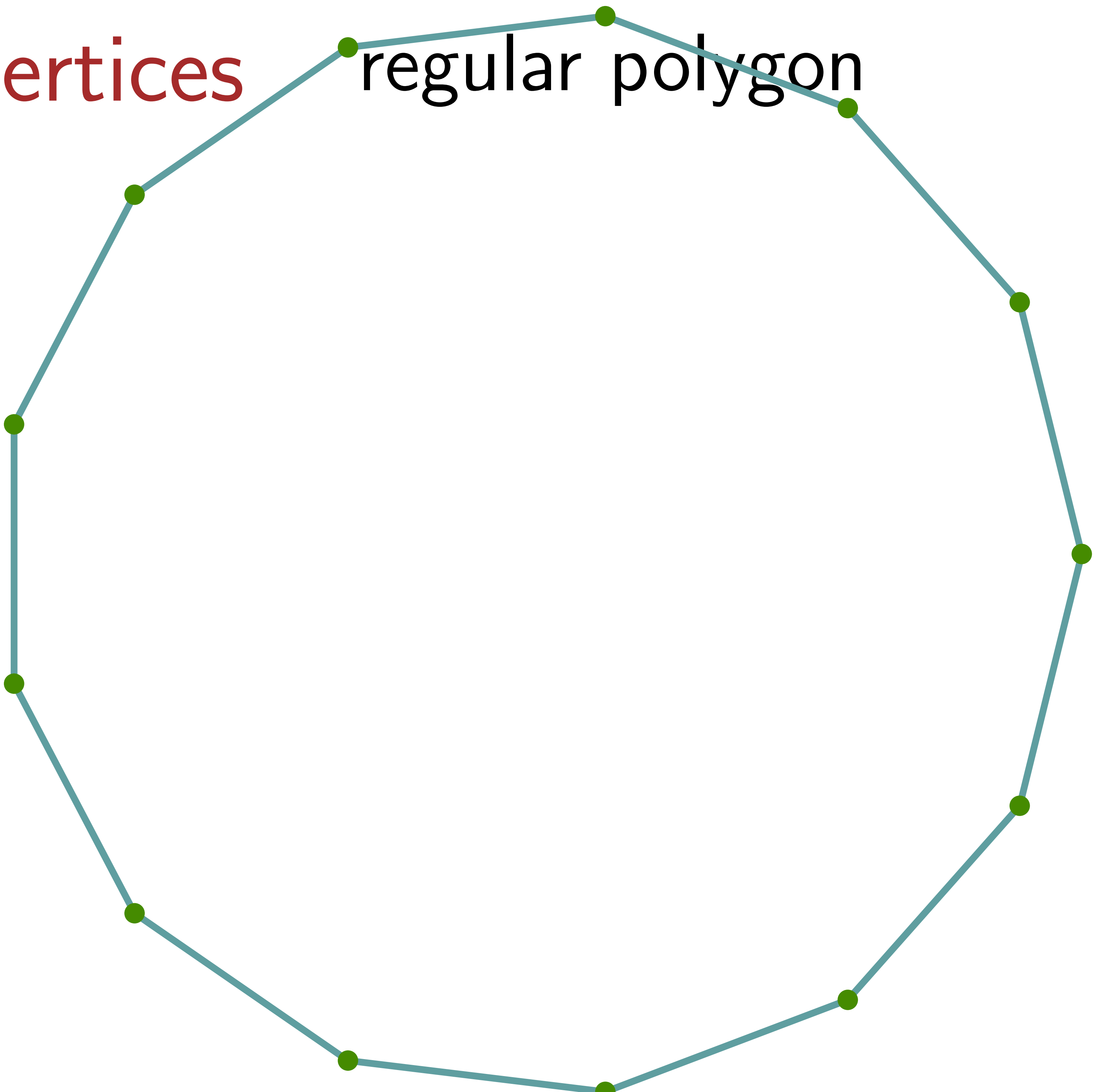
almost linear ?	small ?

Problem in this talk

Worst case analysis

$\sharp$ of convex hull vertices

$\Omega(n)$



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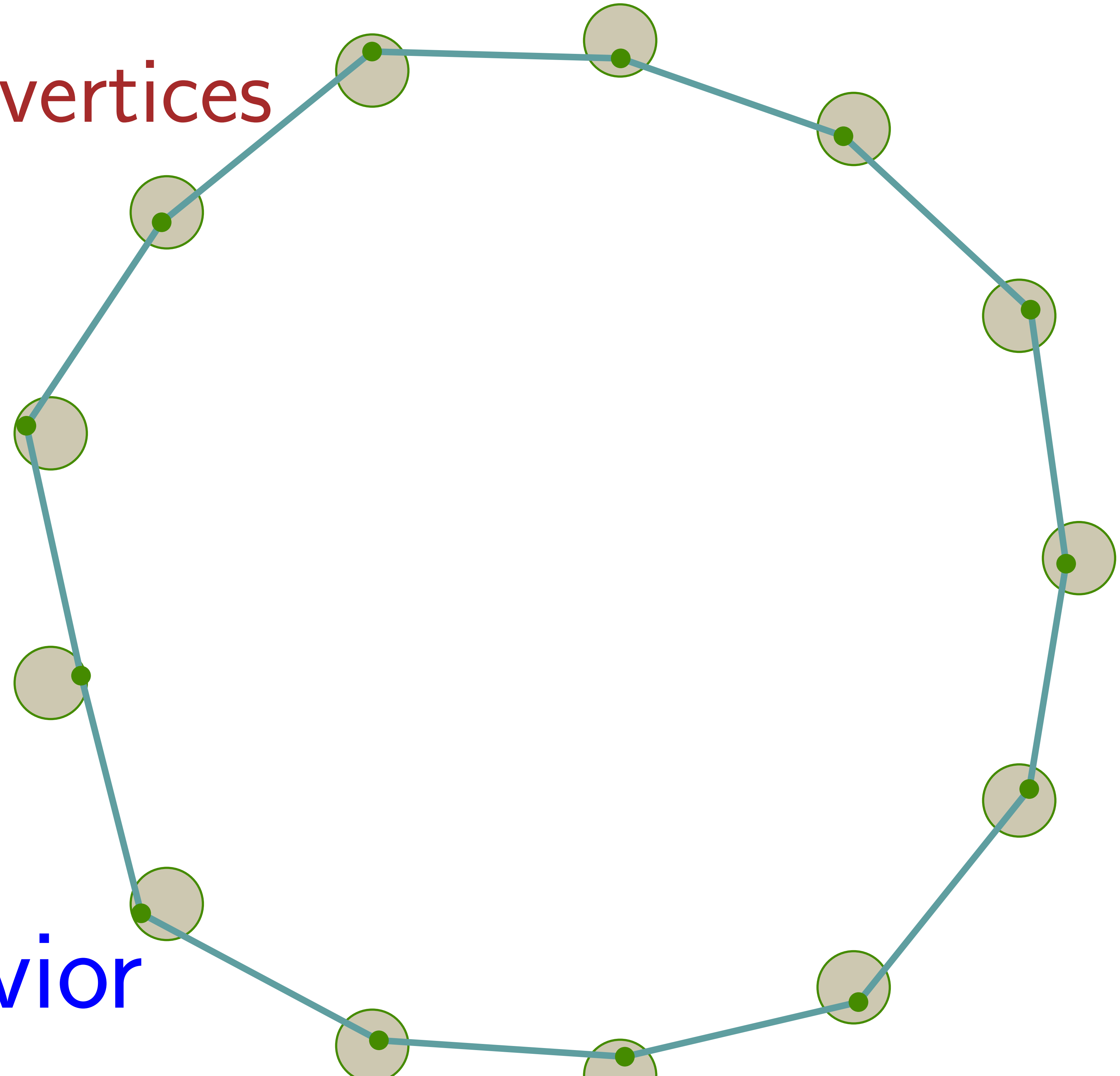

vs noisy behavior

Problem in this talk

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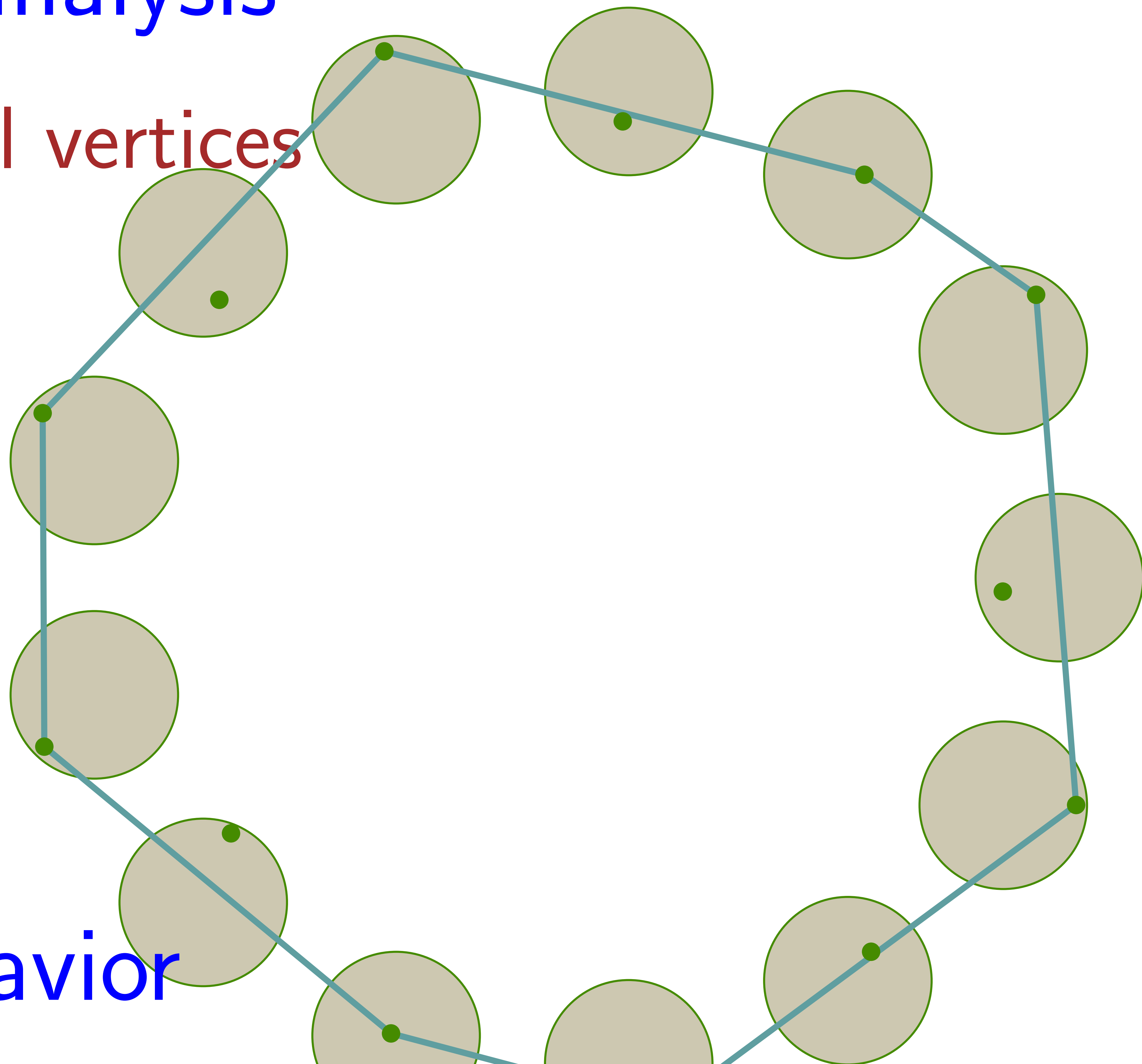
♯ of convex hull vertices

$\Omega(n)$



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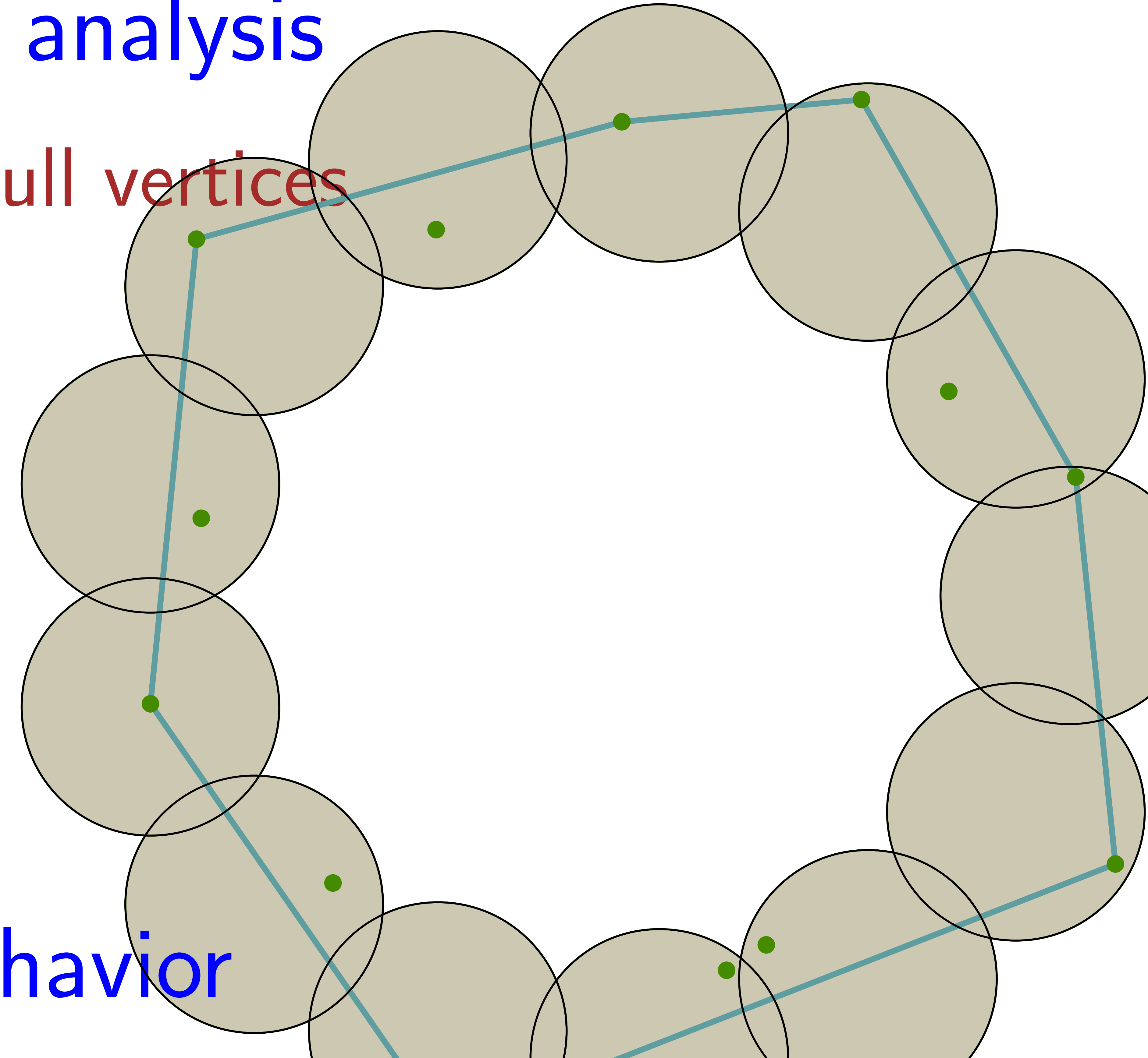

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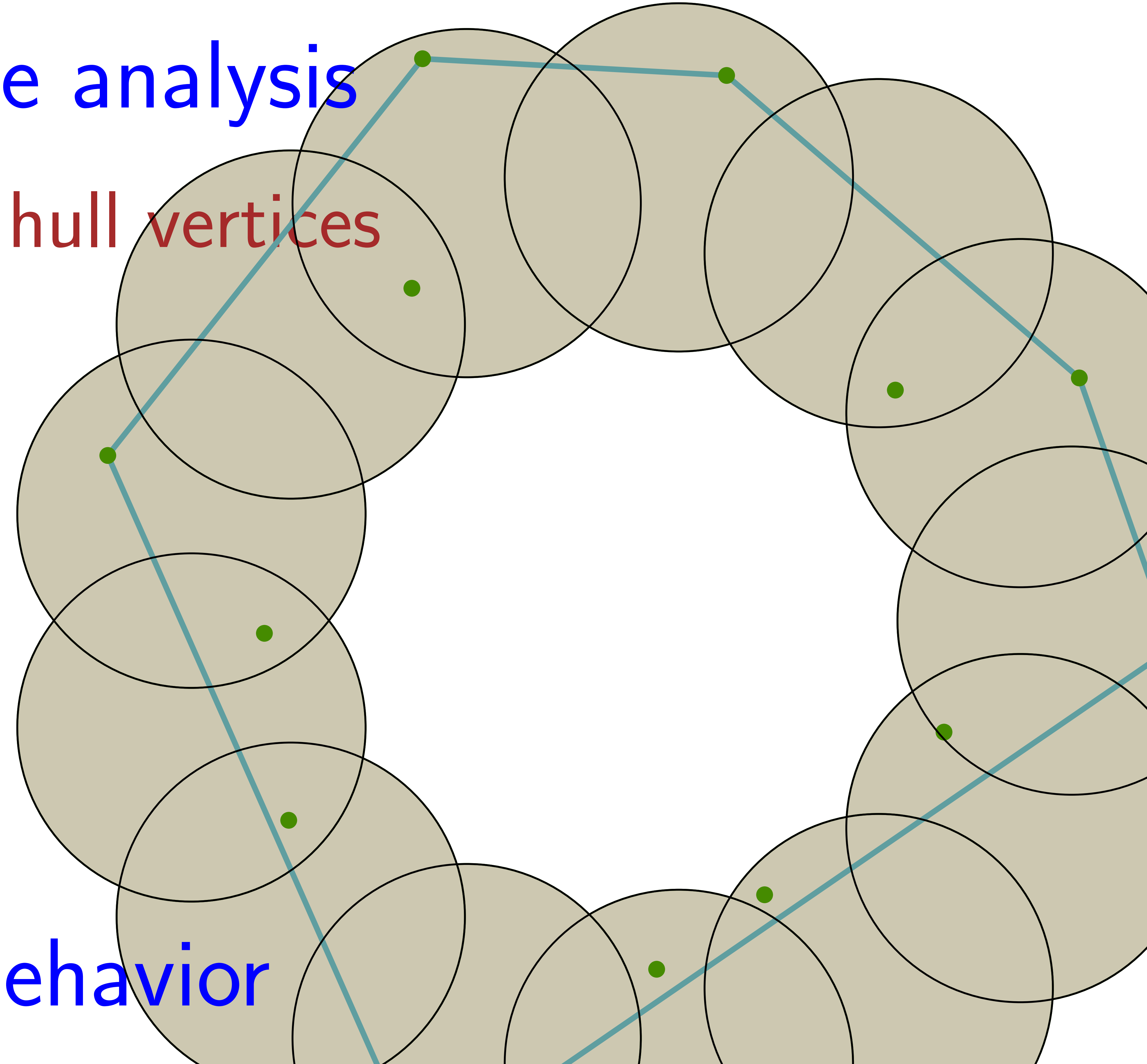

vs noisy behavior

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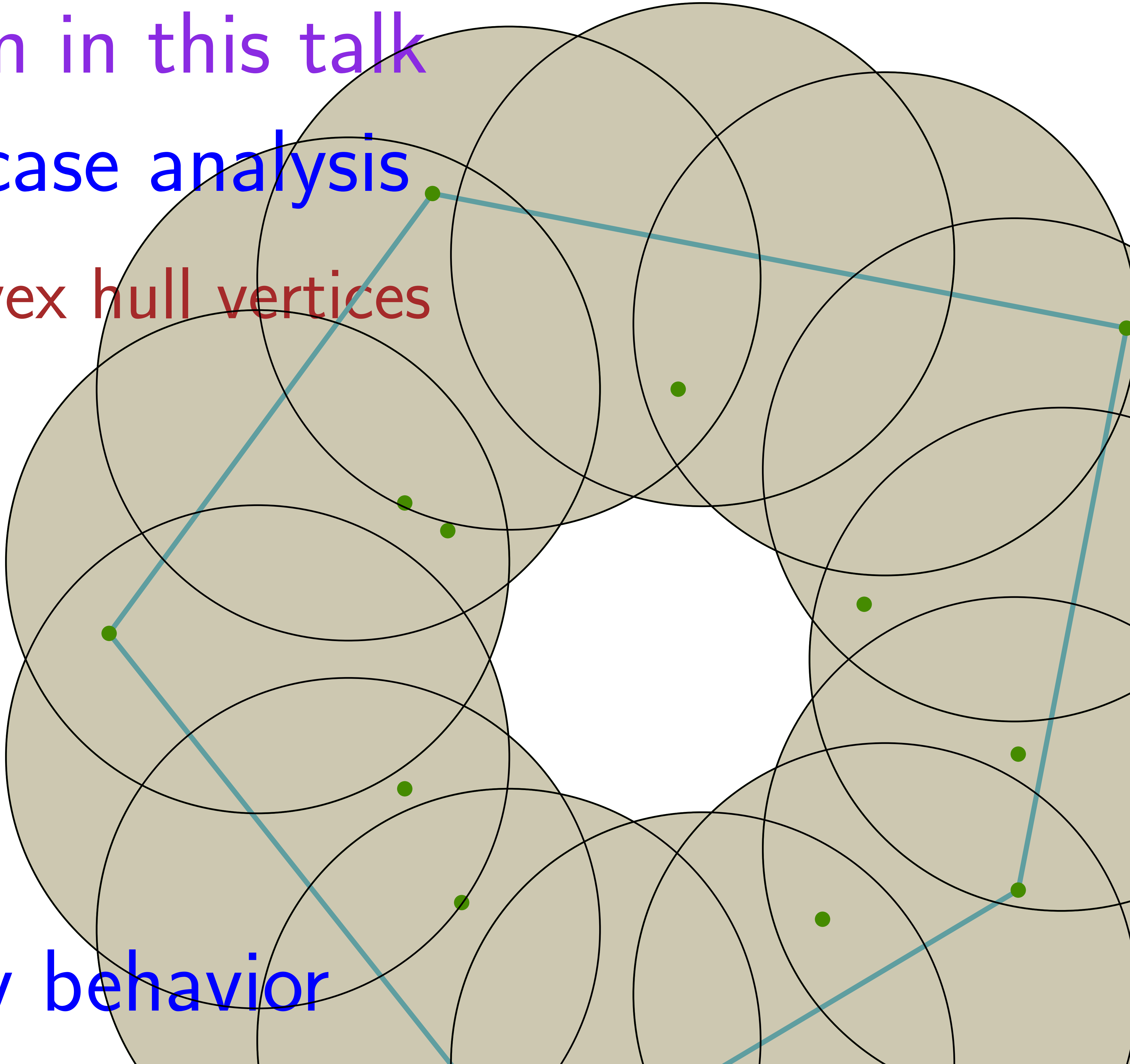

vs noisy behavior

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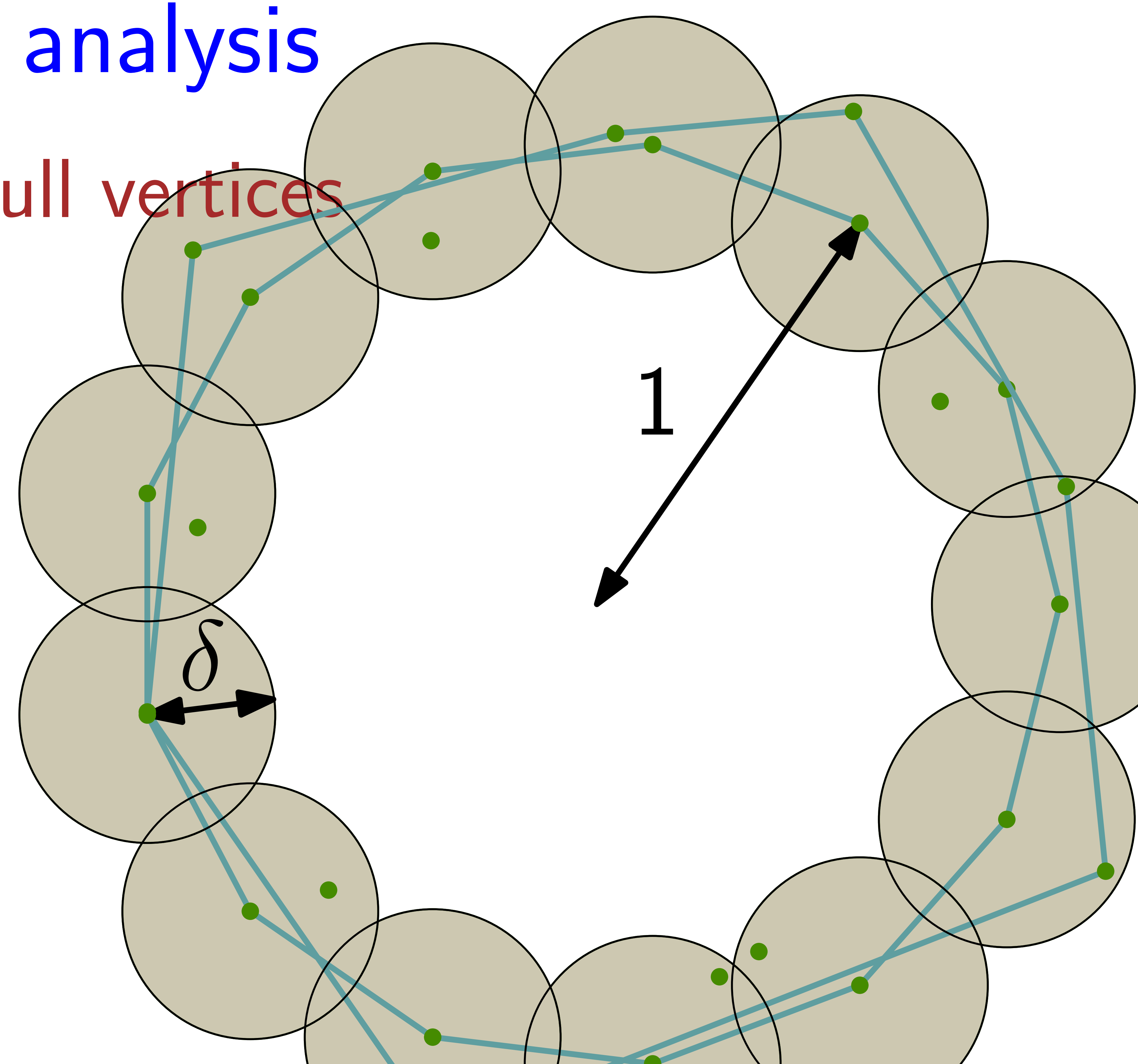
Worst case analysis

♯ of convex hull vertices

$\Omega(n)$

Notations



Problem in this talk

Worst case analysis

♯ of convex hull vertices

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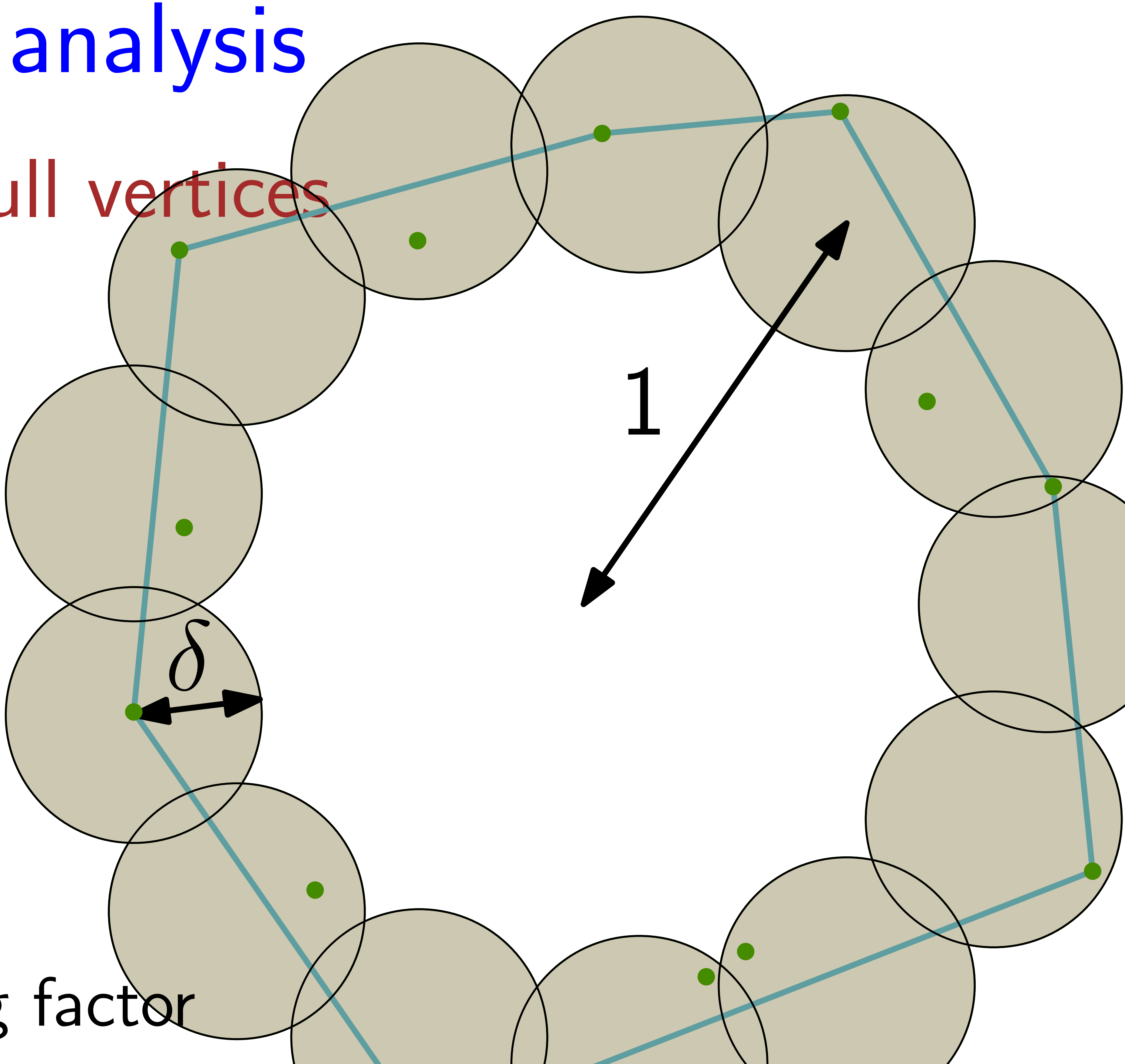
Notations

Result

$\tilde{\Theta}\left(n^{\frac{1}{4}}\delta^{-\frac{3}{8}}\right)$

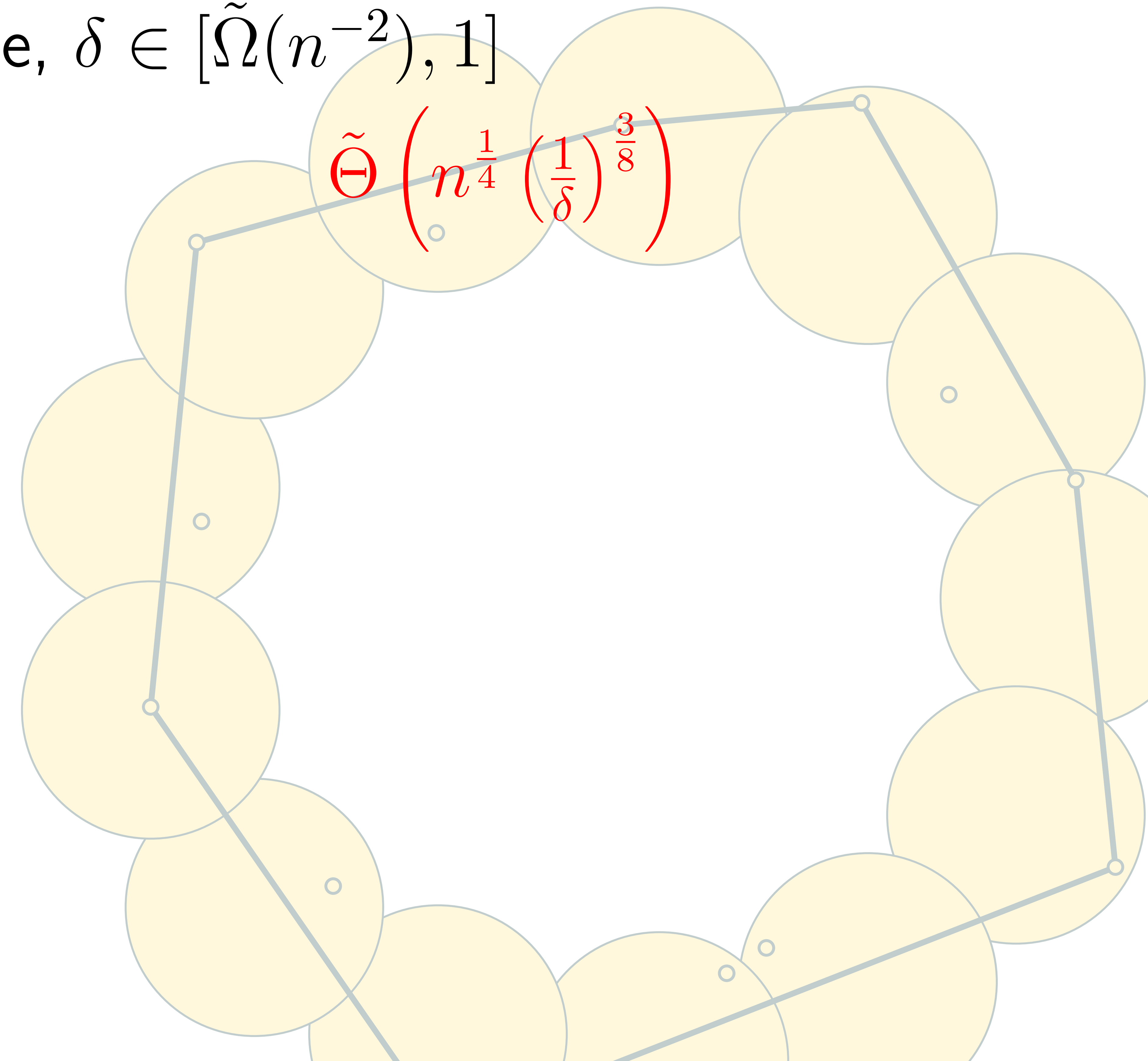


$\tilde{\Theta}$=up to polylog factor

Results

initial position$=(\epsilon, \kappa)$-sample

dim 2, L_2 noise, $\delta \in [\tilde{\Omega}(n^{-2}), 1]$



$$\tilde{\Theta}\left(n^{\frac{1}{4}}\left(\frac{1}{\delta}\right)^{\frac{3}{8}}\right)$$

Results

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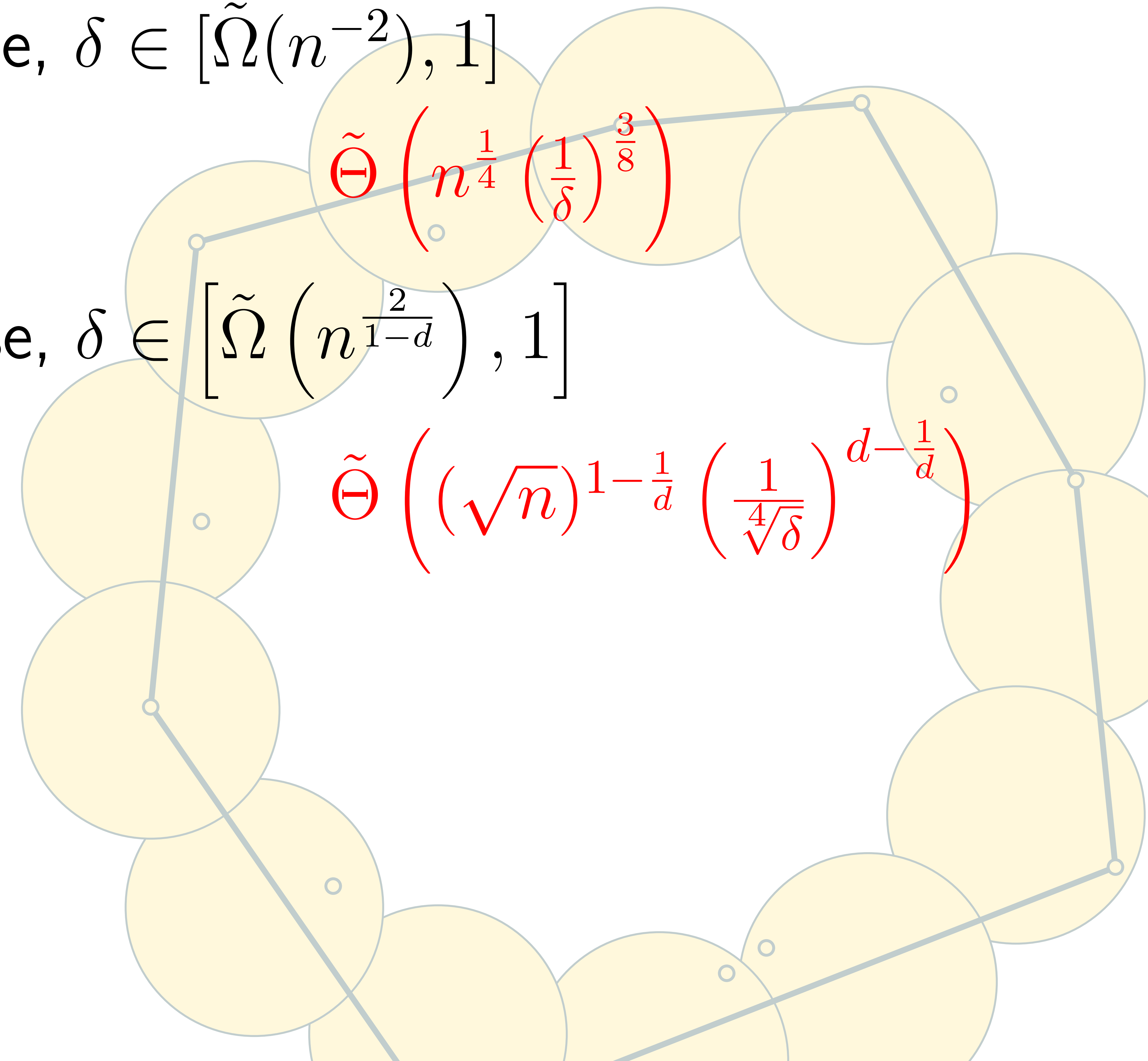
dim d, L_2 noise, $\delta \in \left[\tilde{\Omega}\left(n^{\frac{2}{1-d}}\right), 1\right]$

$$\tilde{\Theta}\left((\sqrt{n})^{1-\frac{1}{d}}\left(\frac{1}{\sqrt[4]{\delta}}\right)^{d-\frac{1}{d}}\right)$$



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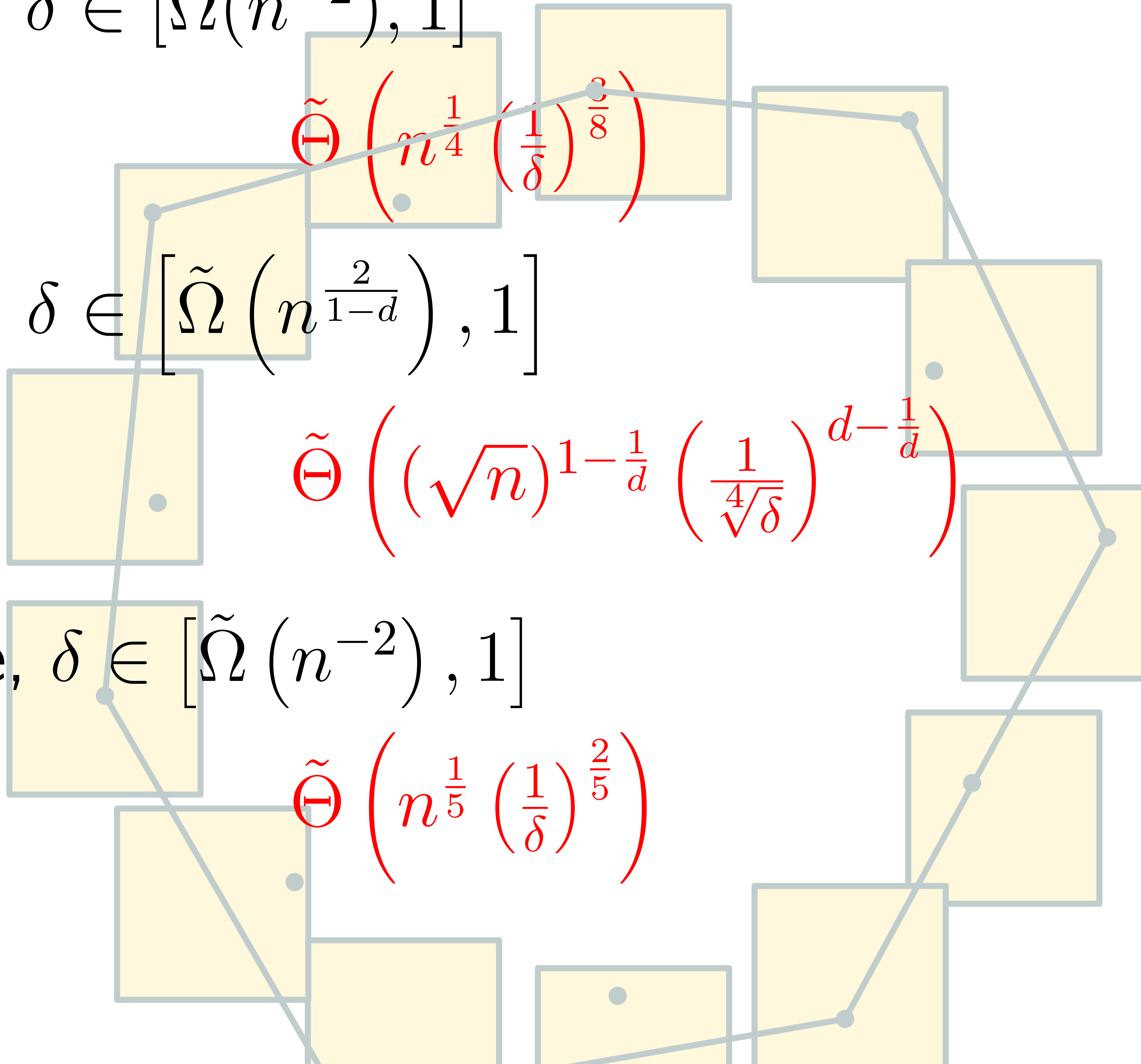
$$\tilde{\Theta}\left(n^{\frac{1}{5}}\left(\frac{1}{\delta}\right)^{\frac{2}{5}}\right)$$



Results

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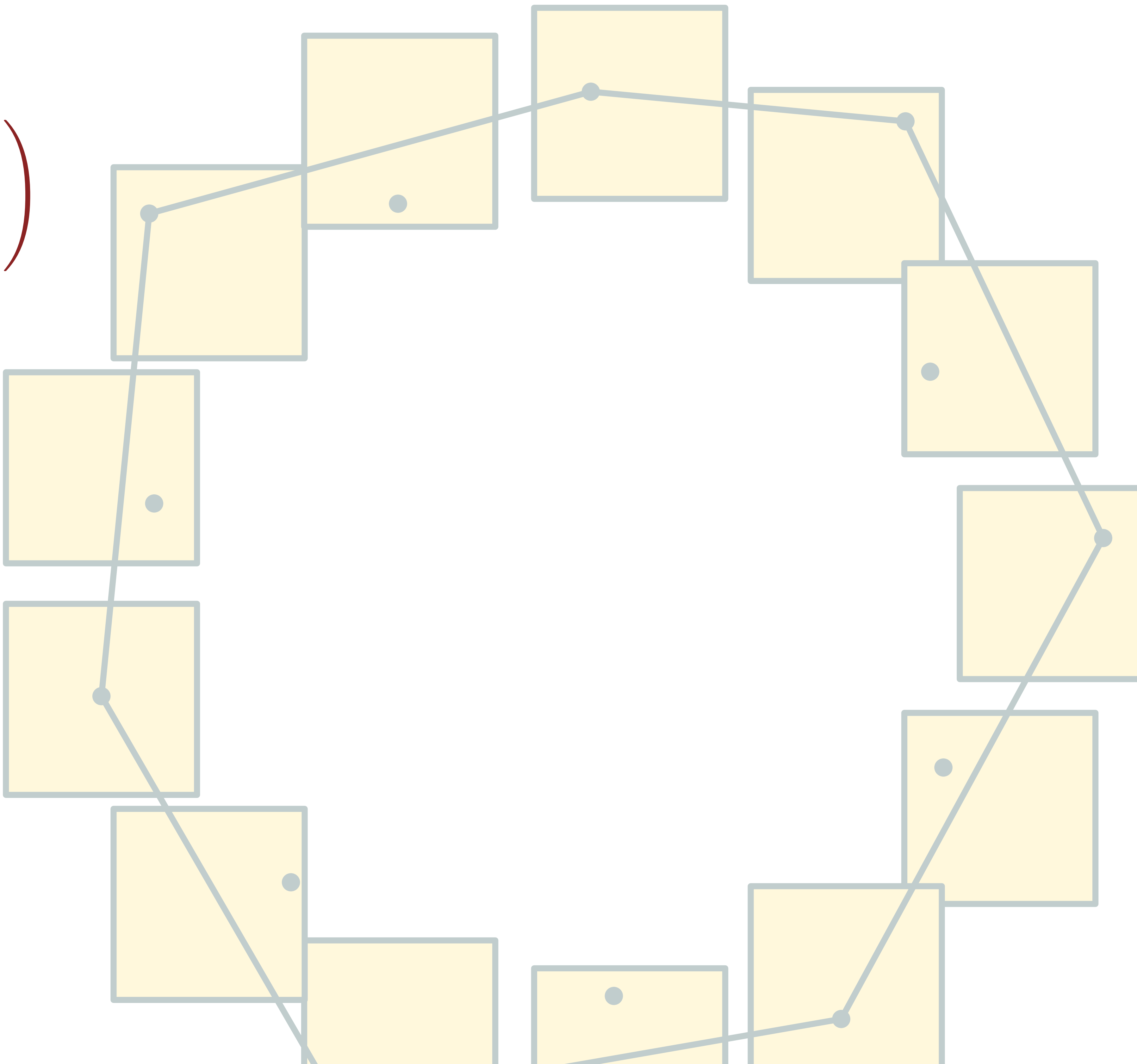
experiments for noise and snap

Related result

[Damerow & Sohler]

dim d, L_∞ noise,

$$O\left(\left(\frac{n \log n}{\delta}\right)^{\frac{d}{d+1}}\right)$$



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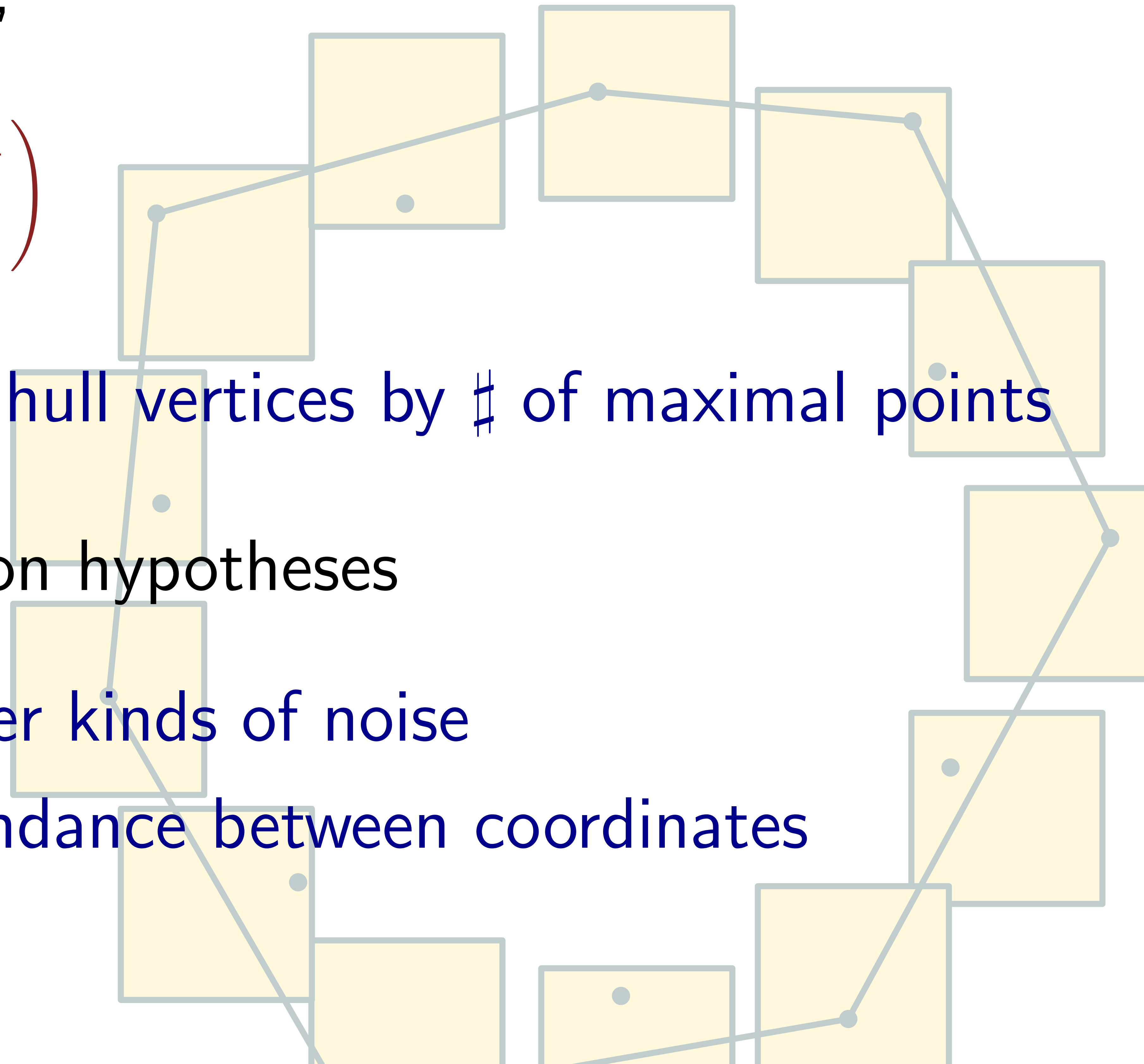


Upper bound ♯ hull vertices by ♯ of maximal points

No inital position hypotheses

Extends to other kinds of noise

independance between coordinates

L_2

Related result

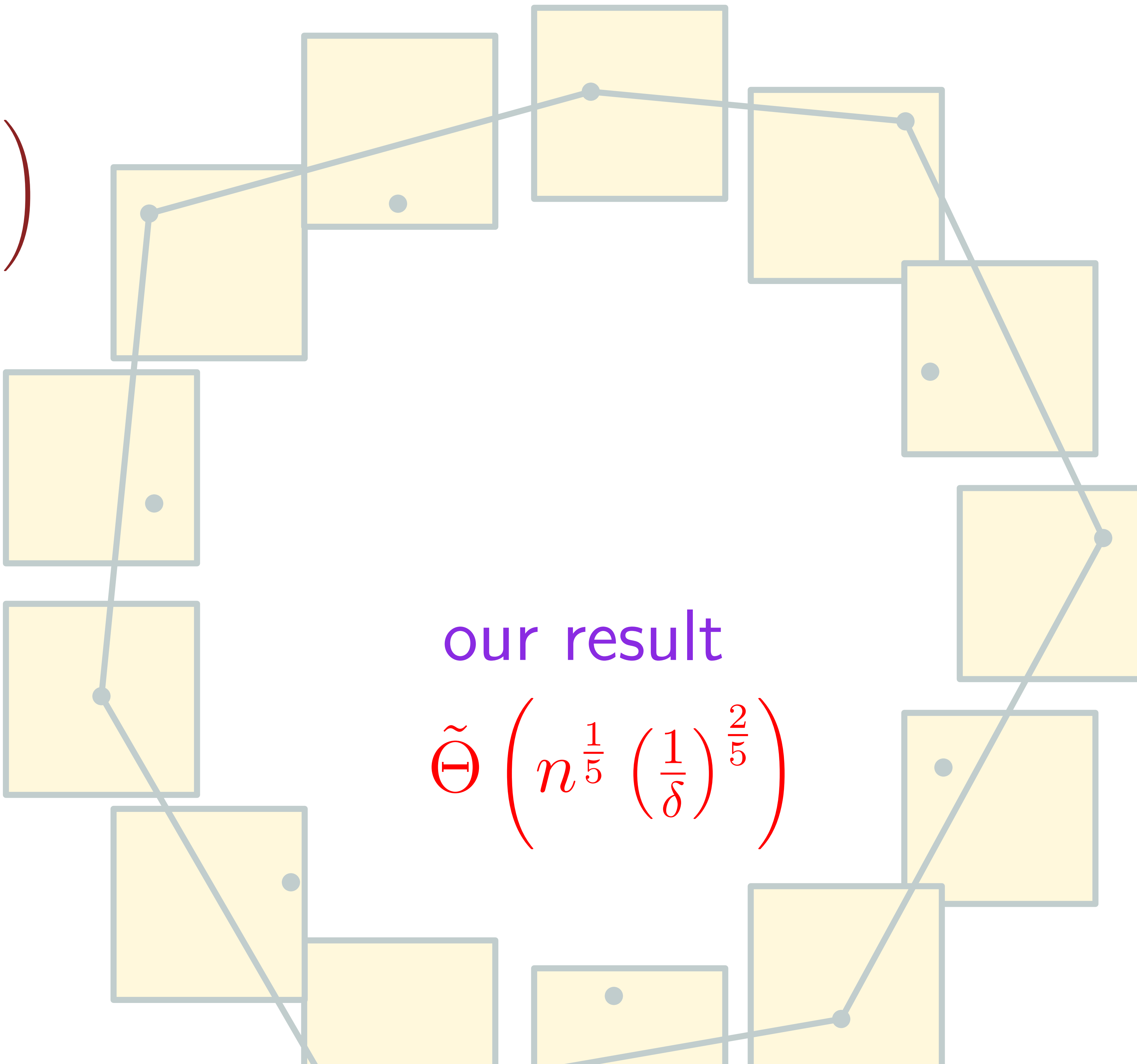
[Damerow & Sohler]

dim d, L_∞ noise,

$$O\left(\left(\frac{n \log n}{\delta}\right)^{\frac{d}{d+1}}\right)$$

dim 2, L_∞ noise,

$$\tilde{O}\left(n^{\frac{2}{3}}\left(\frac{1}{\delta}\right)^{\frac{2}{3}}\right)$$



our result

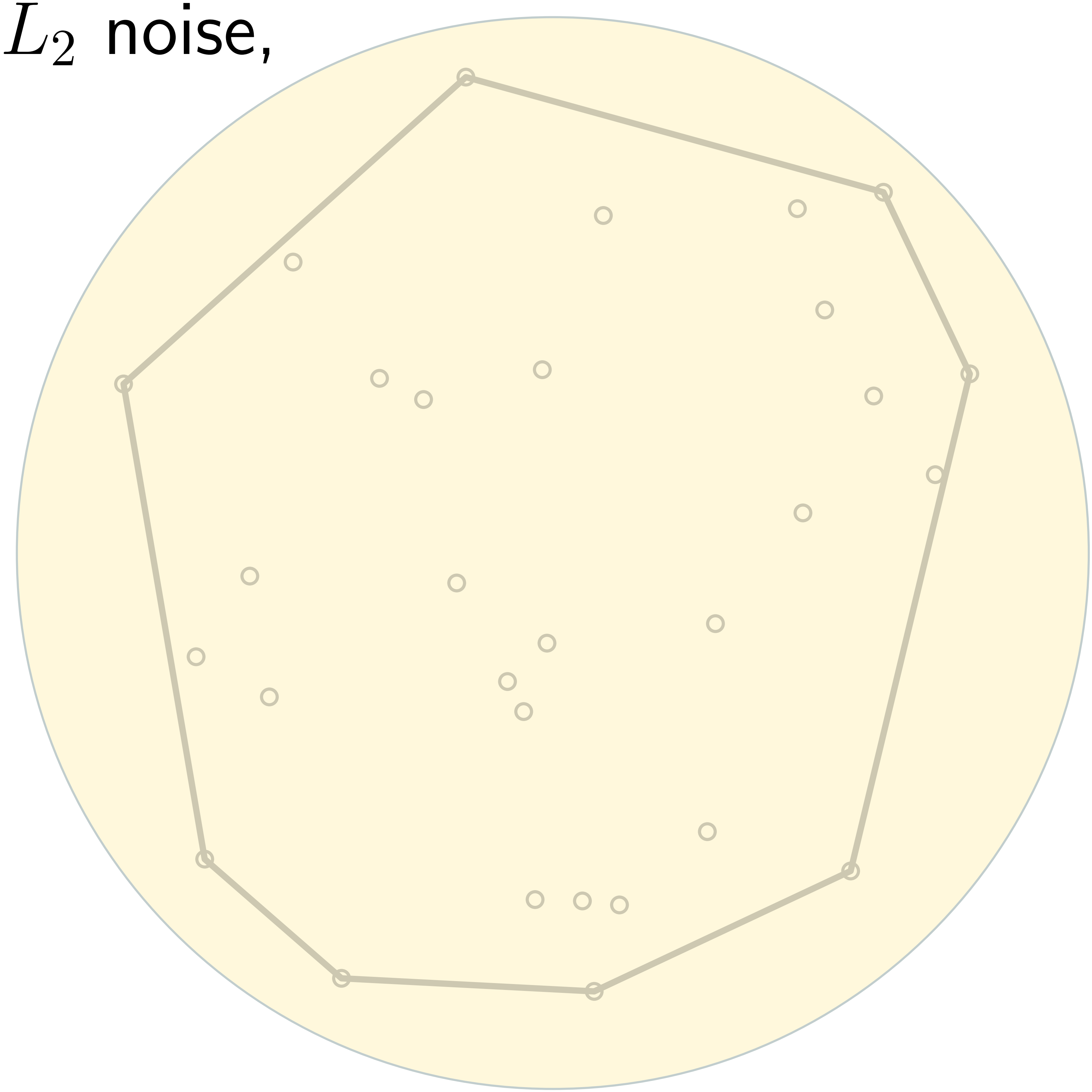
$$\tilde{\Theta}\left(n^{\frac{1}{5}}\left(\frac{1}{\delta}\right)^{\frac{2}{5}}\right)$$

Related result

[Rényi & Sulanke]

dim d, "infinite" L_2 noise,

$$\Theta\left(n^{\frac{d-1}{d+1}}\right)$$



Related result

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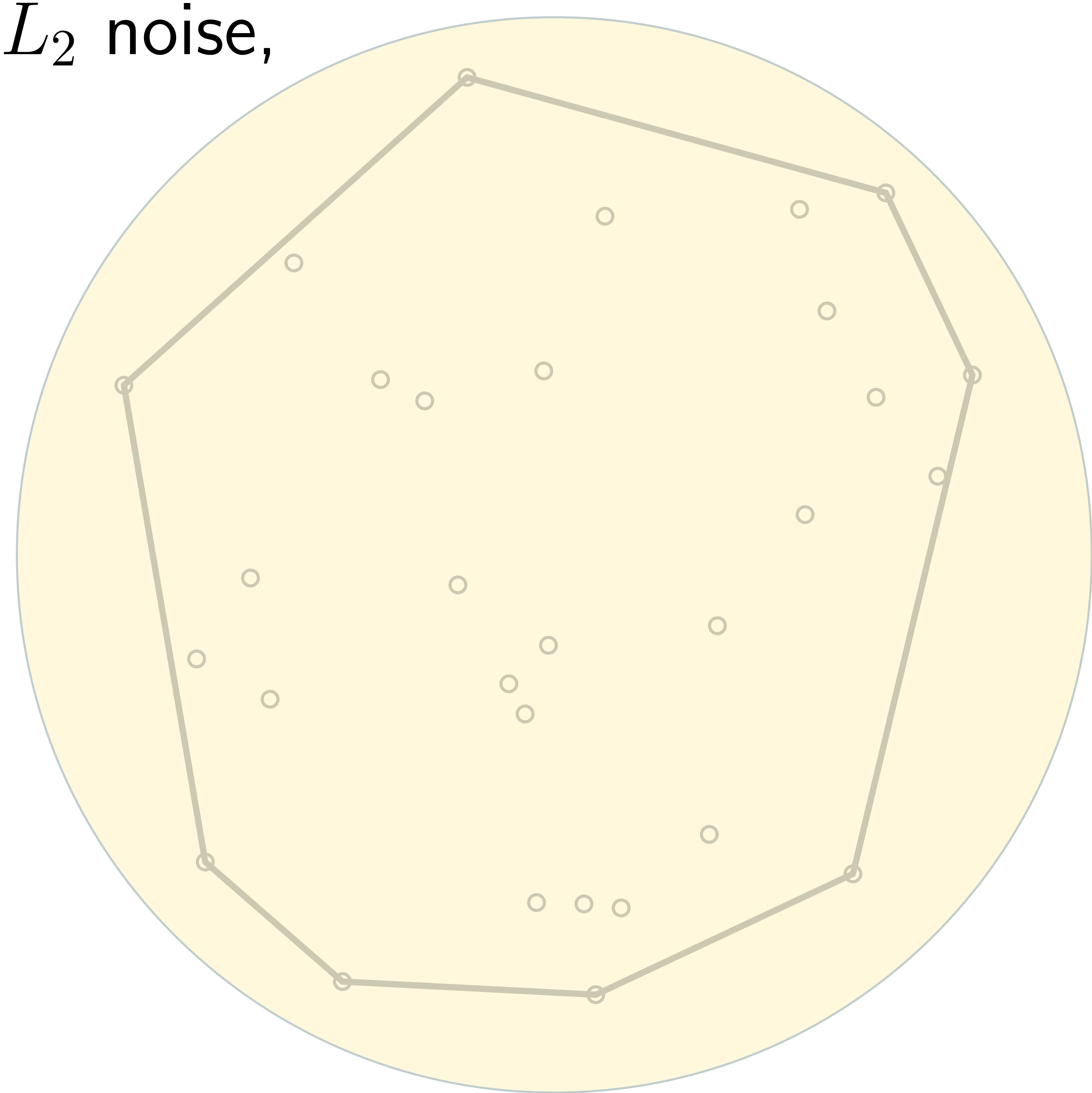
$$\Theta\left(n^{\frac{d-1}{d+1}}\right)$$

$d = 2$

$$\Theta\left(n^{\frac{1}{3}}\right)$$



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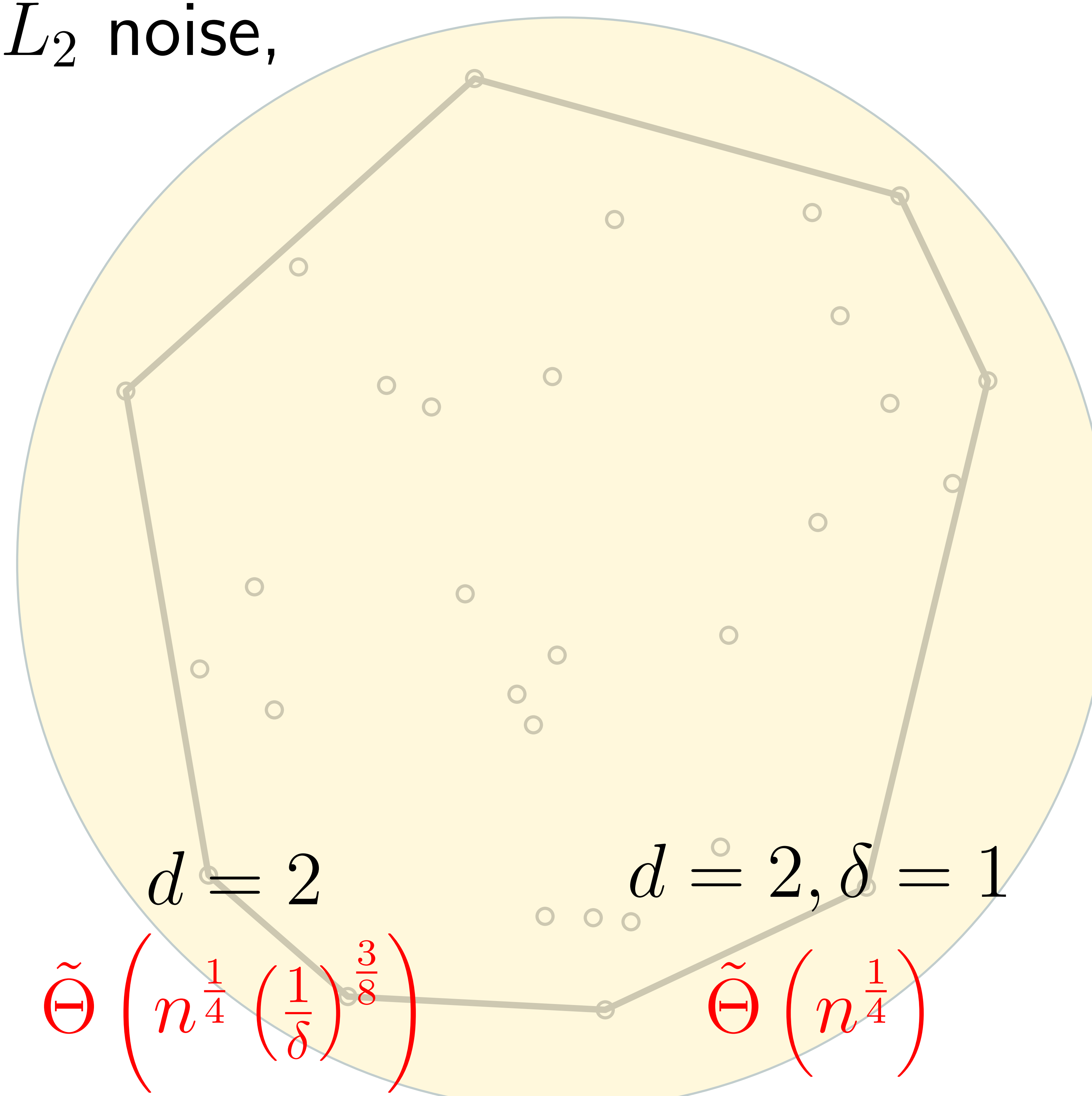


our result

$d = 2, \delta = n^{-2}$

$$\tilde{\Theta}\left(n\right)$$

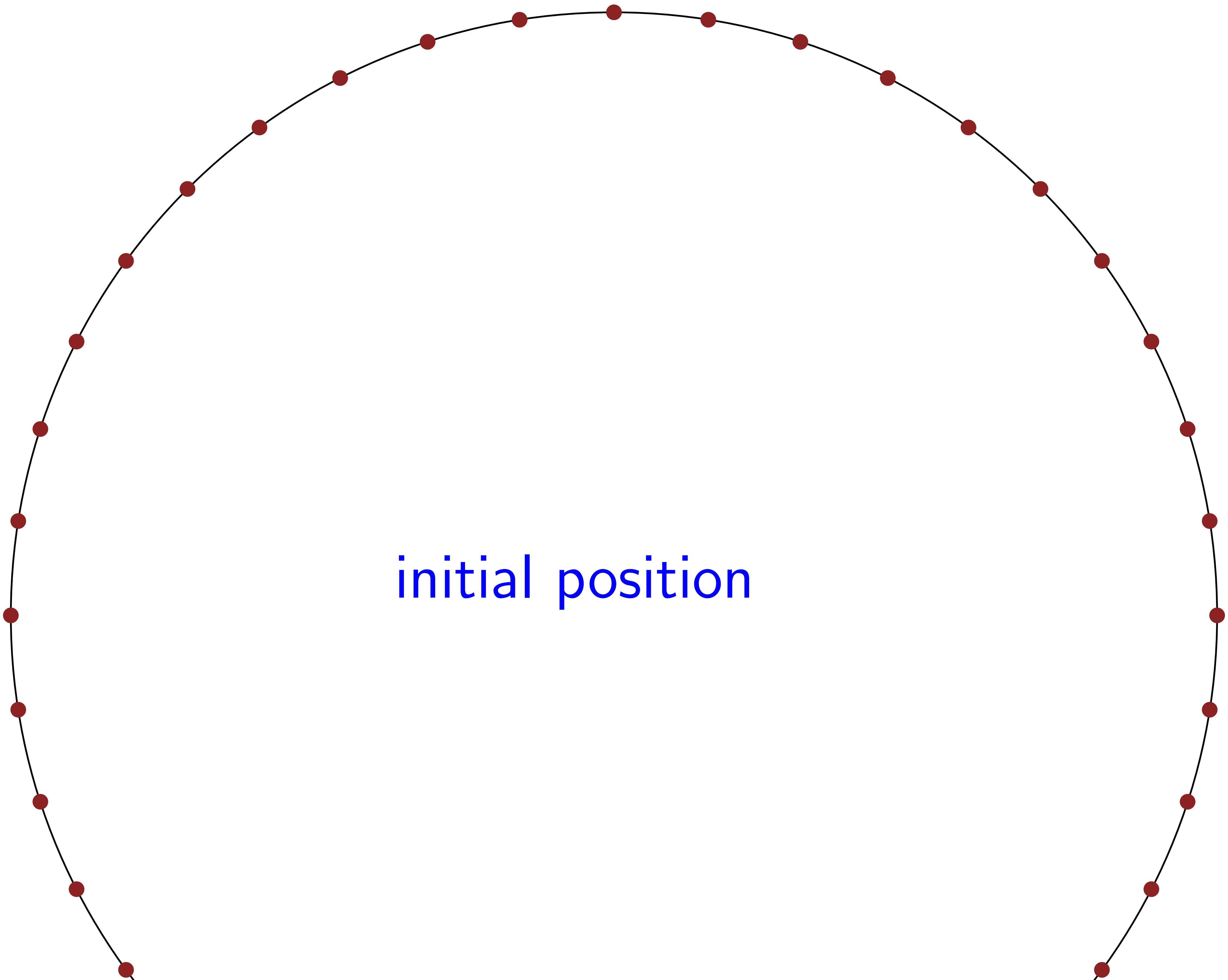
$d = 2$

$$\tilde{\Theta}\left(n^{\frac{1}{4}}\left(\frac{1}{\delta}\right)^{\frac{3}{8}}\right)$$

$d = 2, \delta = 1$

$$\tilde{\Theta}\left(n^{\frac{1}{4}}\right)$$

Proof



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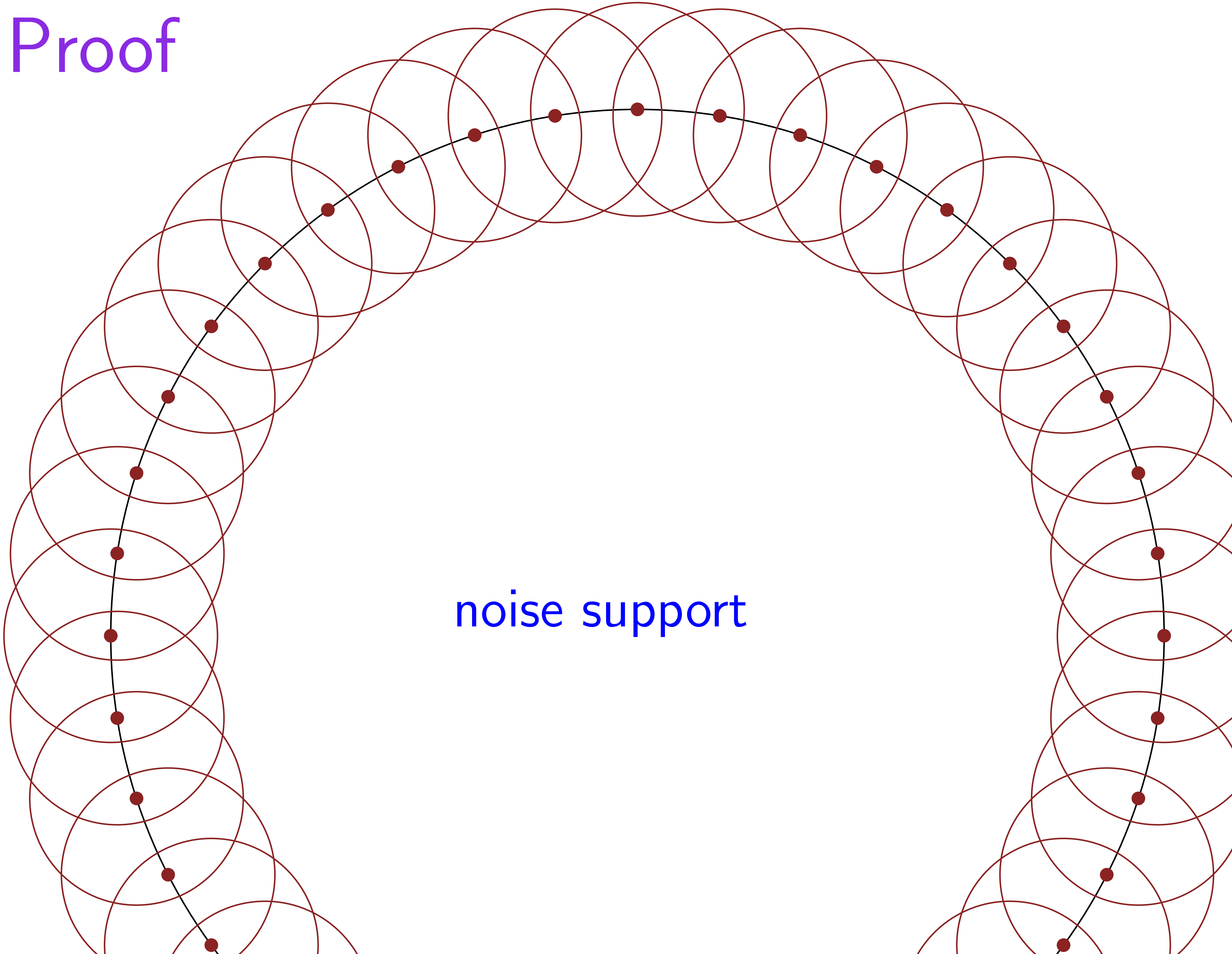

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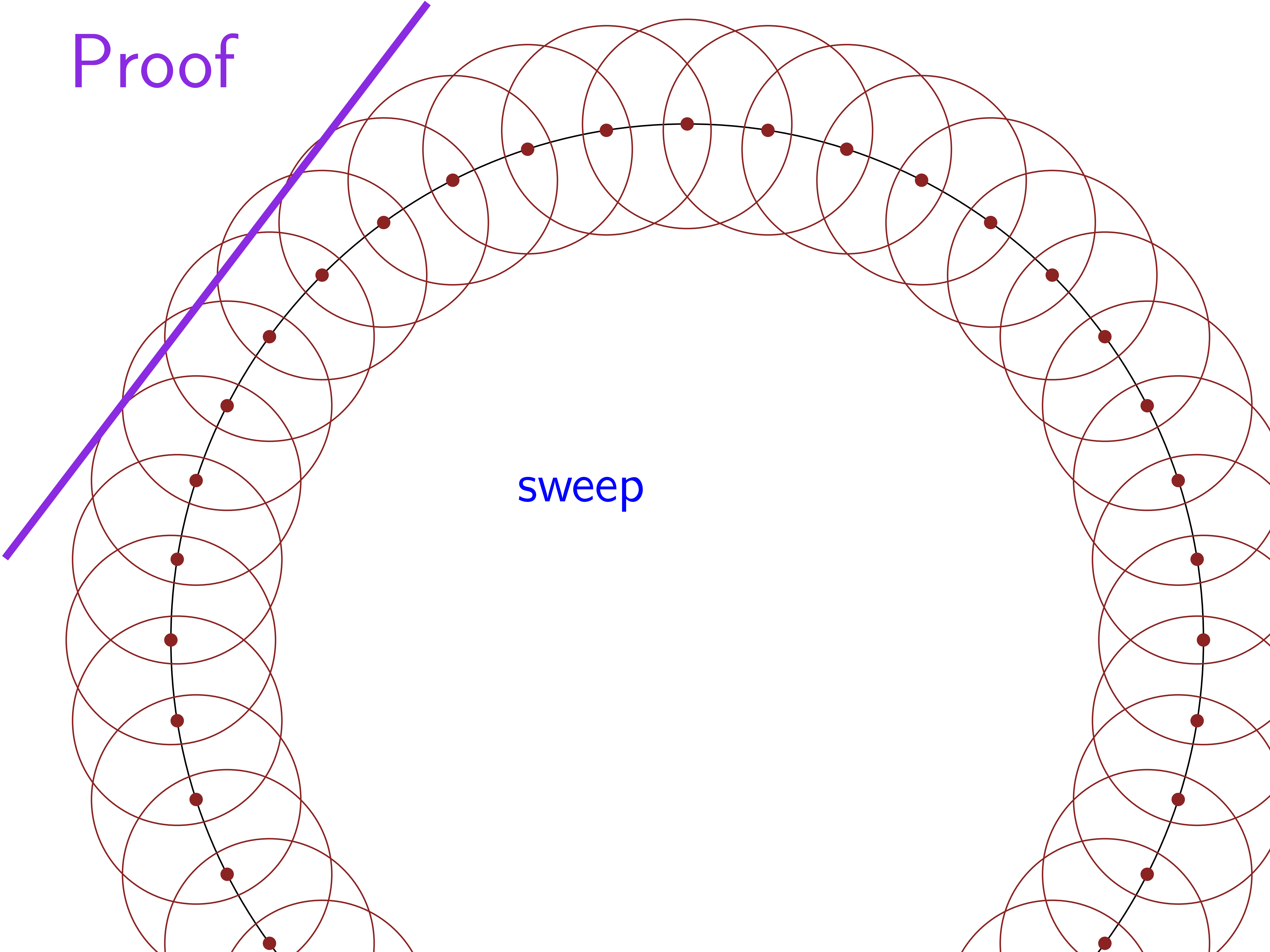
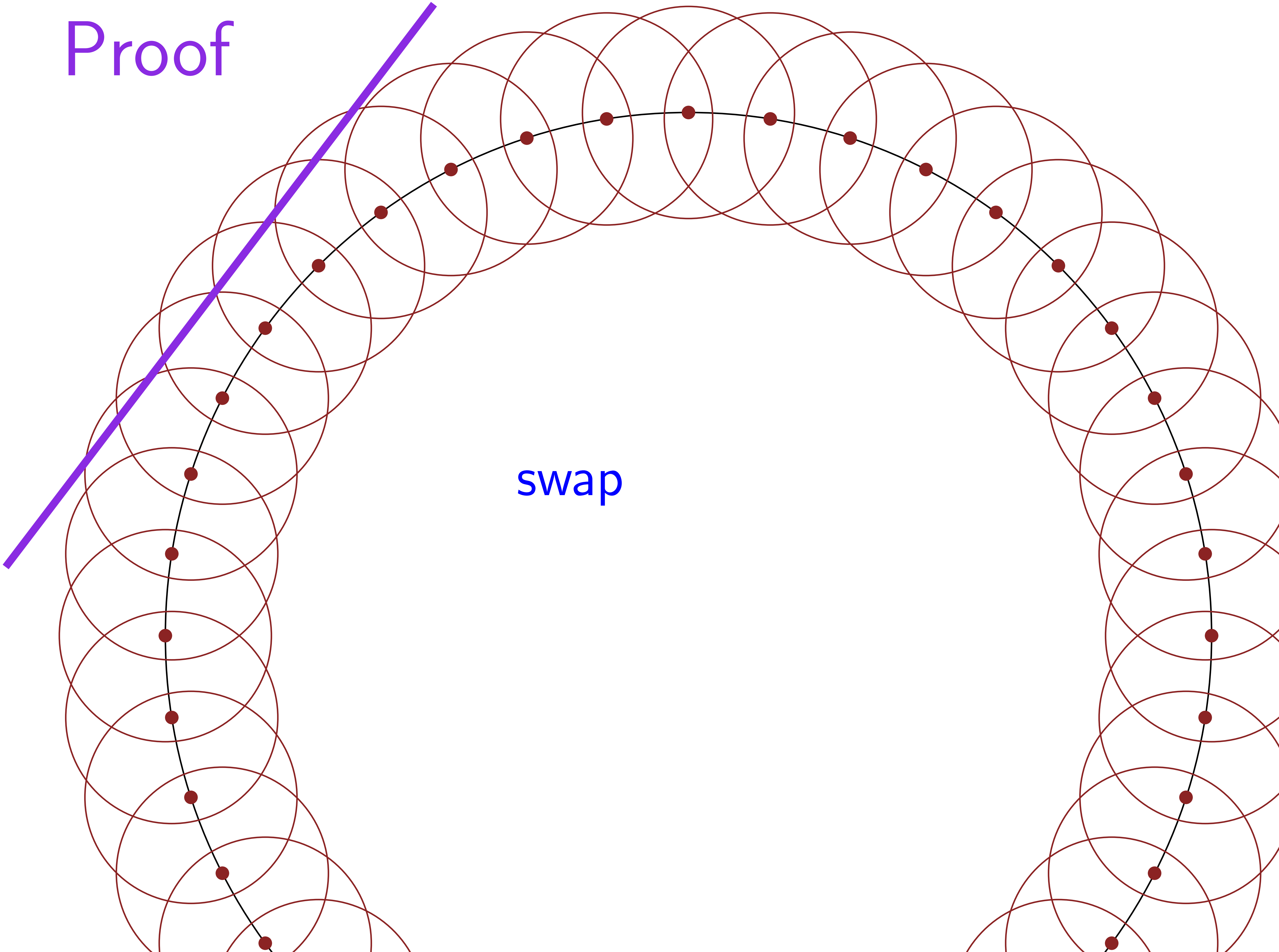


Proof



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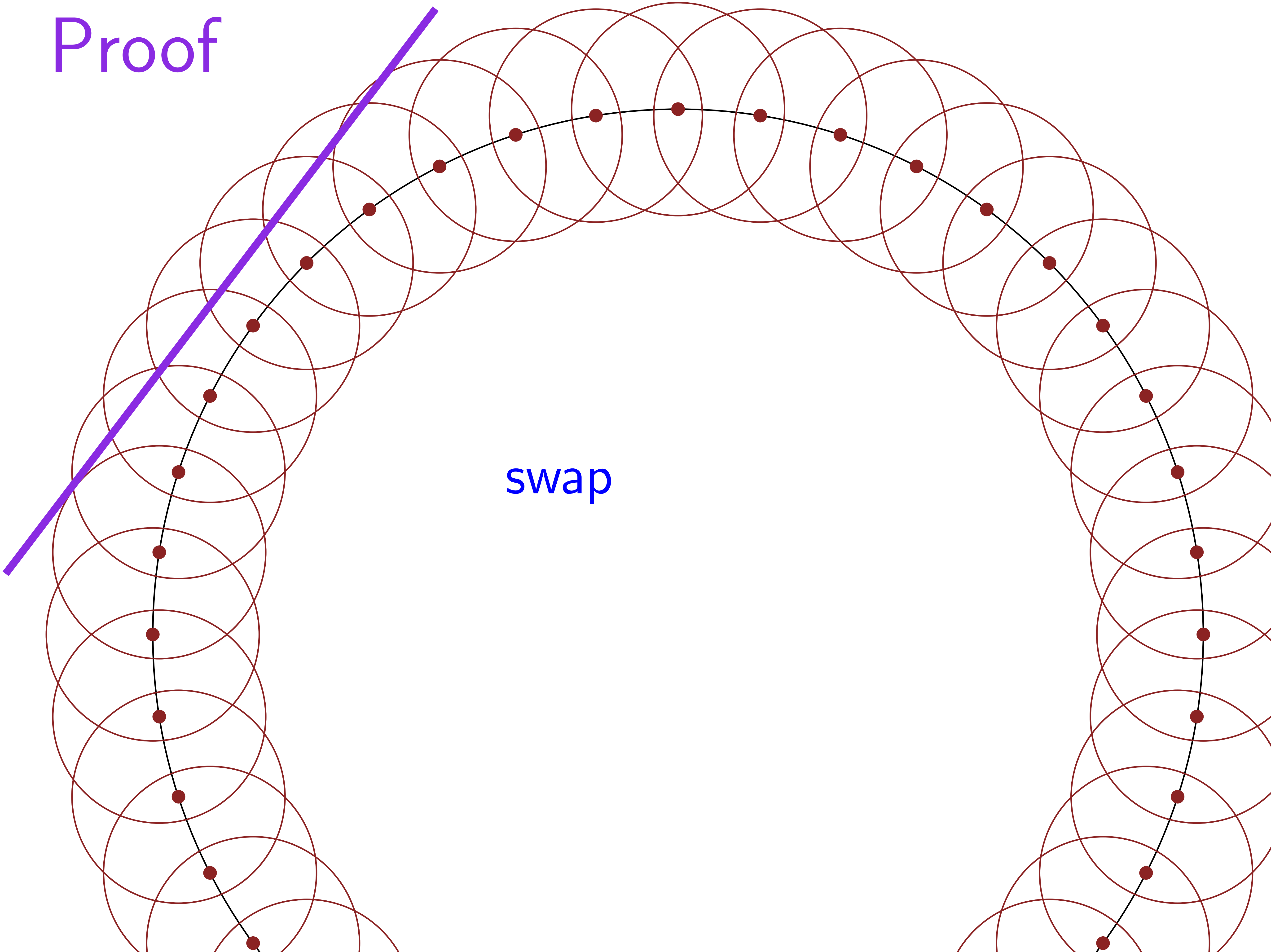

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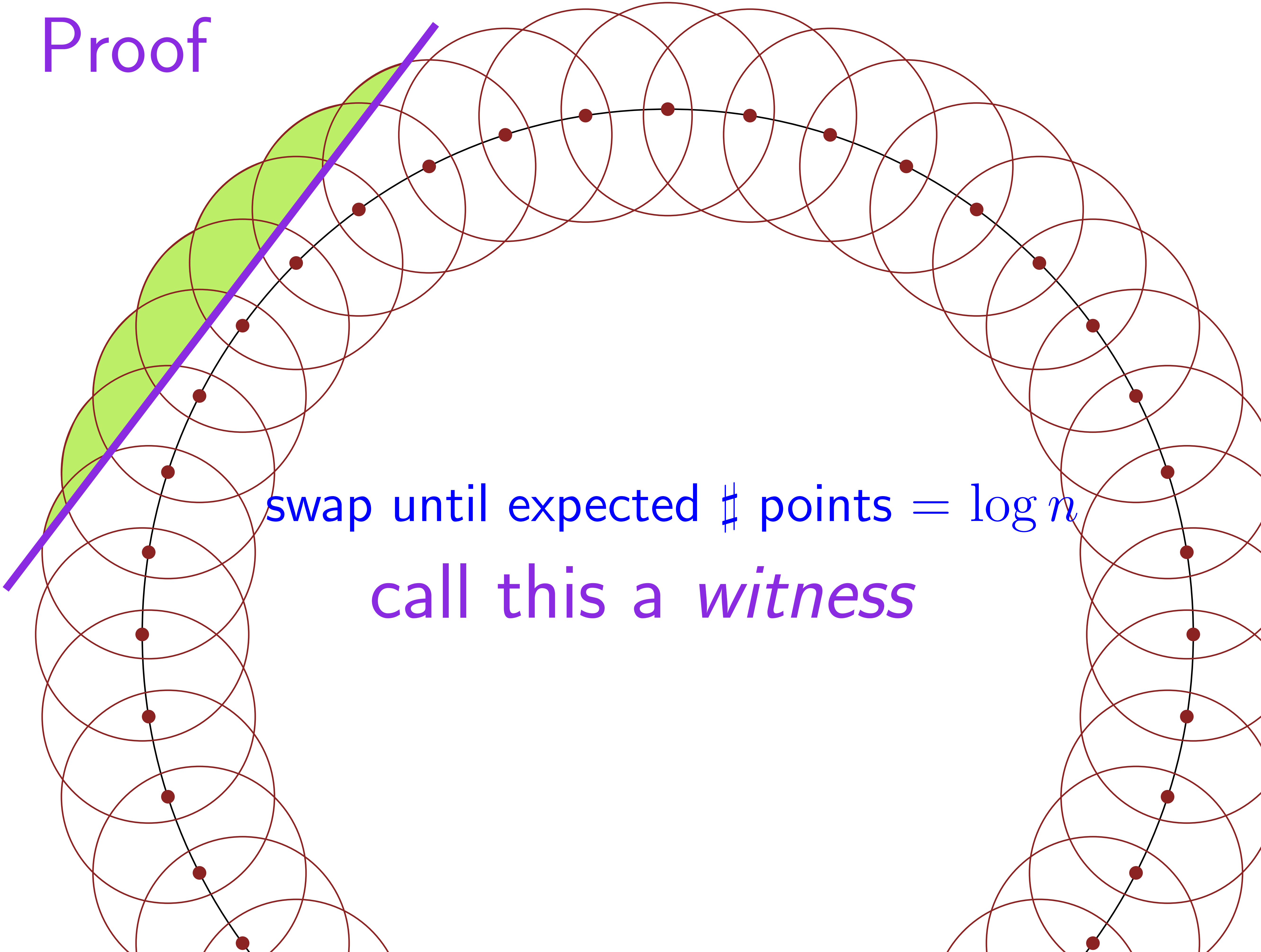
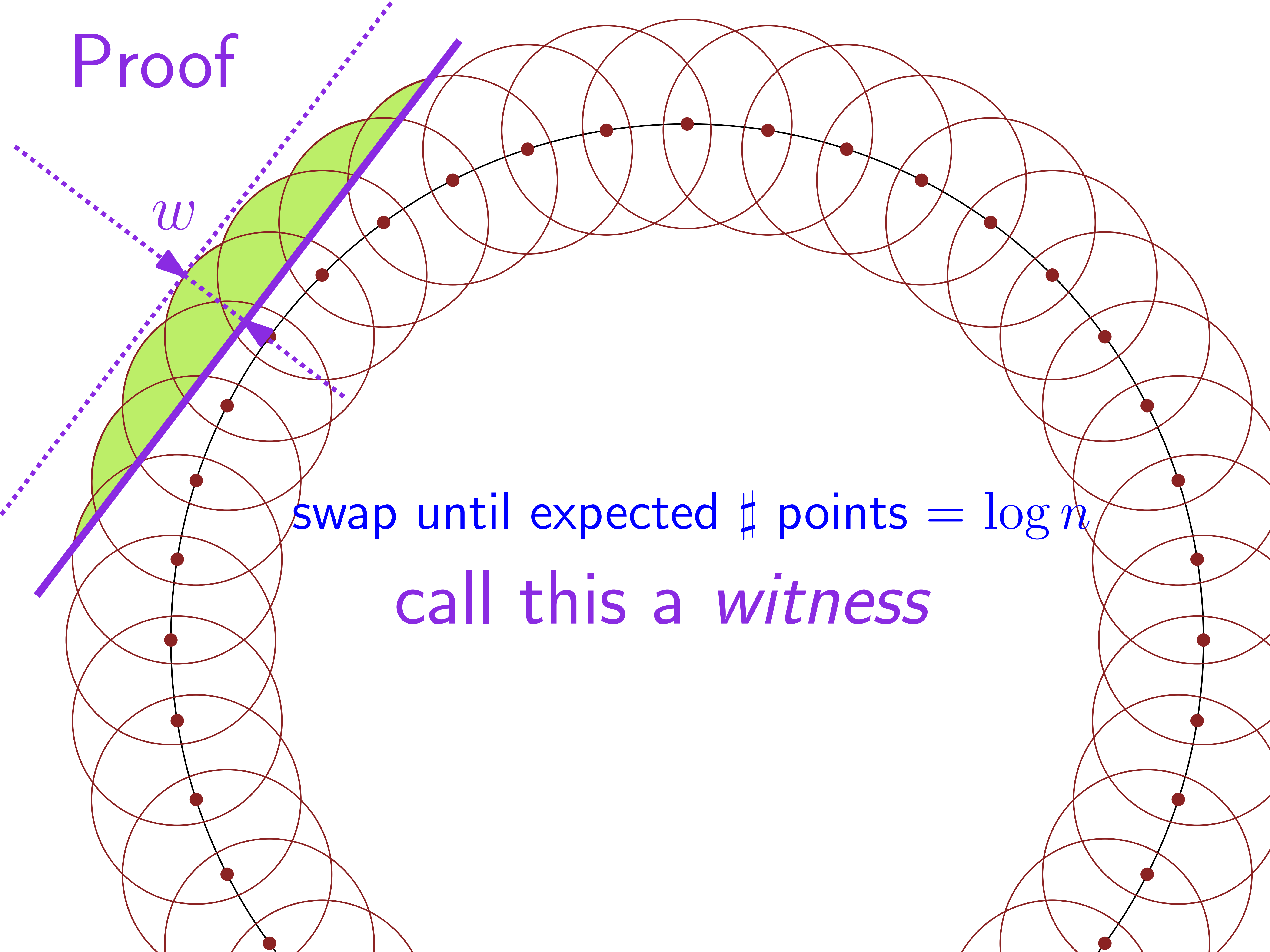


swap until expected $\sharp$ points $= \log n$

call this a *witness*

Proof



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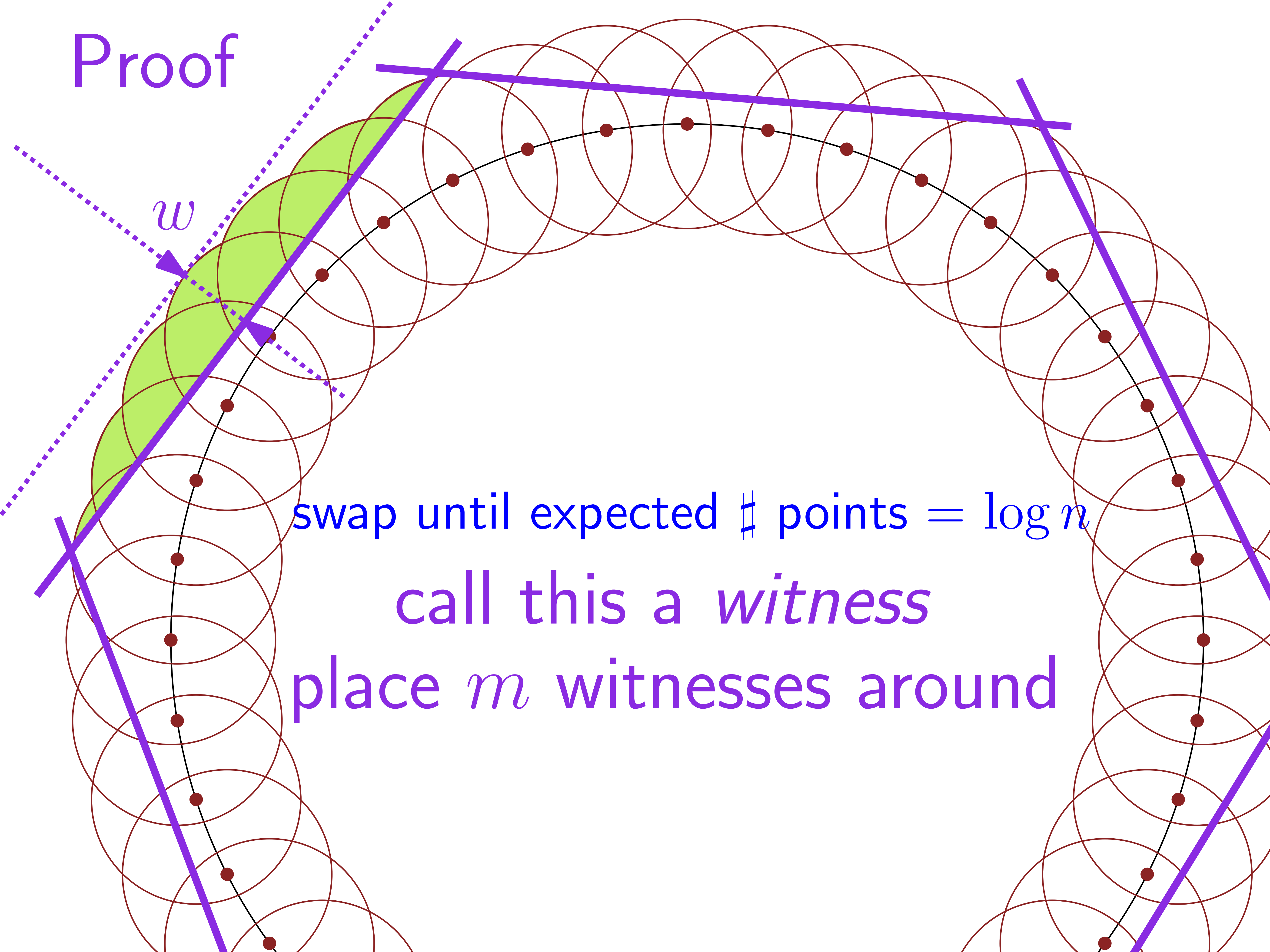

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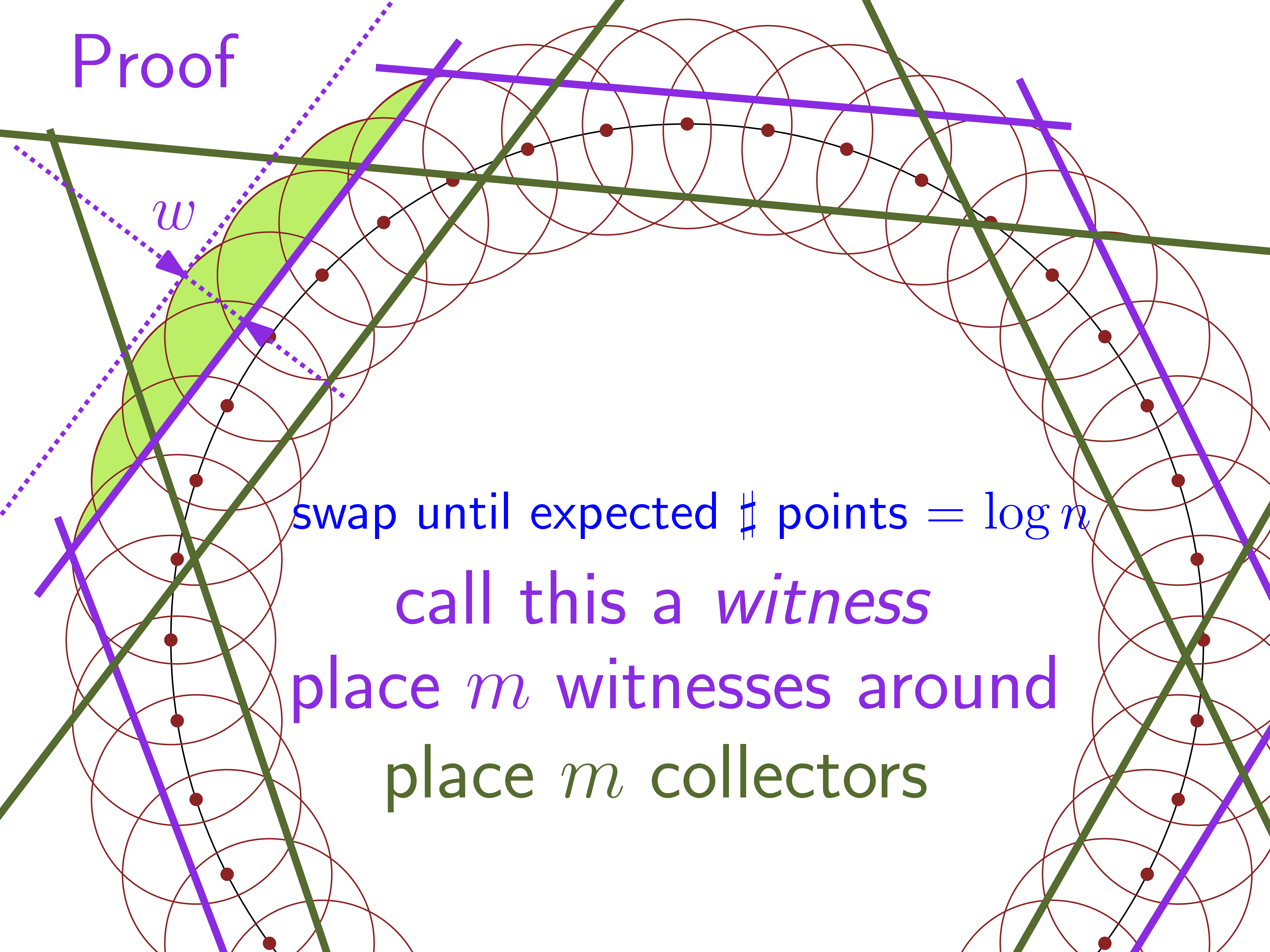
place m witnesses around

Proof



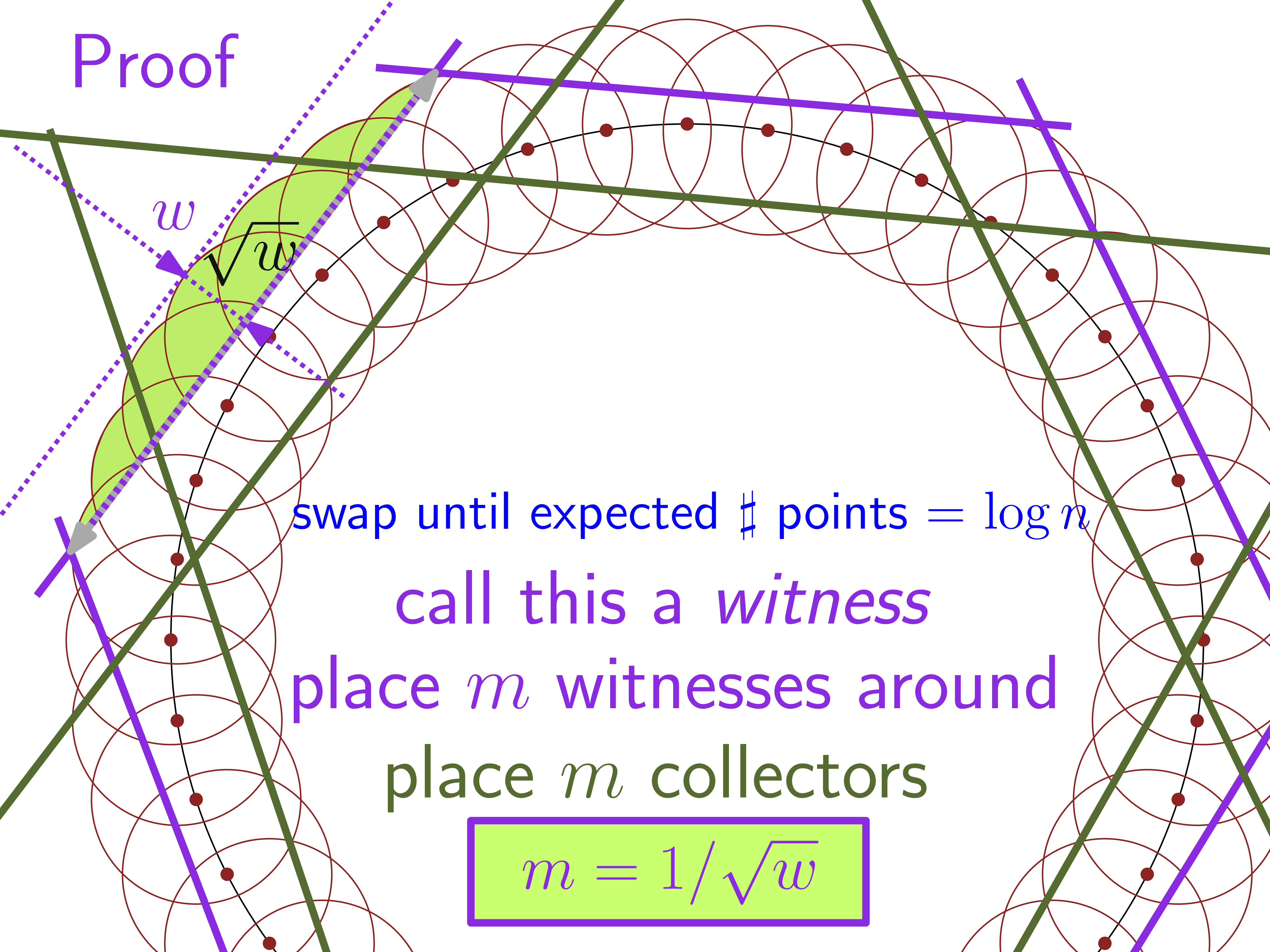
swap until expected $\sharp$ points $= \log n$

call this a *witness*

place m witnesses around

place m collectors

Proof



swap until expected ♯ points $= \log n$

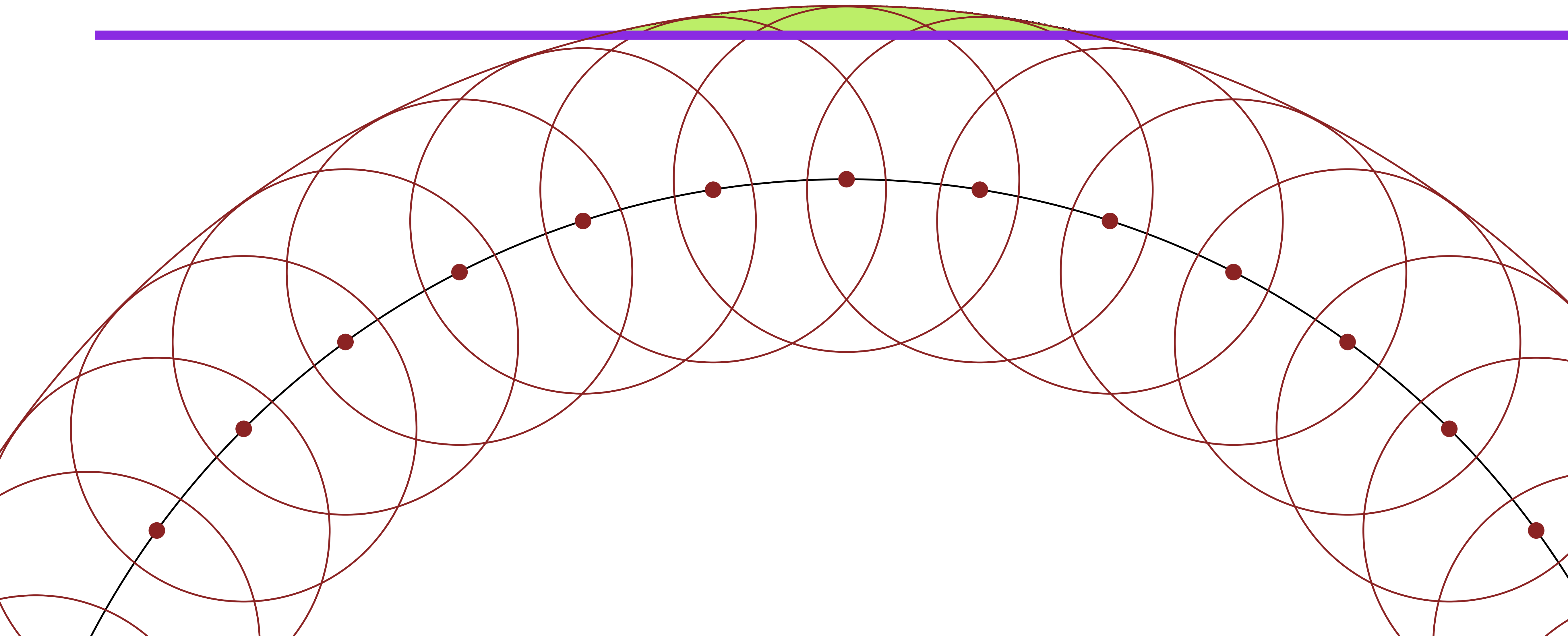
call this a *witness*

place m witnesses around

place m collectors

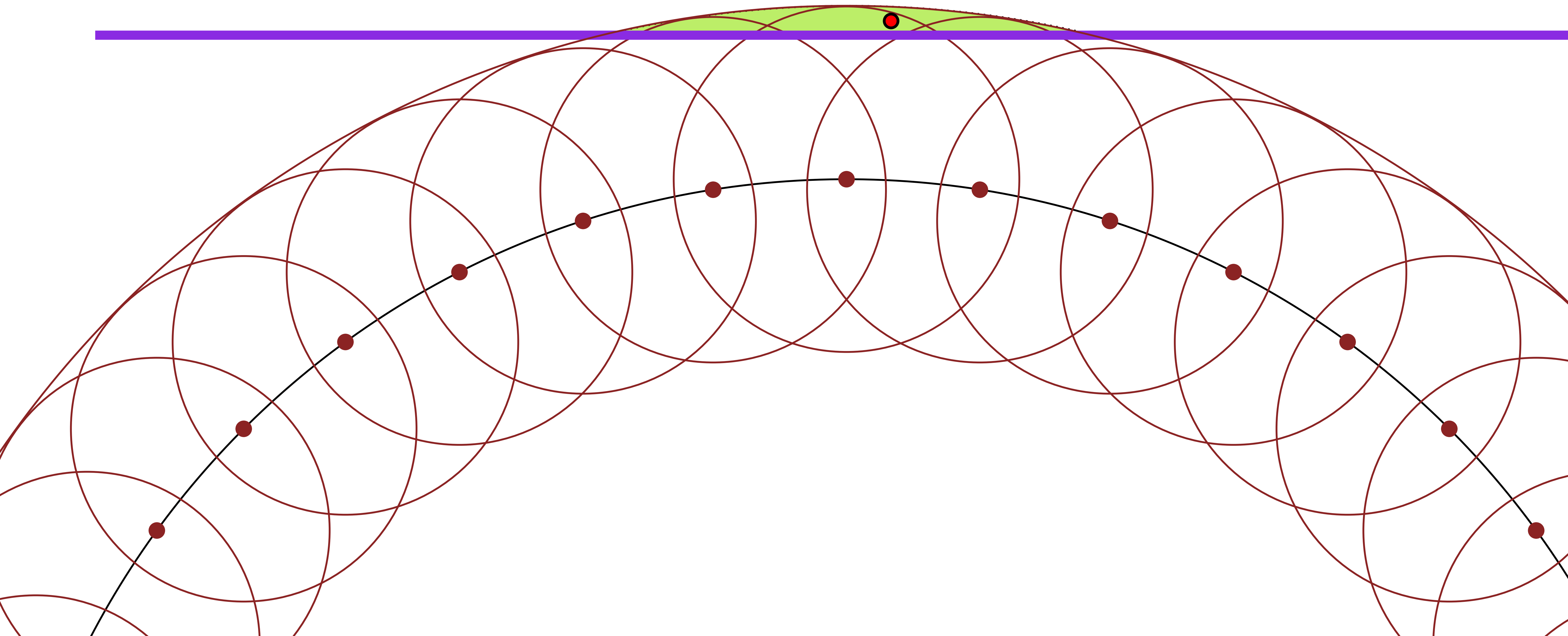
$$m = 1/\sqrt{w}$$

Collector placing rule



Collector placing rule

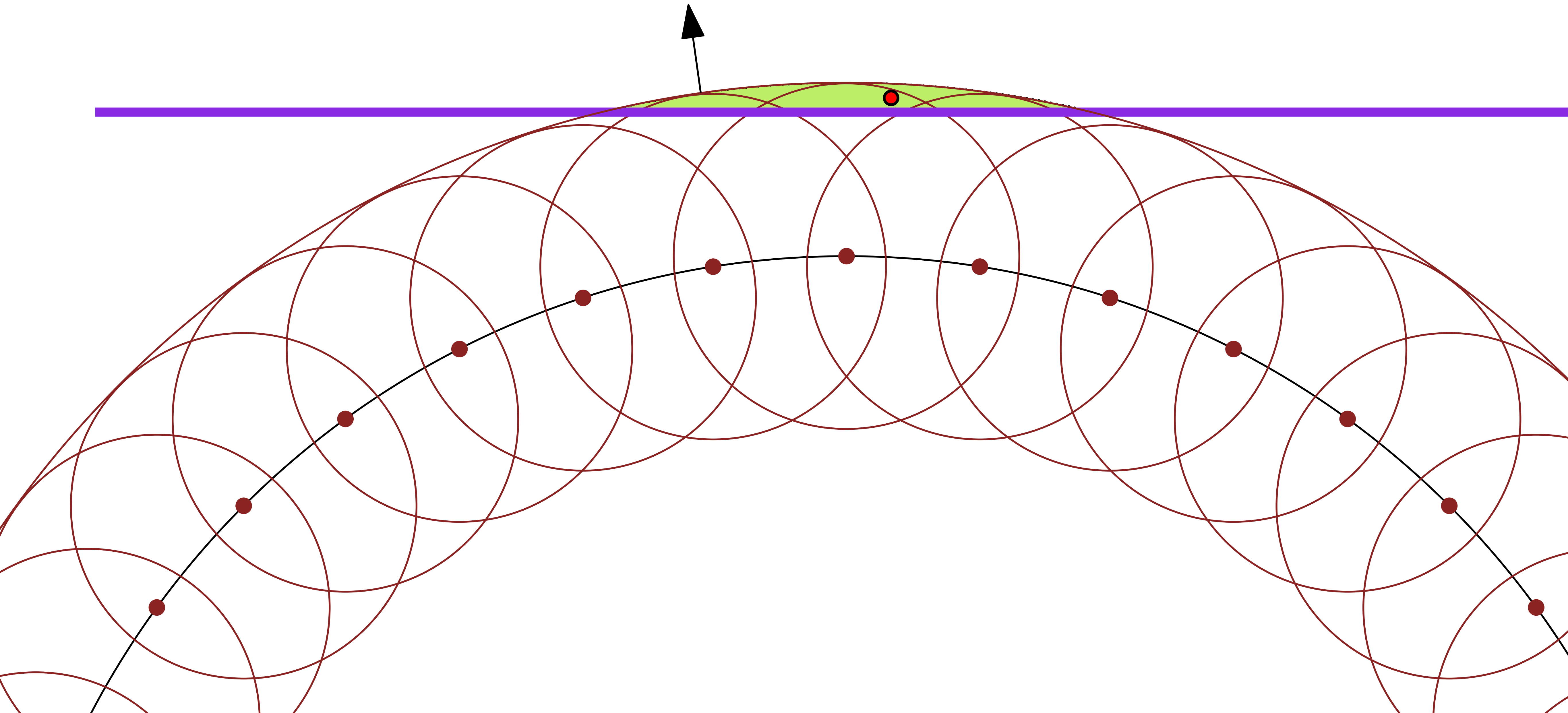
if the witness is non empty



Collector placing rule

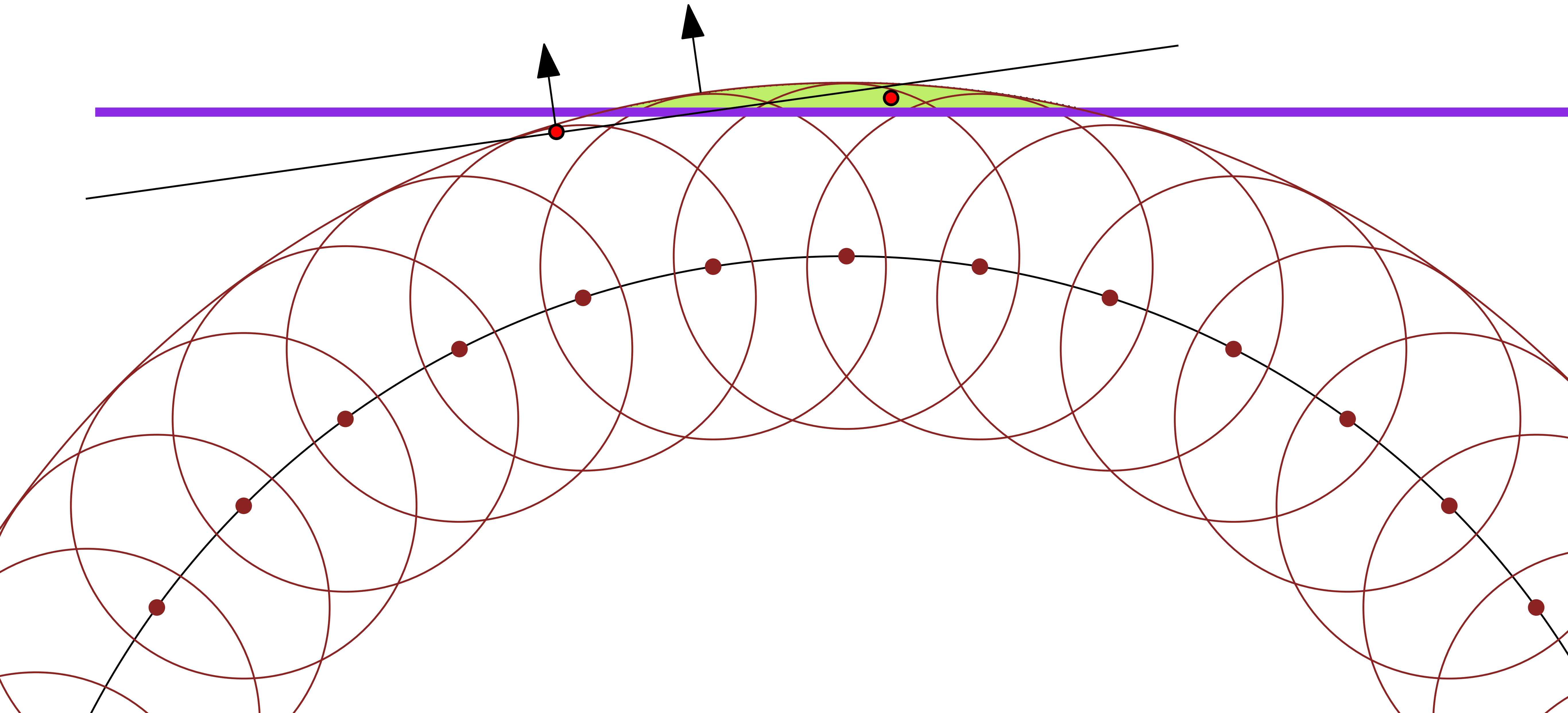
if the witness is non empty

considering a direction "of the witness"



Collector placing rule

if the witness is non empty
considering a direction ”of the witness”
the collector must contain the extremal point in that direction



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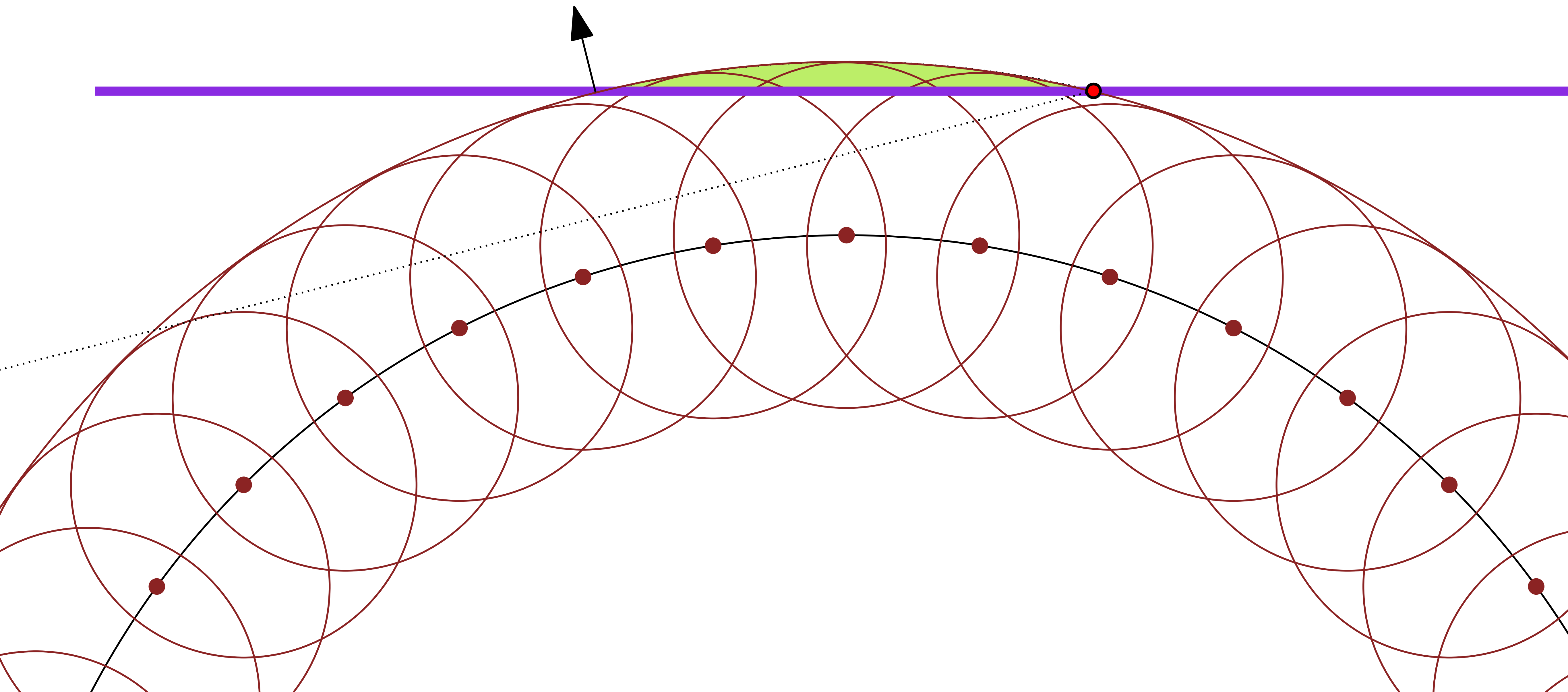

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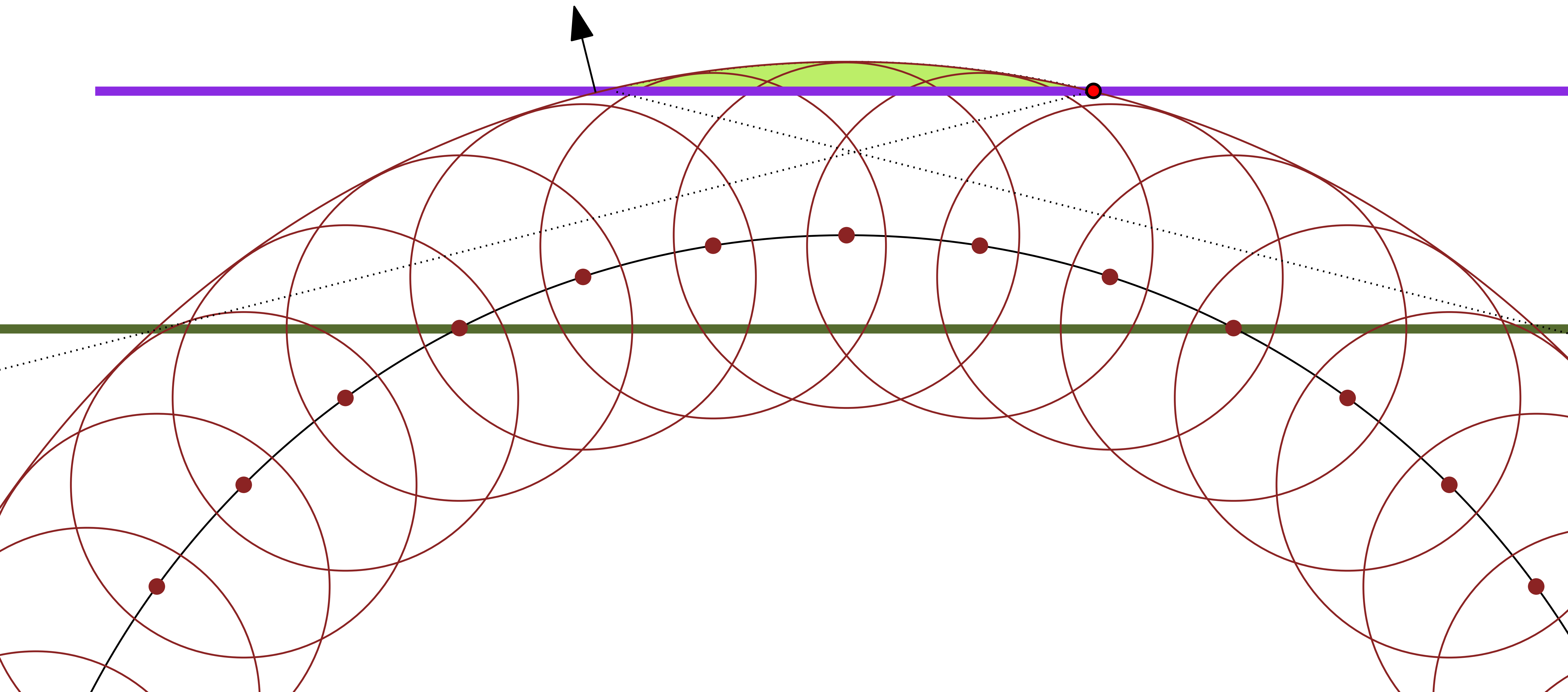
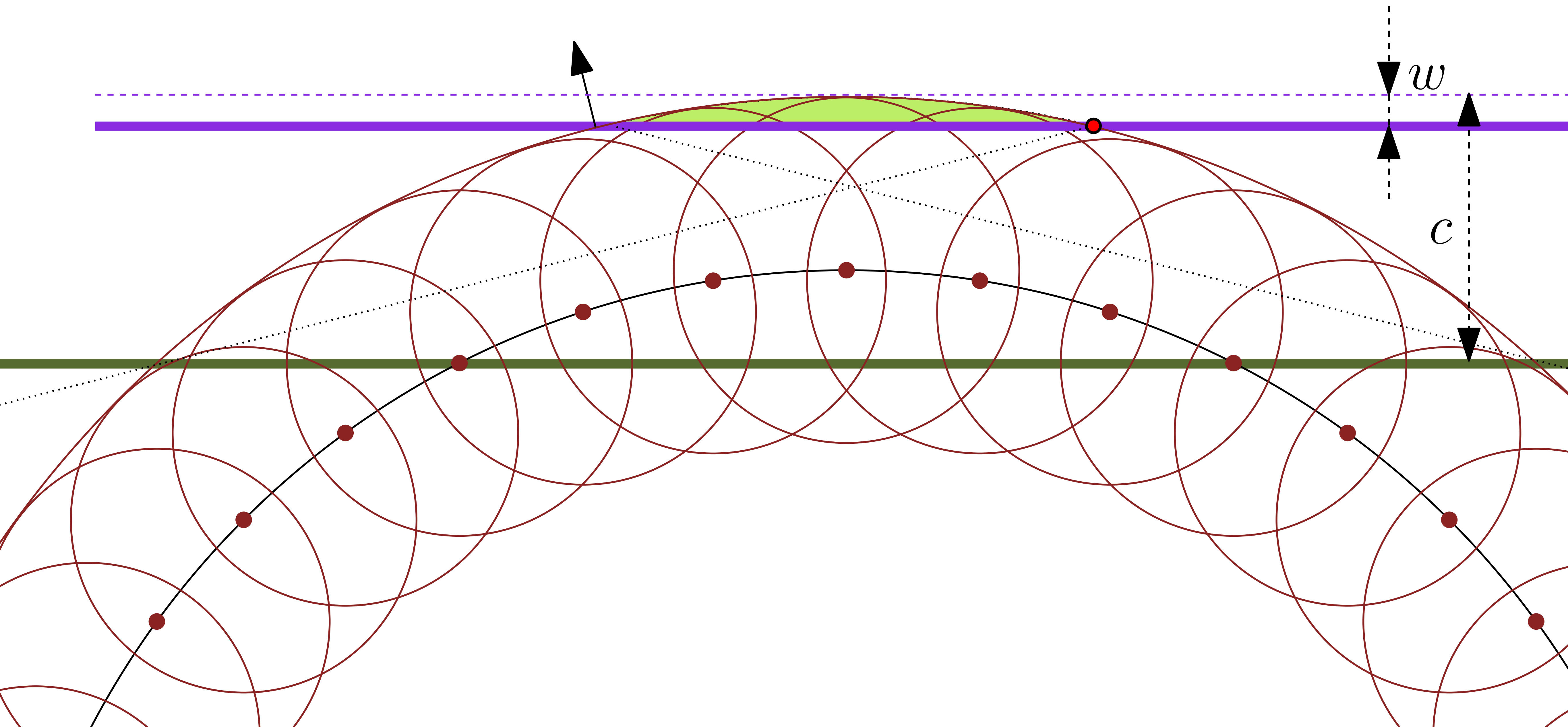
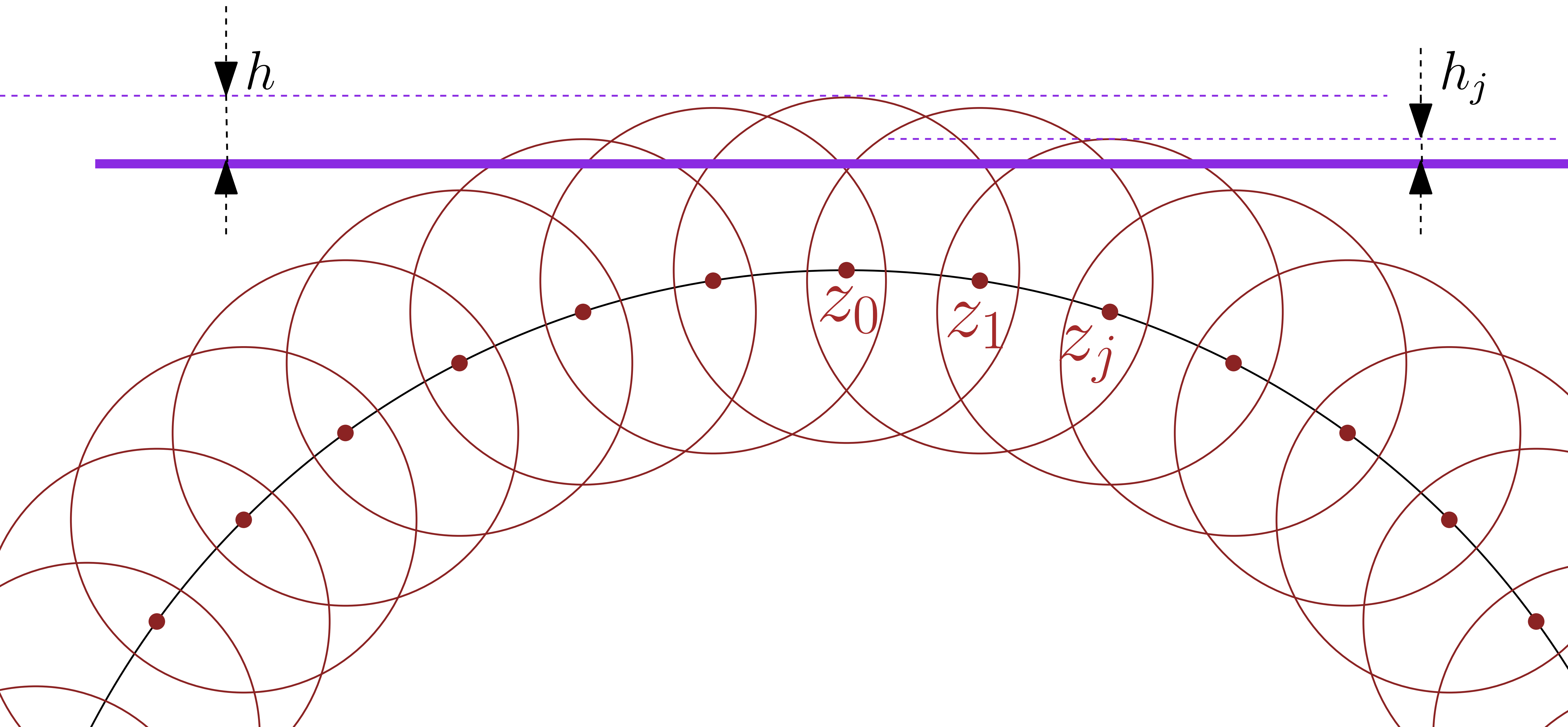


Collector placing rule

$$c \simeq 9w$$

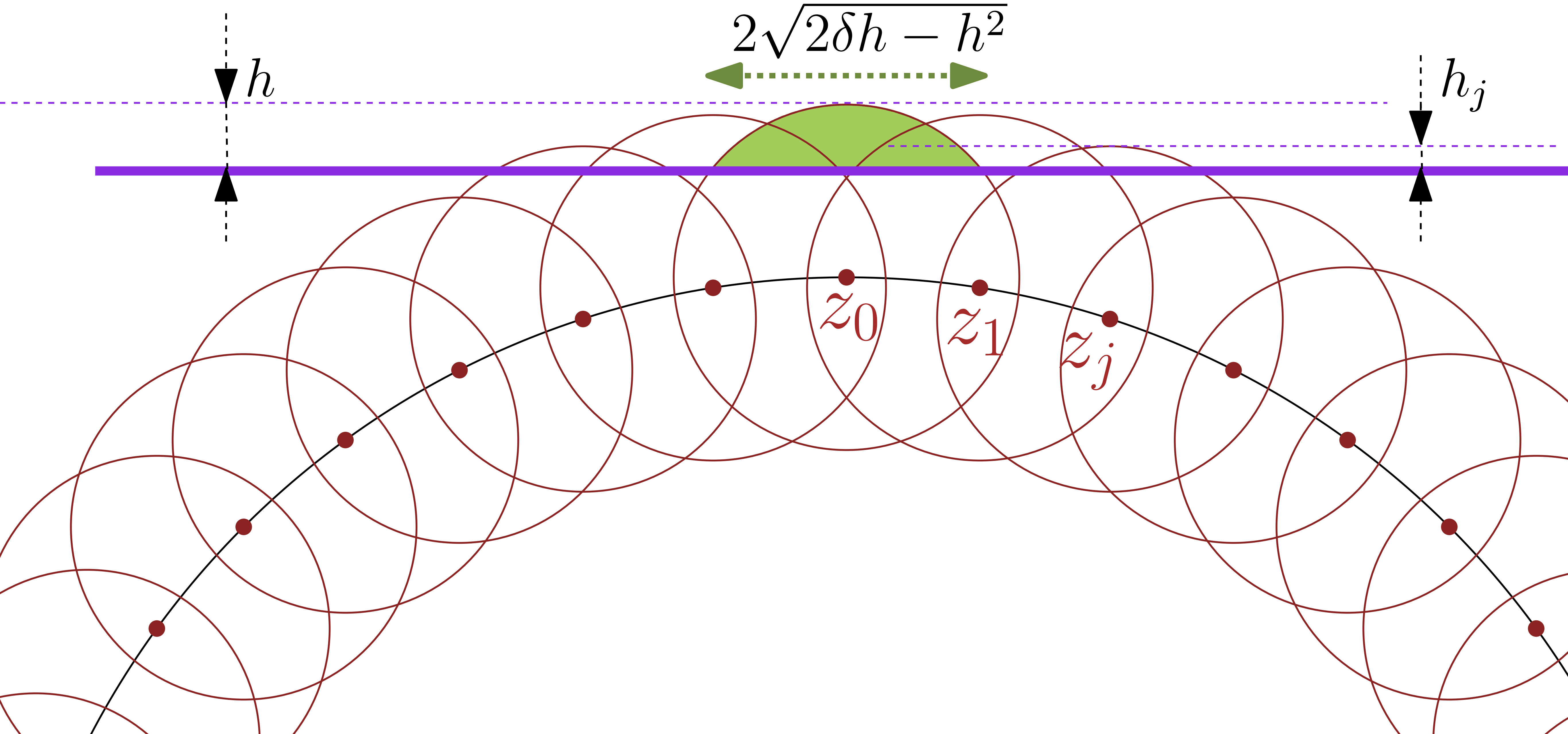


Computing w



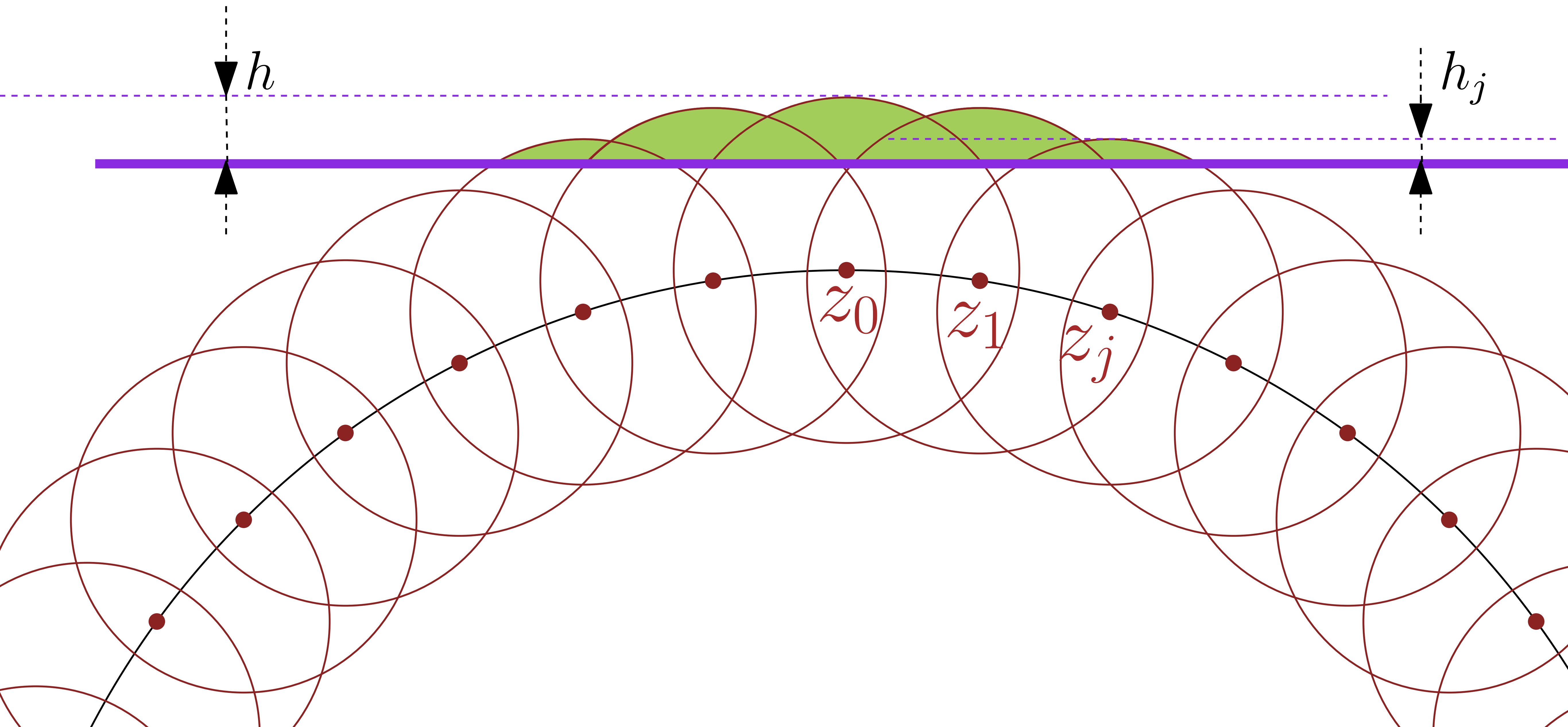
Computing w

$$E(\sharp\text{points}) \sim \frac{\delta^{1/2}h^{3/2}}{\pi\delta^2}$$



Computing w

$$E(\sharp\text{points}) \sim \sum_{j=0} \frac{\delta^{1/2} h_j^{3/2}}{\pi \delta^2}$$



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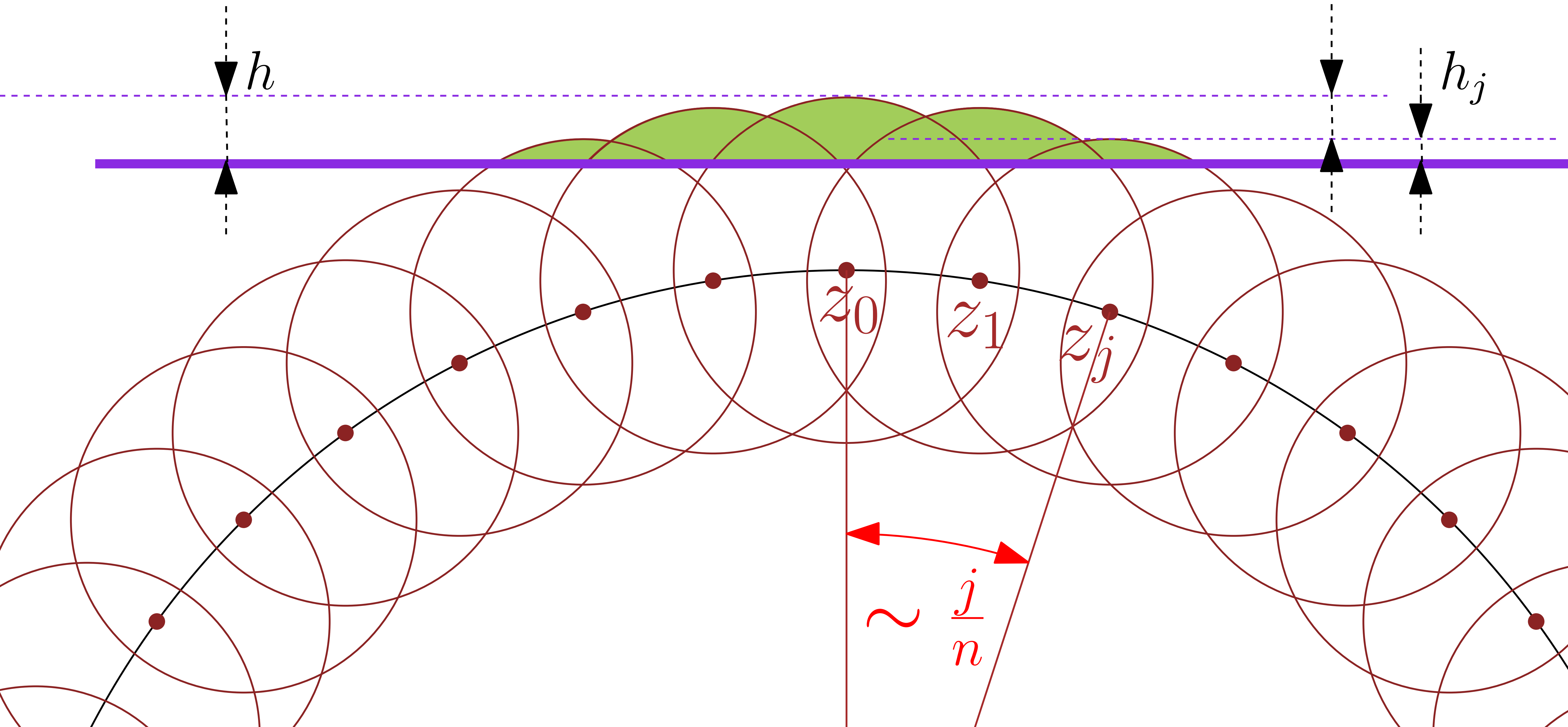

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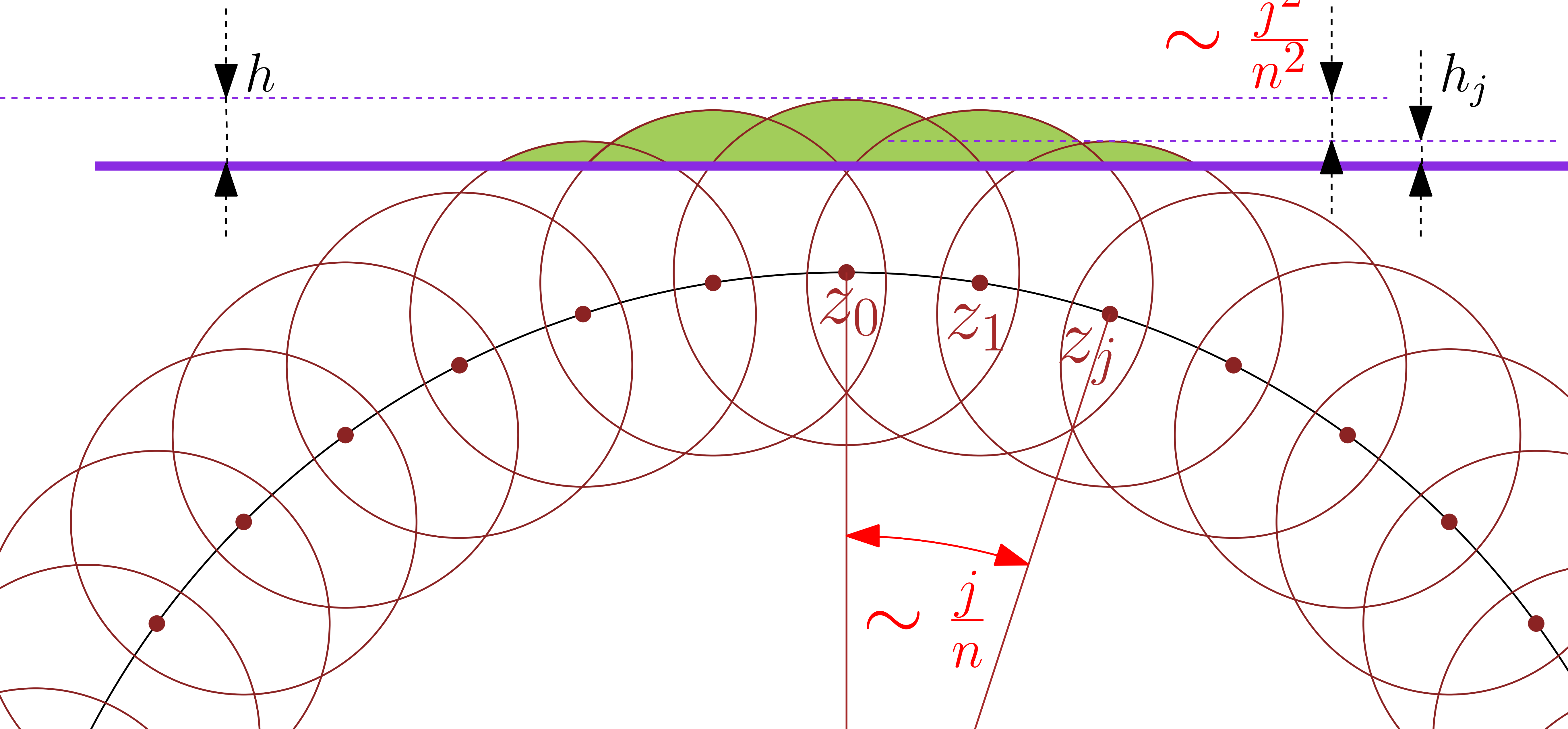
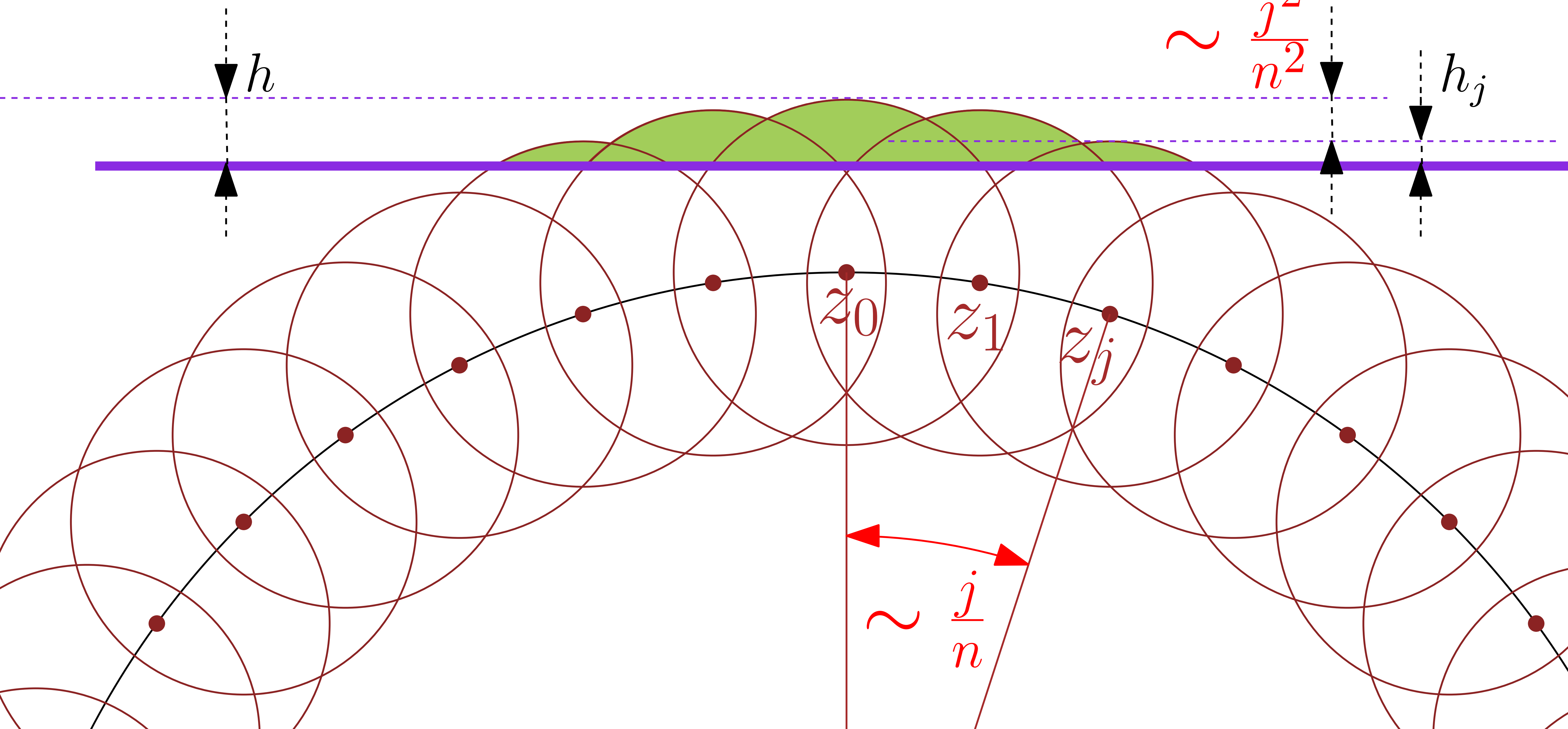


Computing w

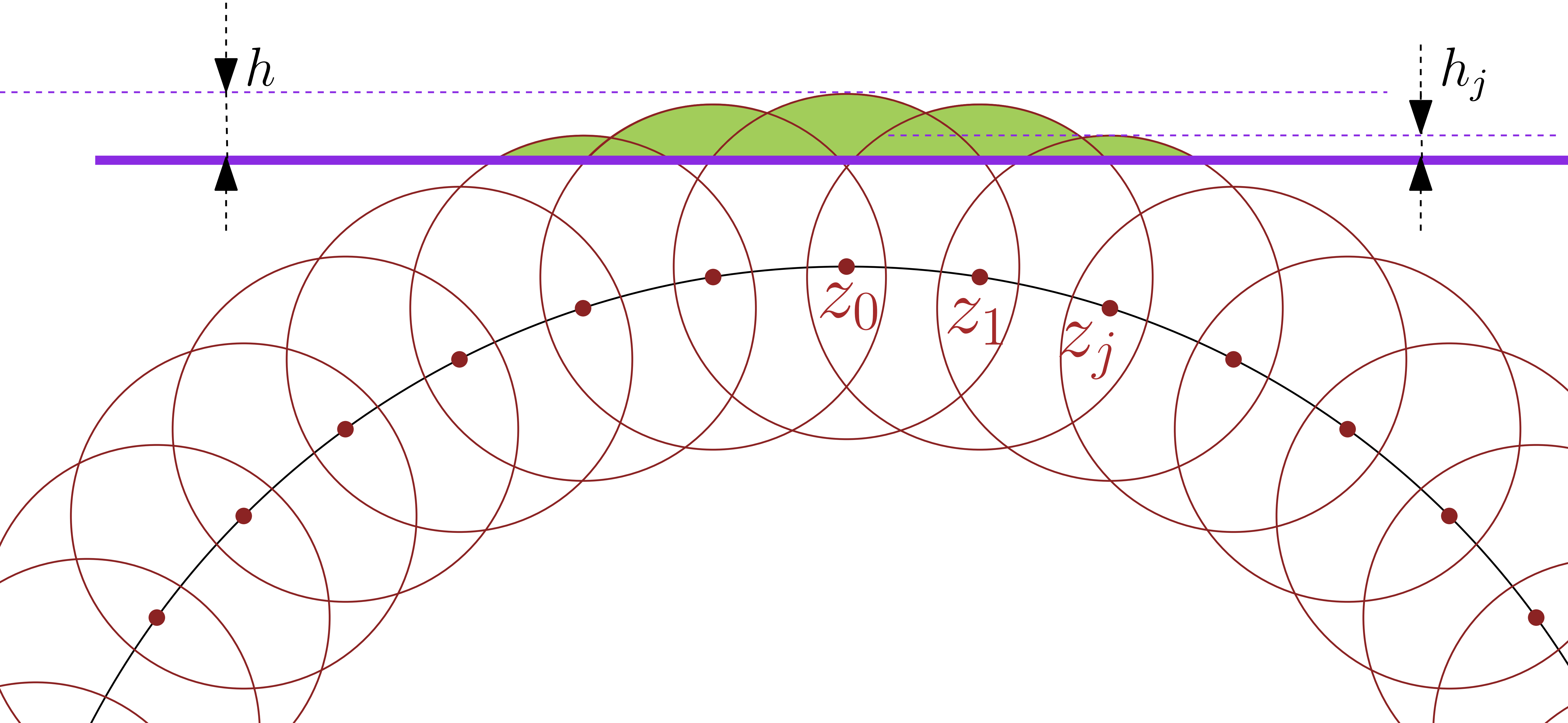
$$E(\sharp\text{points}) \sim \sum_{j=0} \frac{\delta^{1/2} h_j^{3/2}}{\pi \delta^2}$$

$$h_j \geq 0 \iff j \leq n h^{1/2}$$



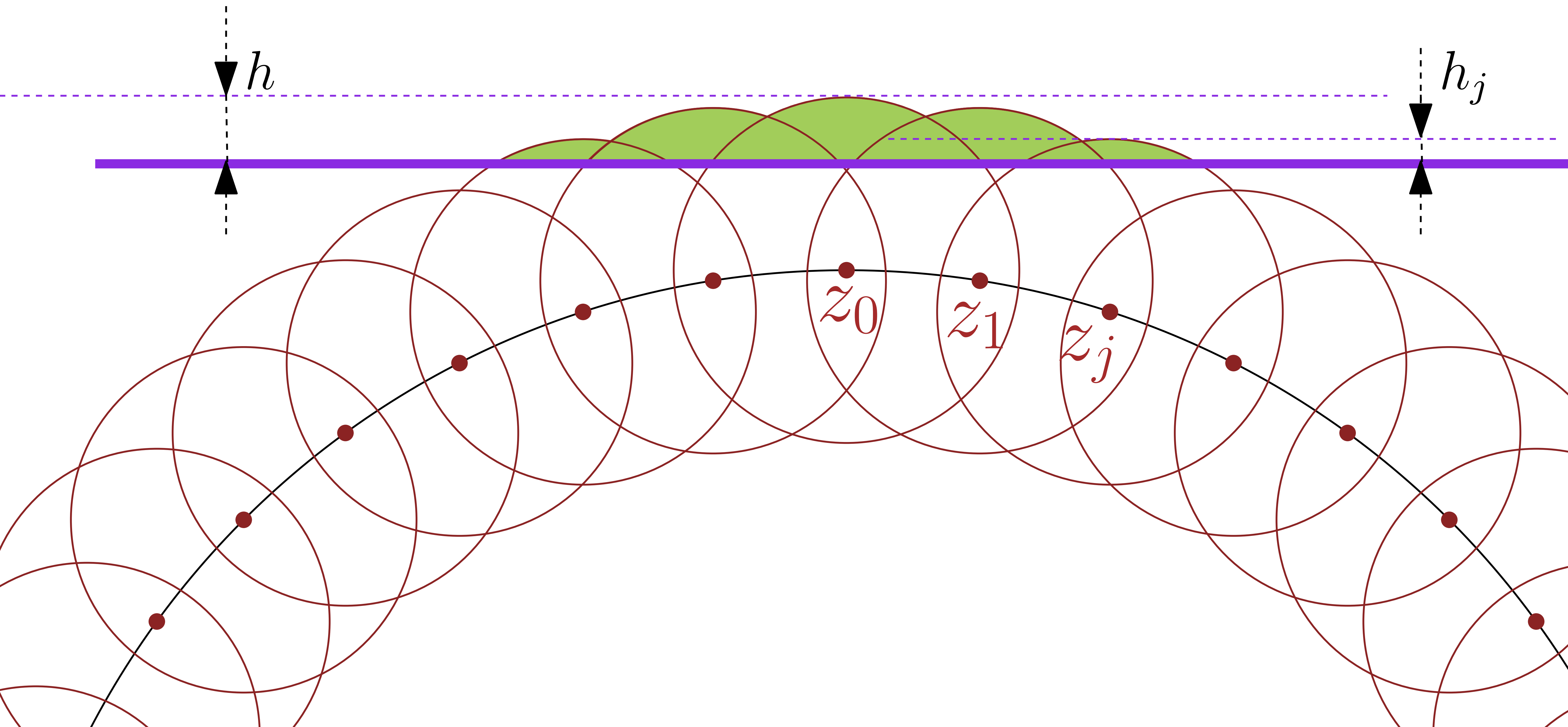
Computing w

$$E(\sharp\text{points}) \sim \sum_{j=0}^{nh^{1/2}} \frac{\delta^{1/2} h_j^{3/2}}{\pi \delta^2}$$



Computing w

$$E(\sharp\text{points}) \sim \sum_{j=0}^{nh^{1/2}} \frac{\delta^{1/2} h_j^{3/2}}{\pi \delta^2} = nh^2 \delta^{-3/2}$$



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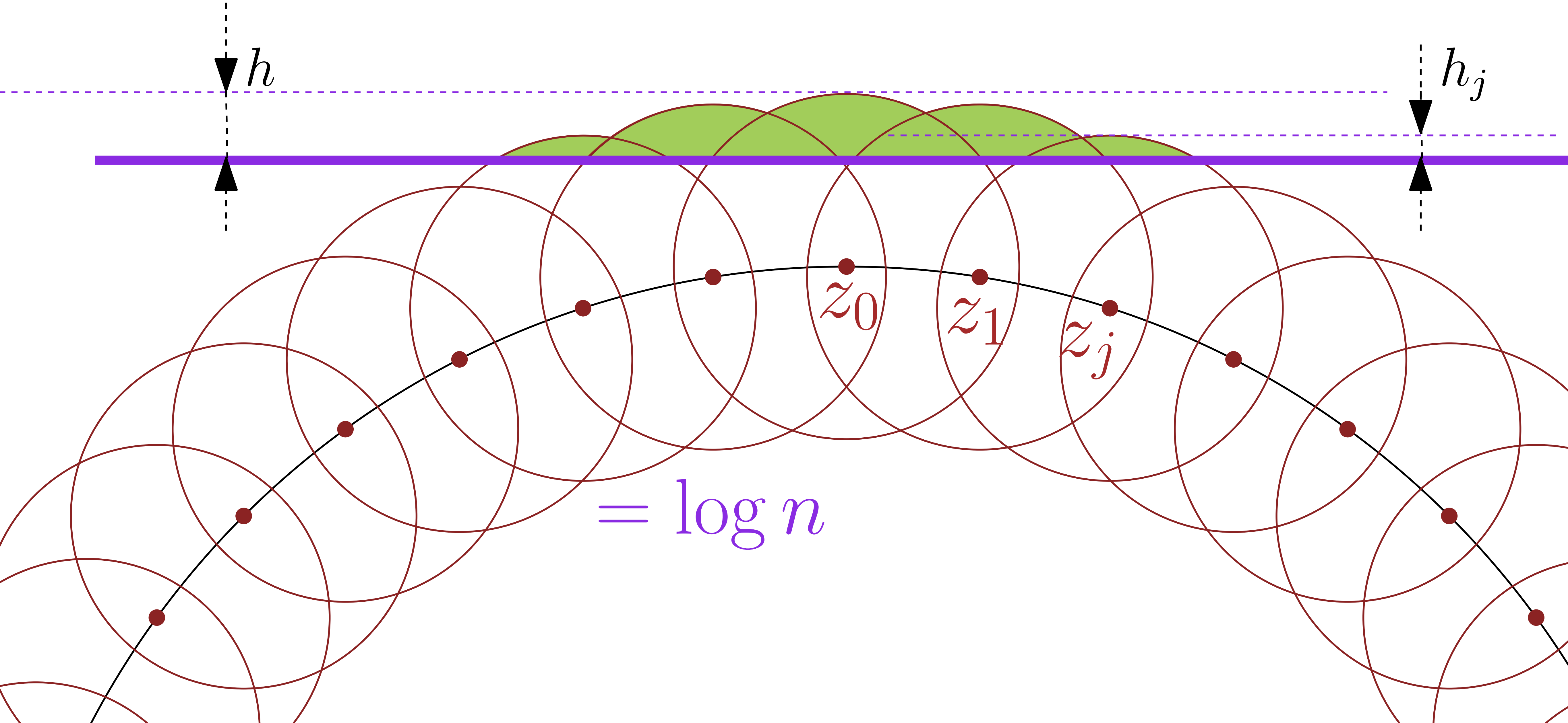

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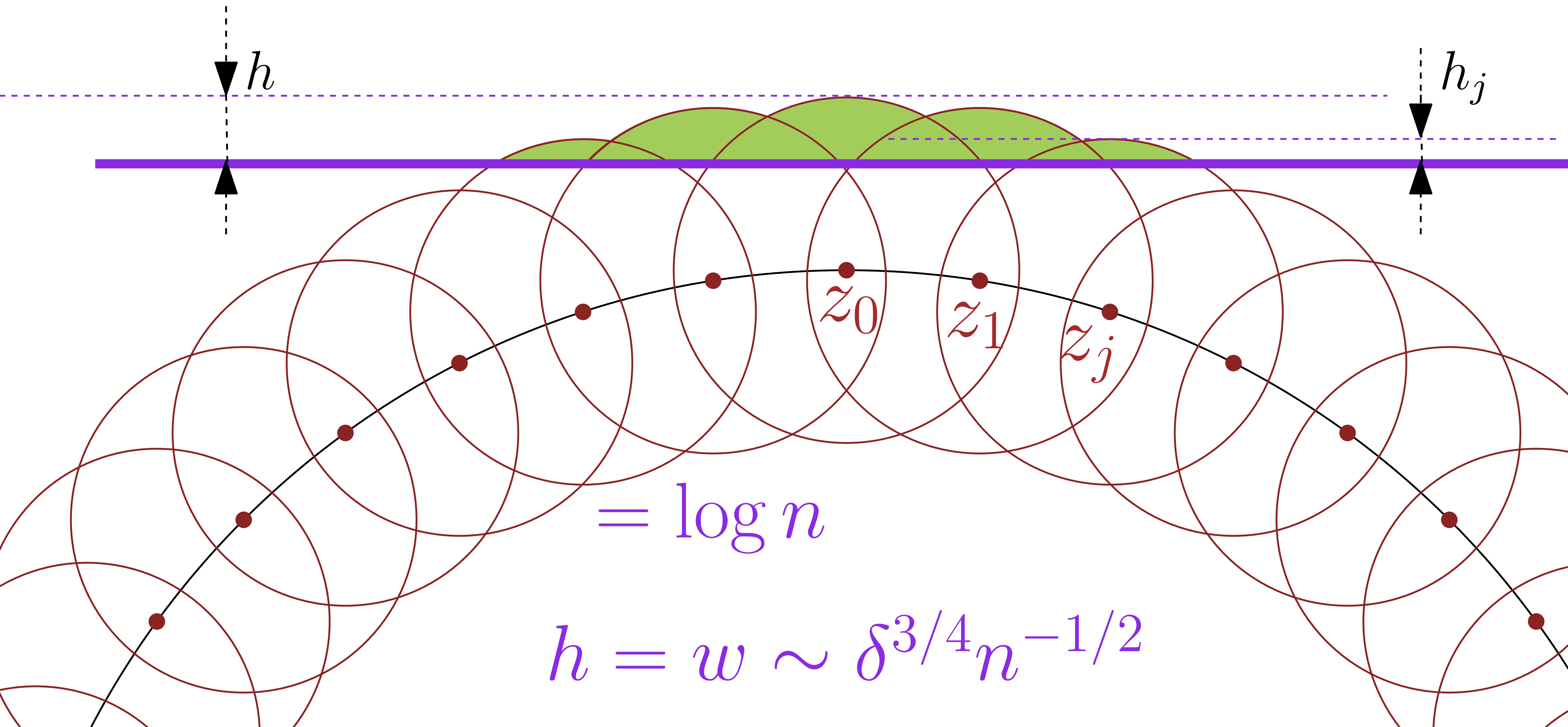
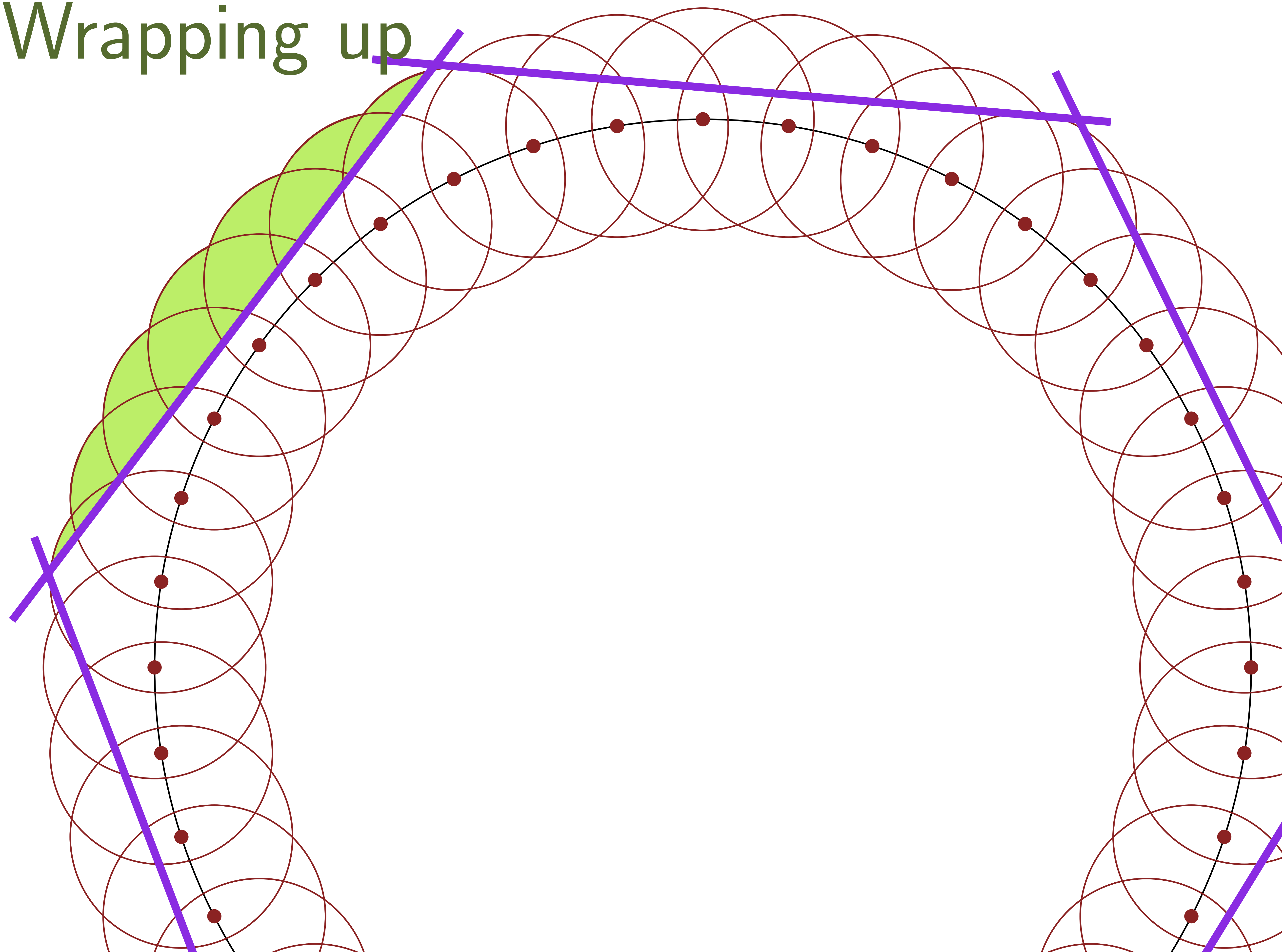
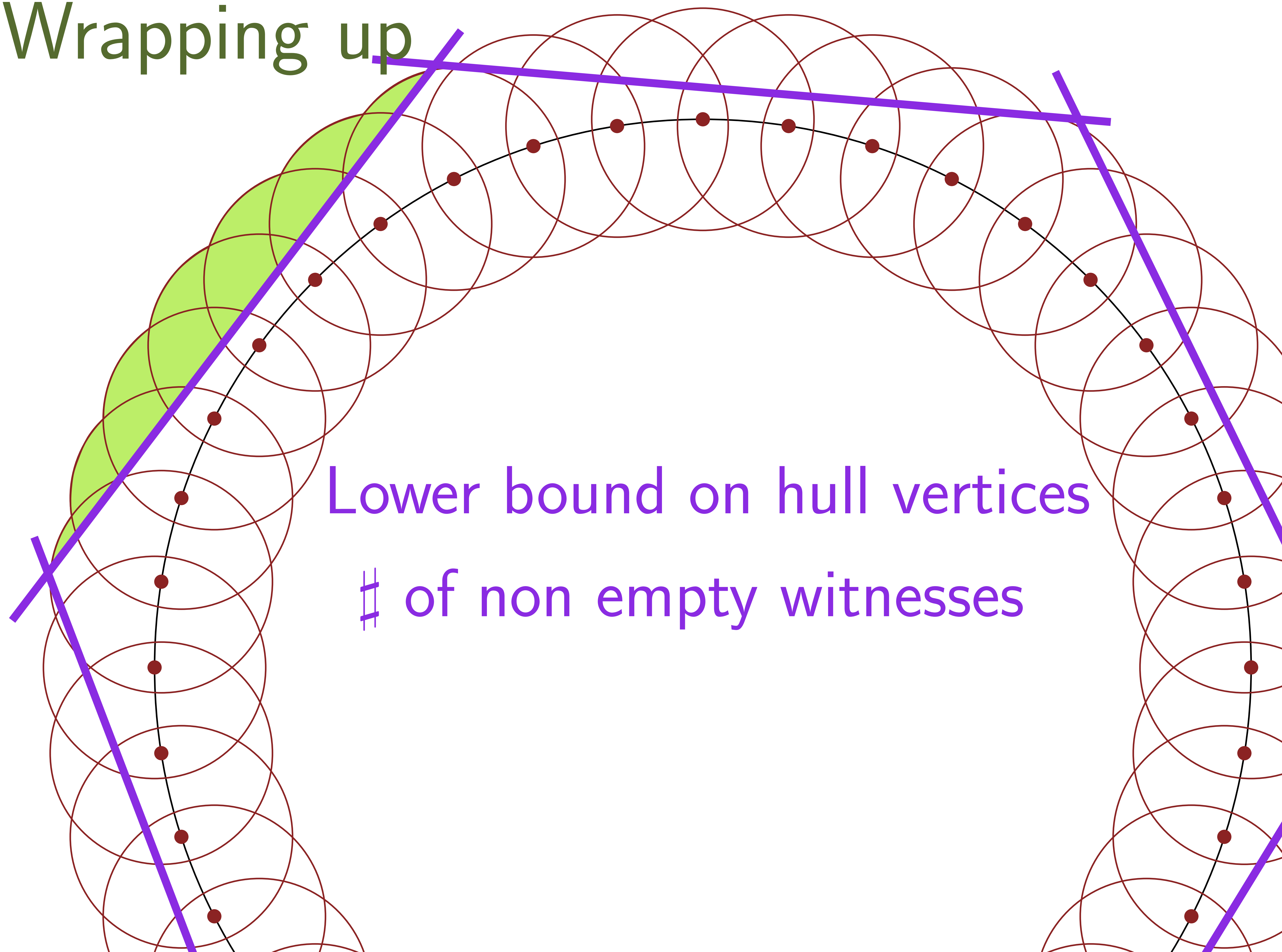


Wrapping up



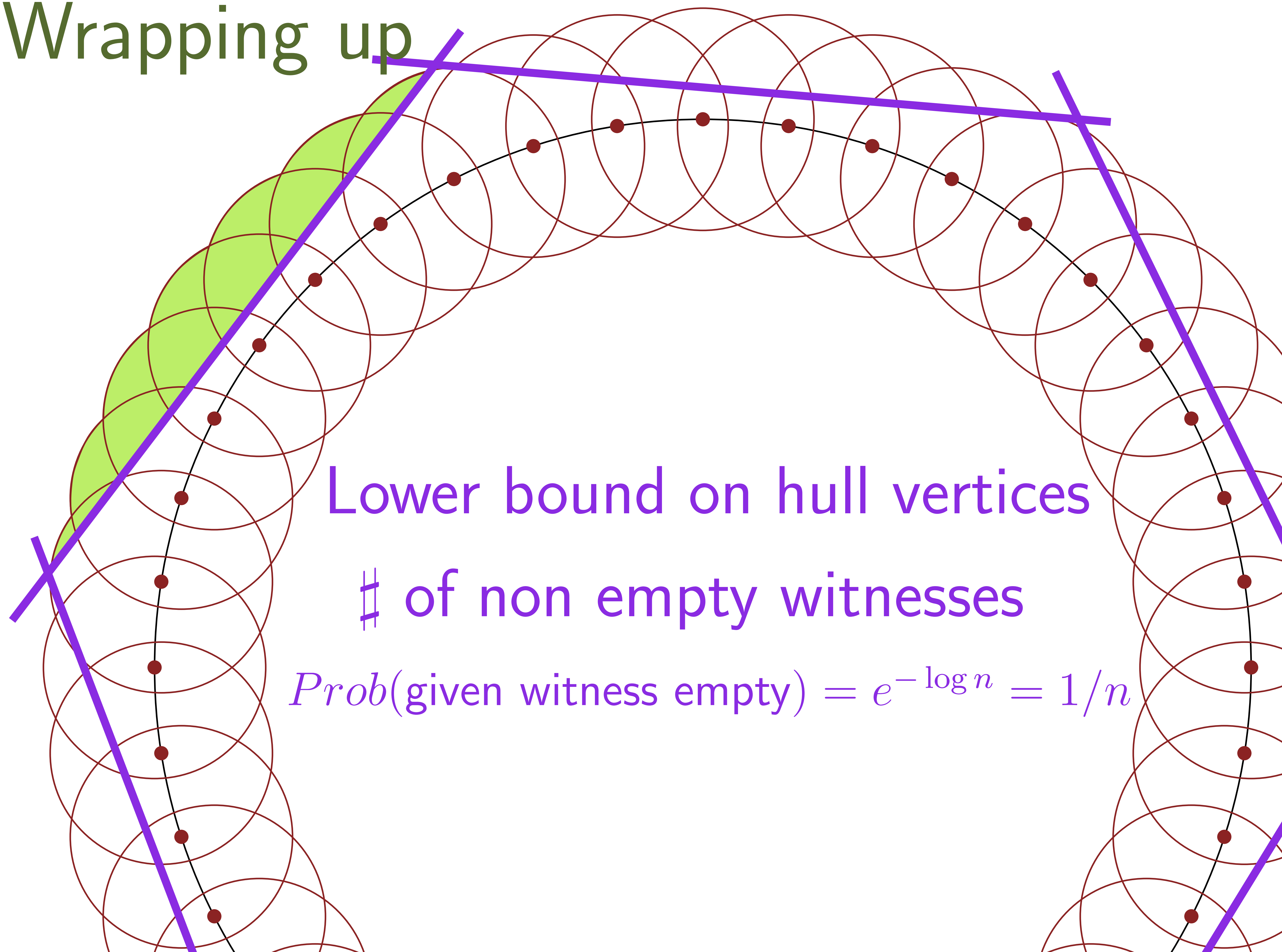
Wrapping up



Lower bound on hull vertices

$\sharp$ of non empty witnesses

Wrapping up

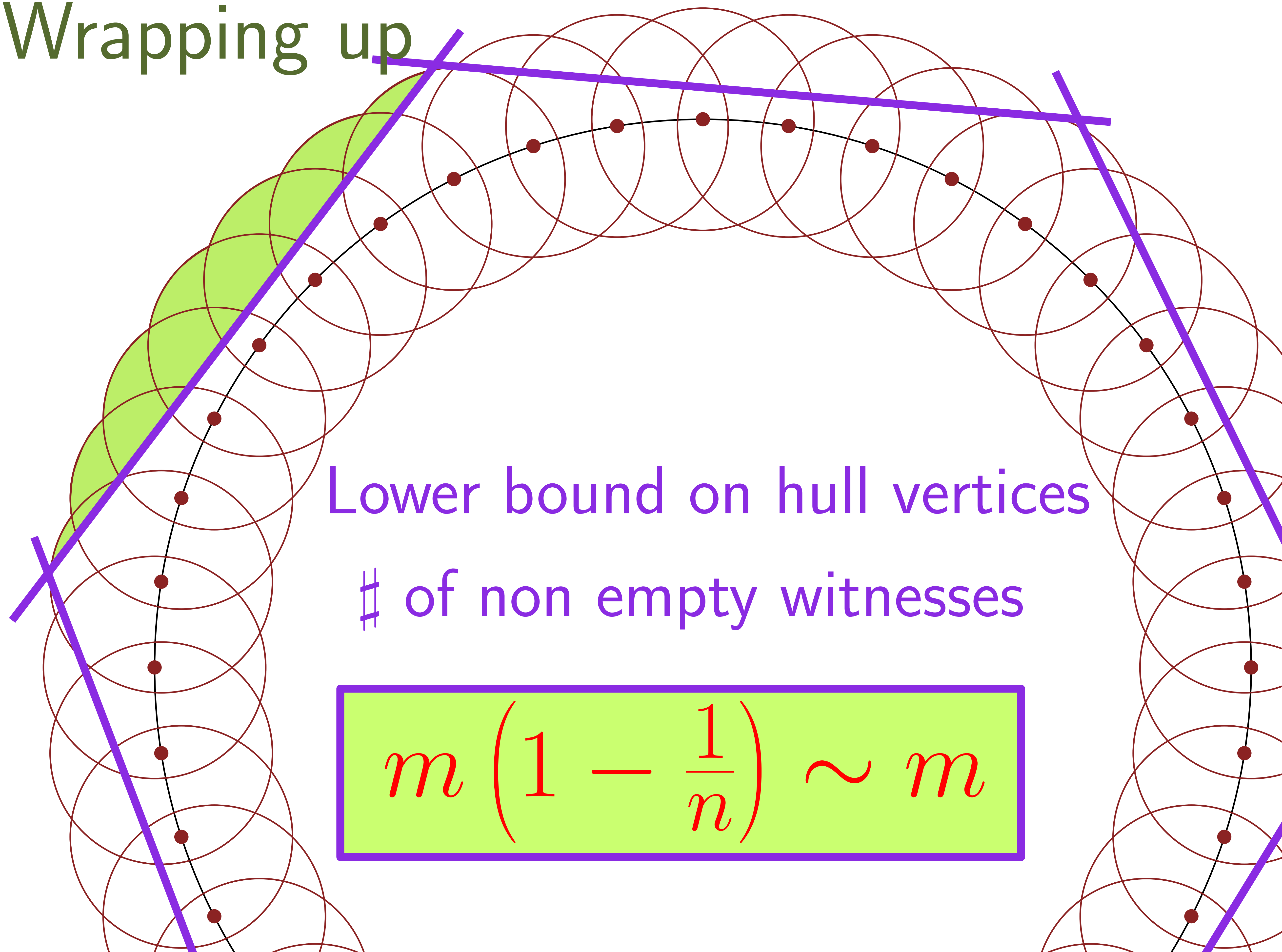


Lower bound on hull vertices

$\sharp$ of non empty witnesses

$Prob(\text{given witness empty}) = e^{-\log n} = 1/n$

Wrapping up

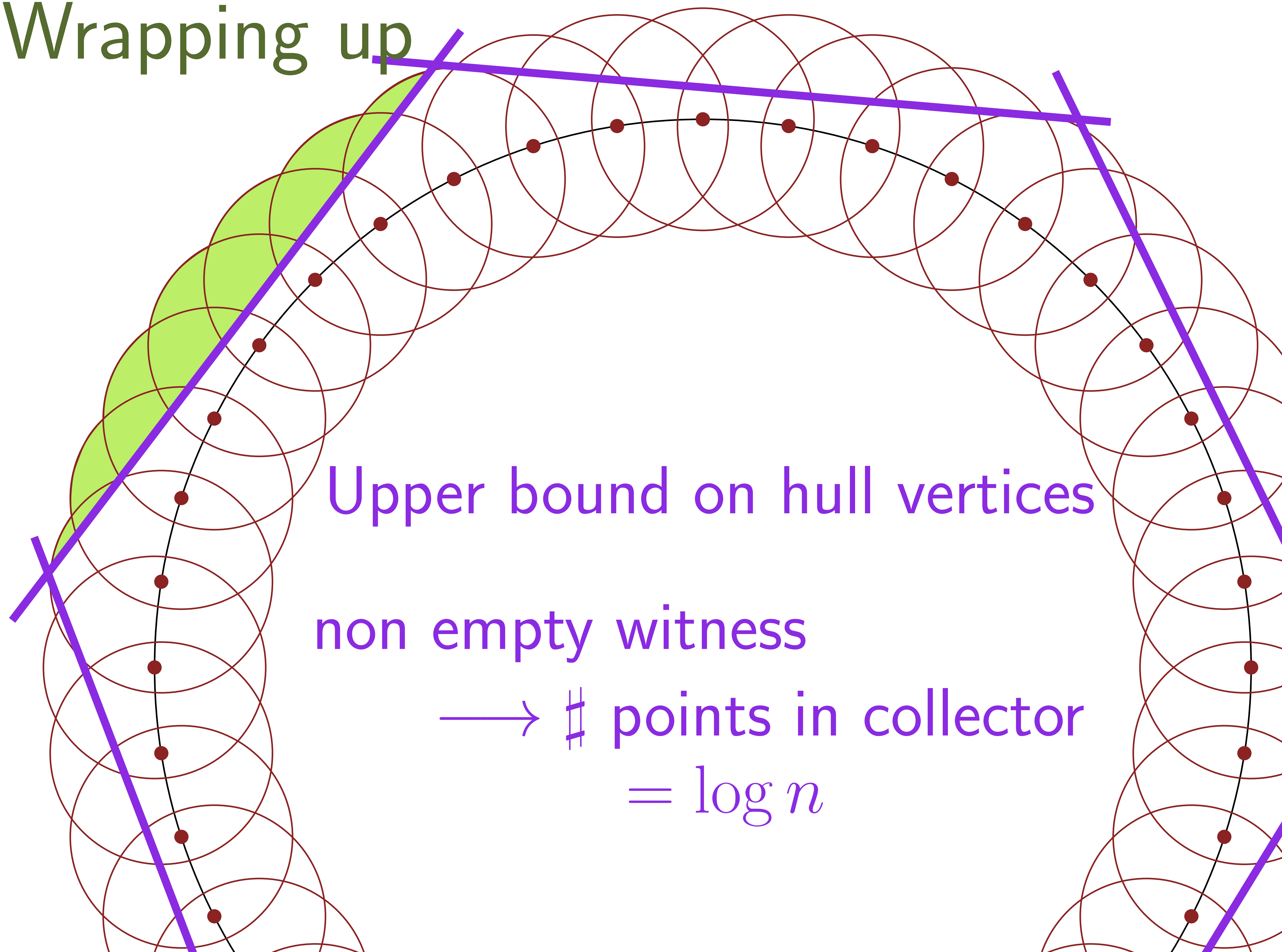


Lower bound on hull vertices

♯ of non empty witnesses

$$m\left(1-\frac{1}{n}\right)\sim m$$

Wrapping up



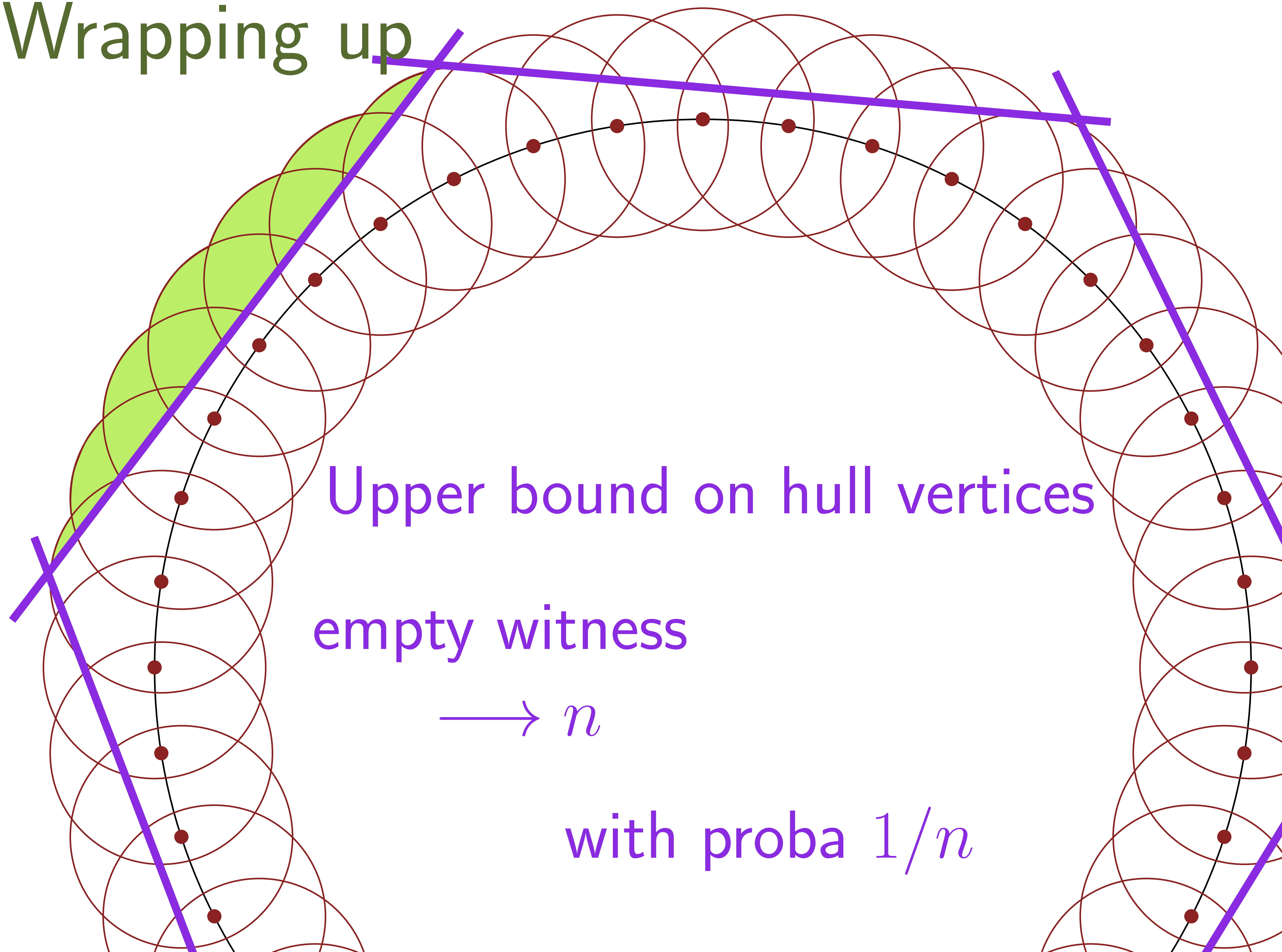
Upper bound on hull vertices

non empty witness

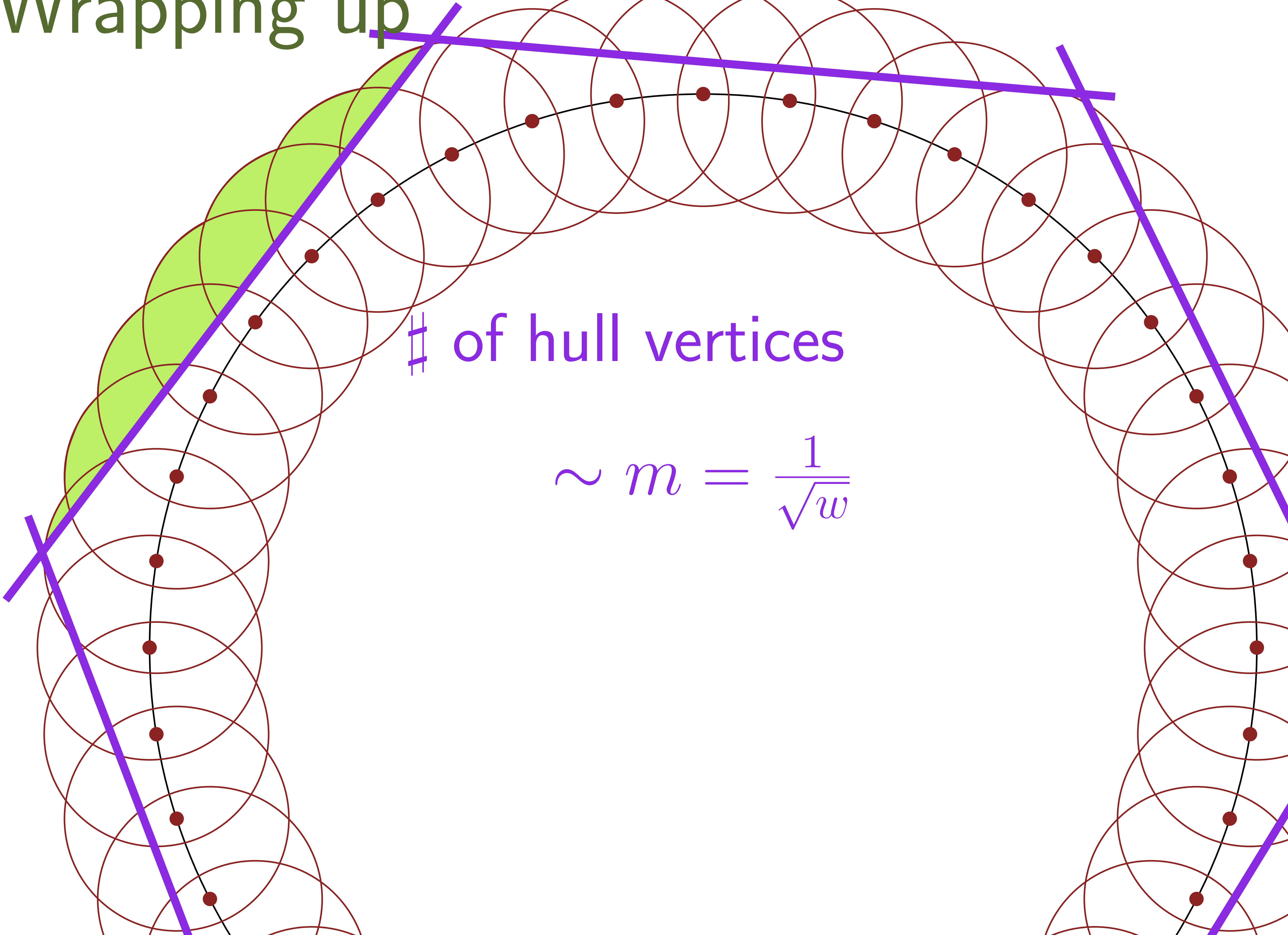
$\longrightarrow \sharp$ points in collector

$= \log n$

Wrapping up


Upper bound on hull vertices
empty witness
$\longrightarrow n$
with proba $1/n$

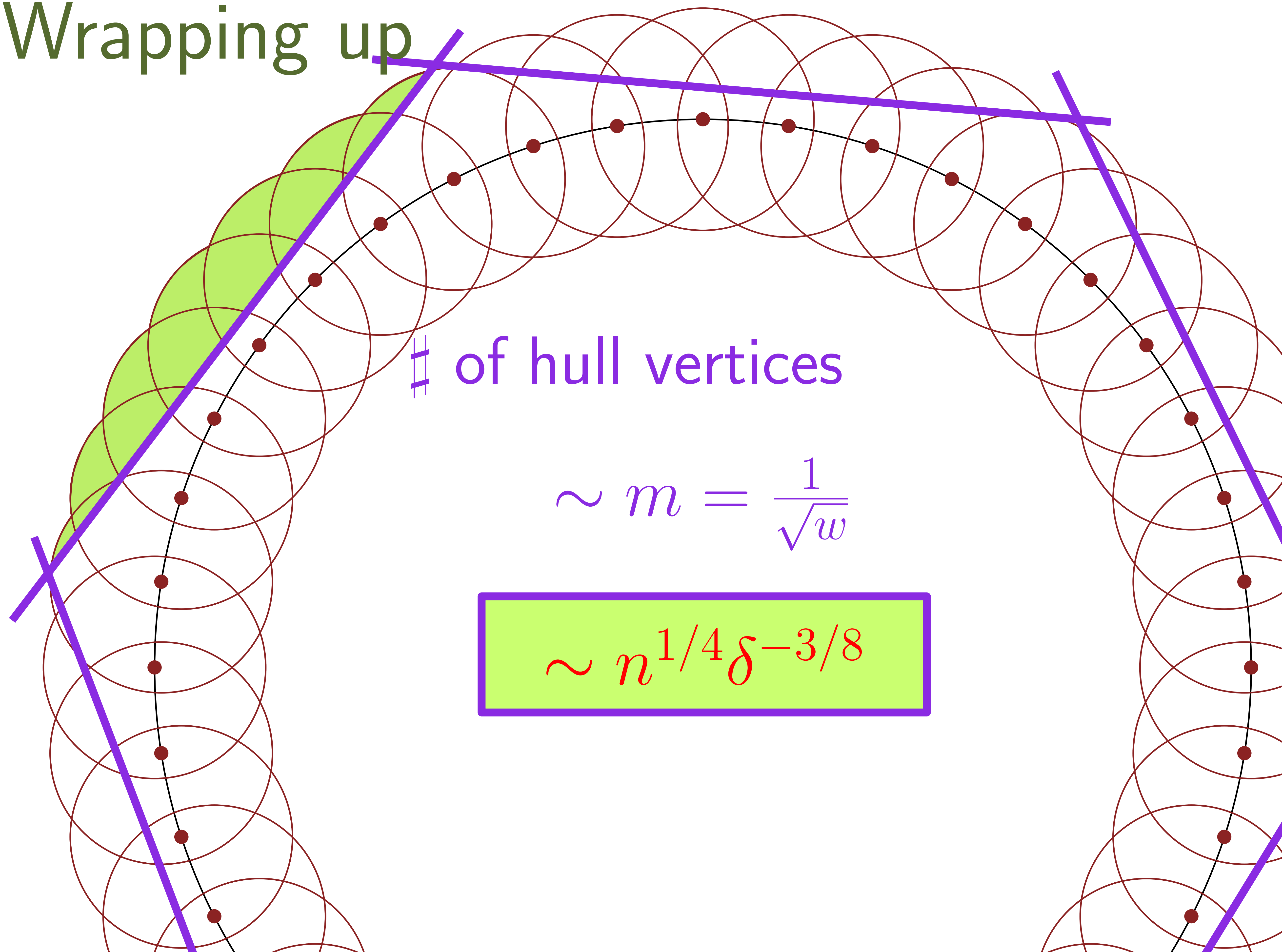
Wrapping up



$\sharp$ of hull vertices

$$\sim m = \frac{1}{\sqrt{w}}$$

Wrapping up

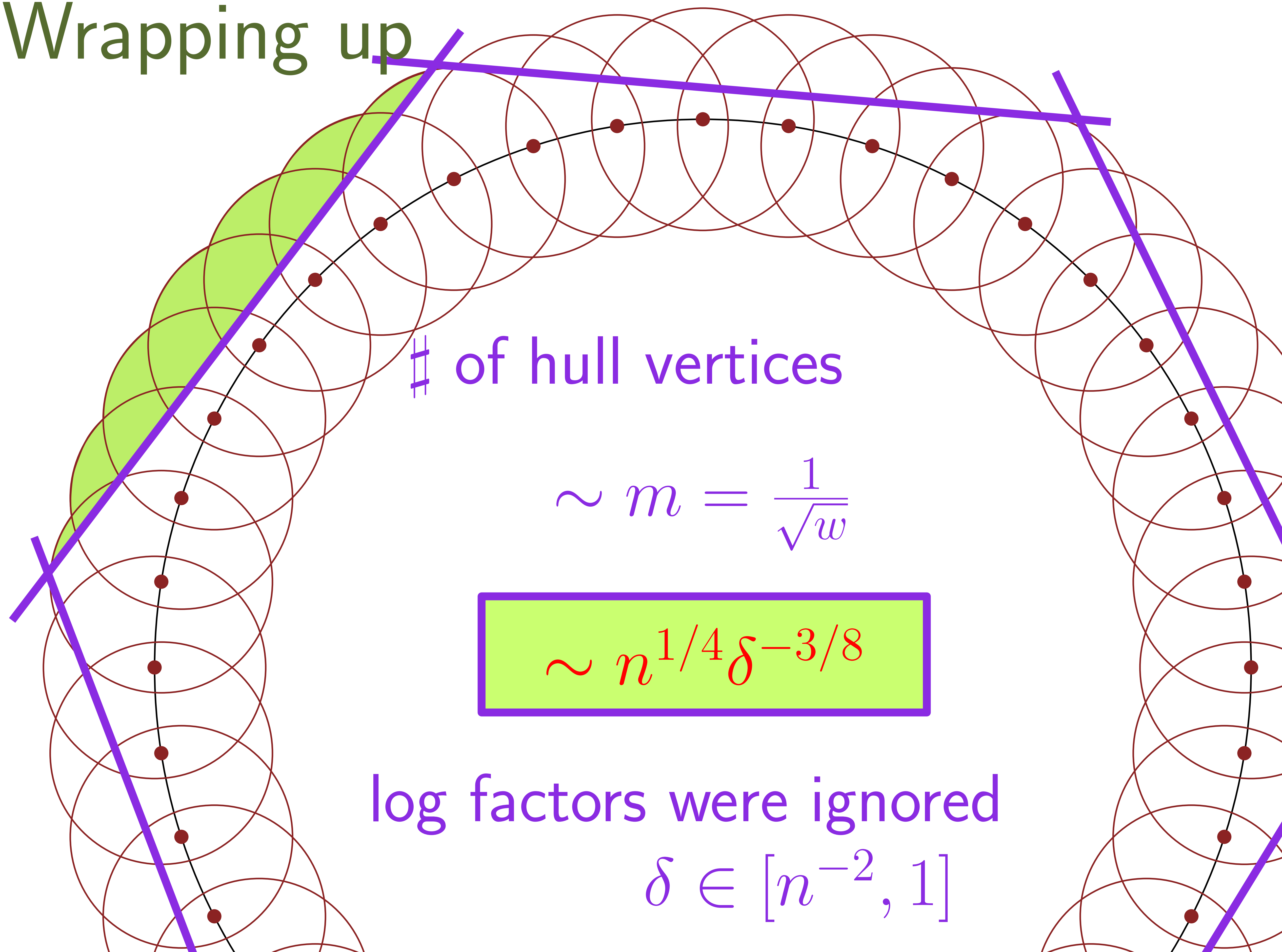


$\sharp$ of hull vertices

$$\sim m = \frac{1}{\sqrt{w}}$$

$$\sim n^{1/4}\delta^{-3/8}$$

Wrapping up



$\sharp$ of hull vertices

$$\sim m = \frac{1}{\sqrt{w}}$$

$$\boxed{\sim n^{1/4}\delta^{-3/8}}$$

log factors were ignored

$$\delta \in [n^{-2}, 1]$$

Higher dimensions

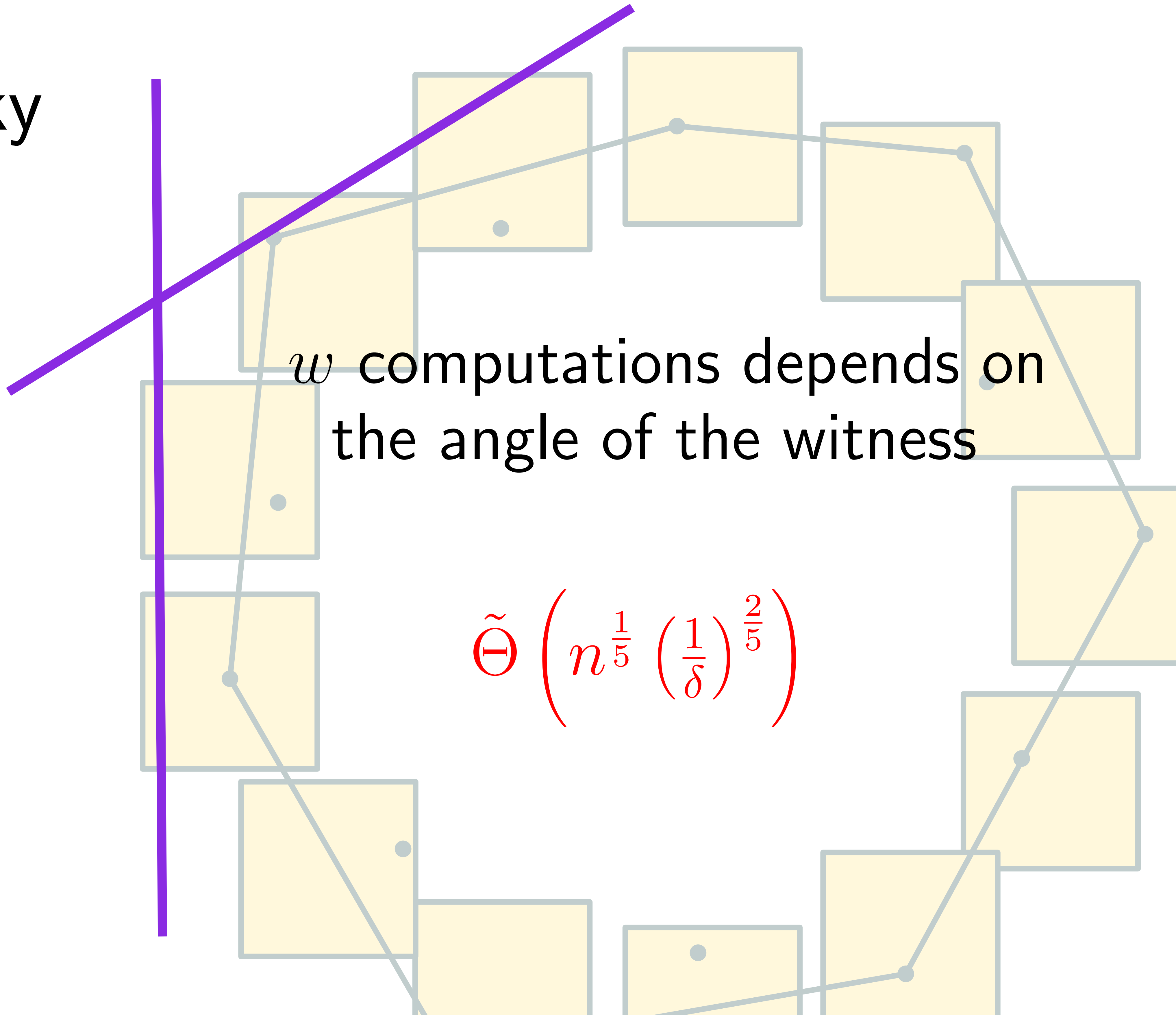
same ideas

initial position$=(\epsilon, \kappa)$-sample

$$\tilde{\Theta}\left((\sqrt{n})^{1-\frac{1}{d}}\left(\frac{1}{\sqrt[4]{\delta}}\right)^{d-\frac{1}{d}}\right)$$

Noise with squared support

more tricky



w computations depends on the angle of the witness

$$\tilde{\Theta}\left(n^{\frac{1}{5}}\left(\frac{1}{\delta}\right)^{\frac{2}{5}}\right)$$

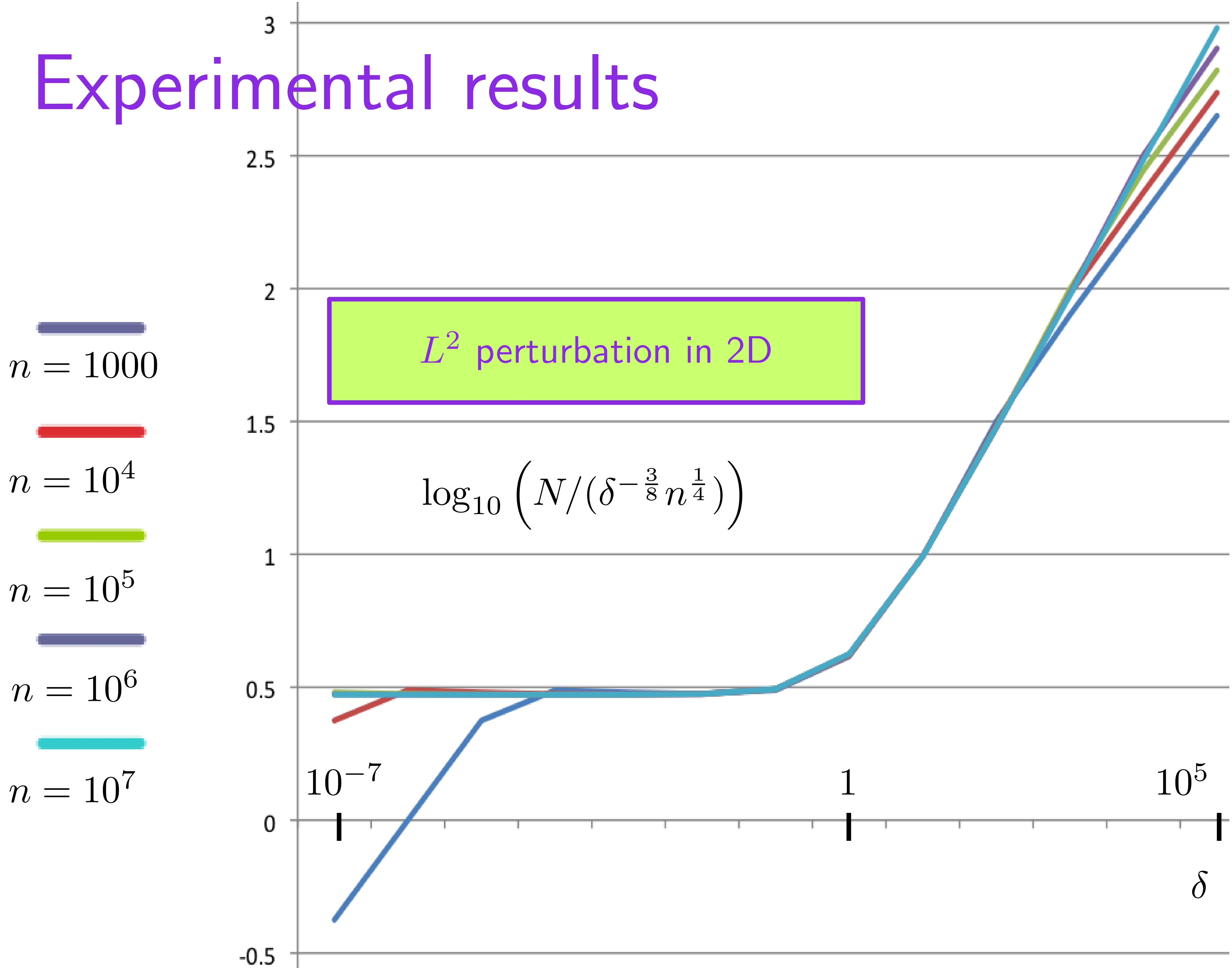
Experimental results

Experimental results

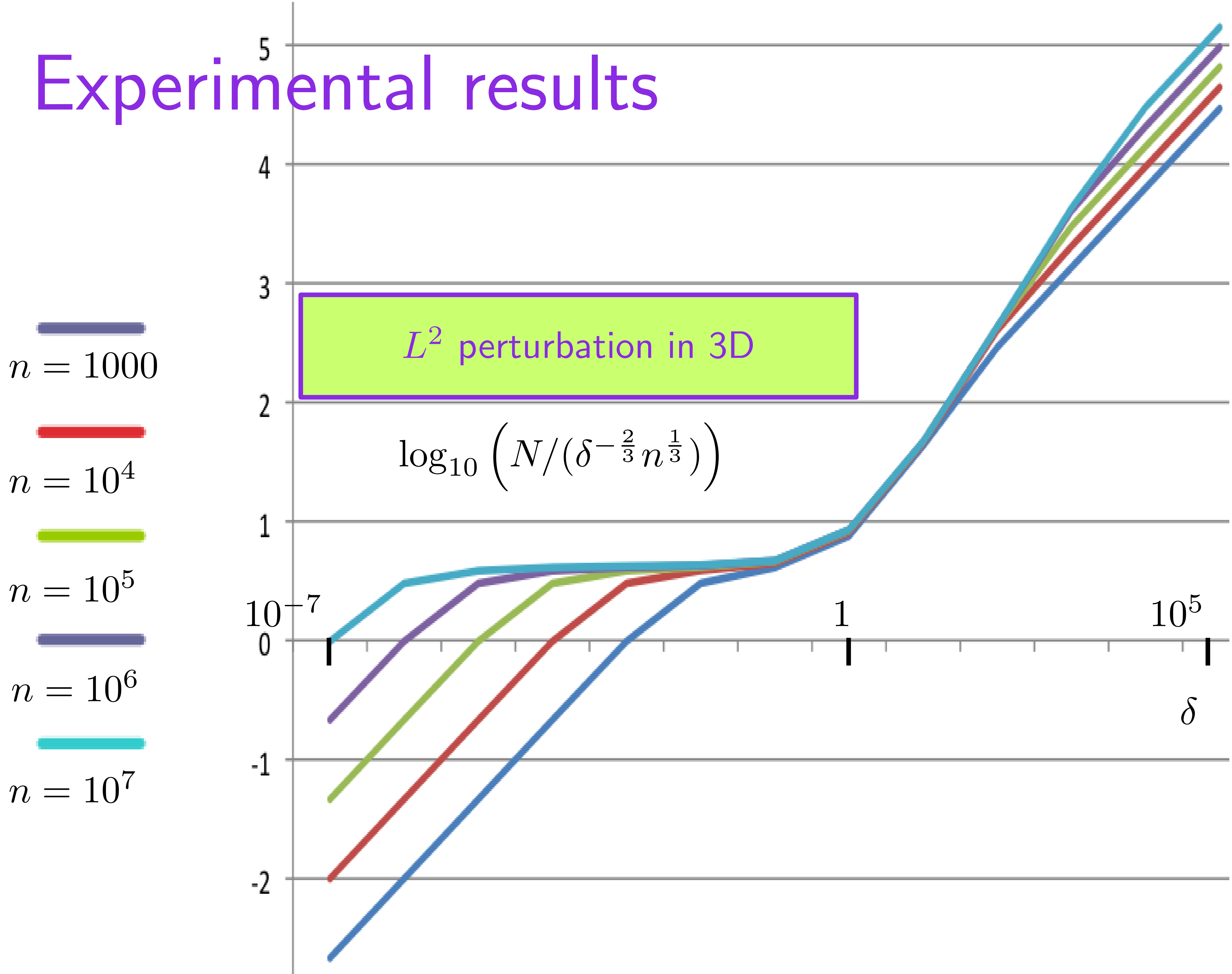
$$N \sim n^{1/4} \delta^{-3/8}$$

$$\log \frac{N}{n^{1/4} \delta^{-3/8}} \simeq cte$$

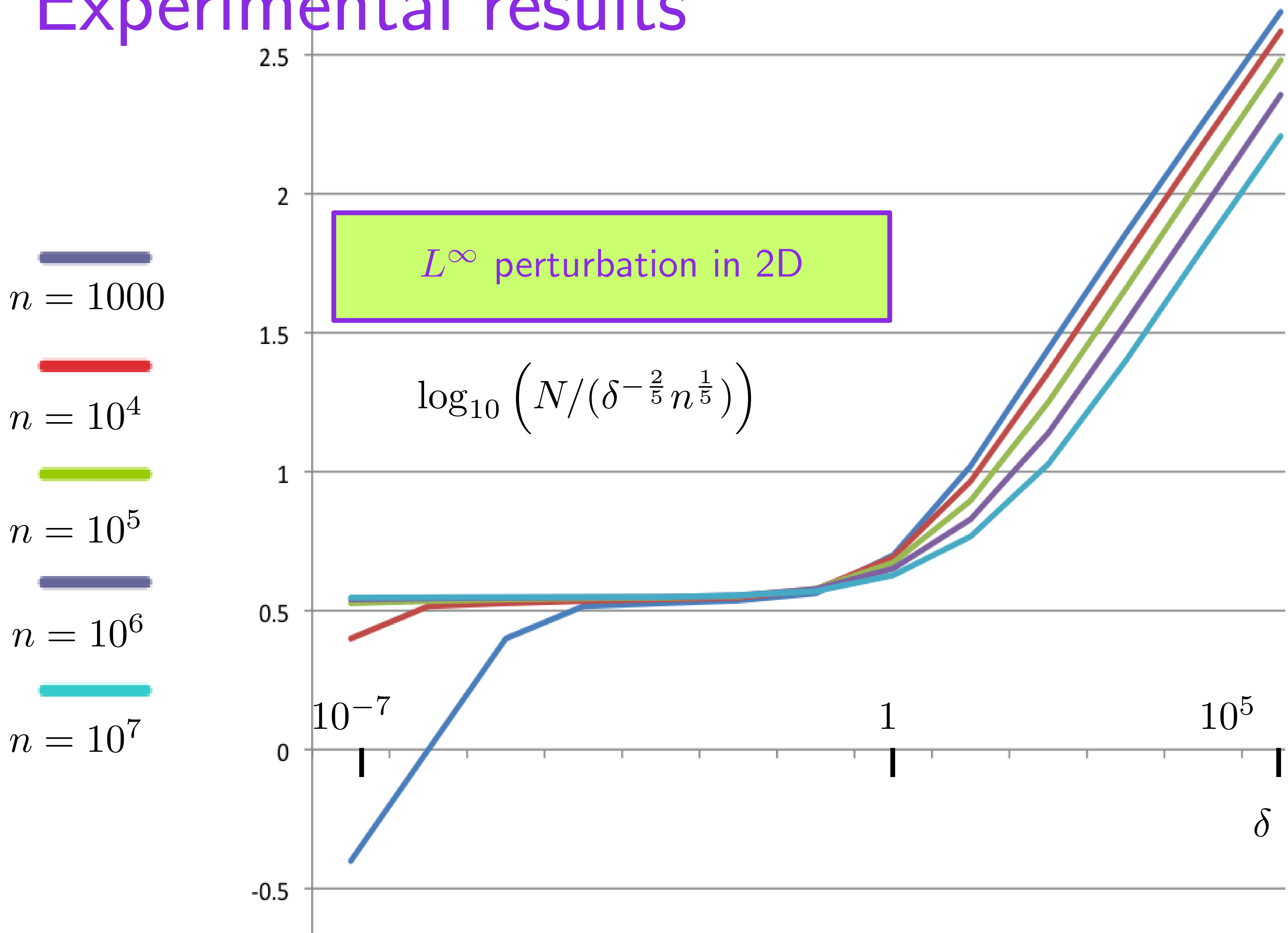
Experimental results



Experimental results



Experimental results

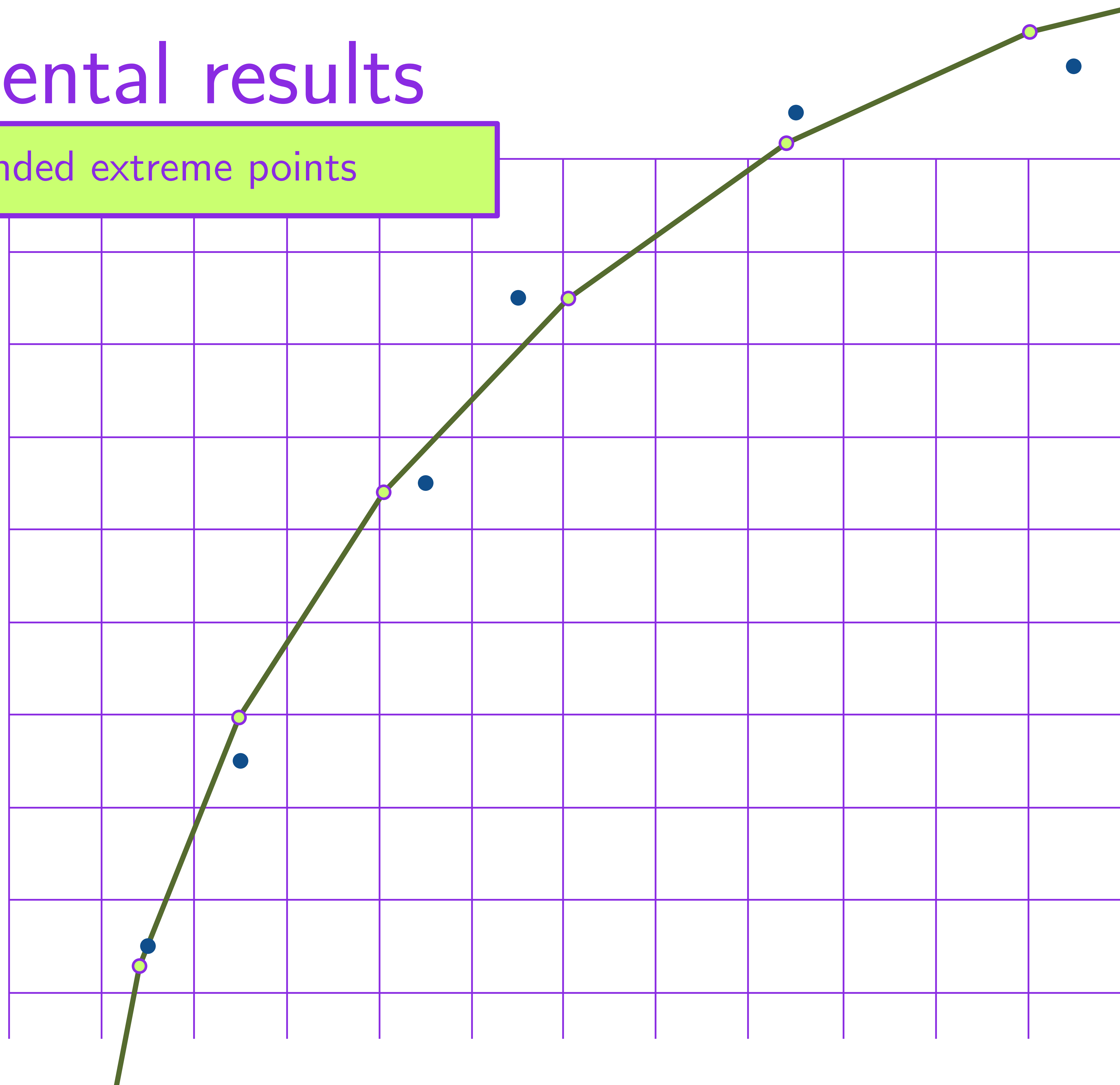


Experimental results

Snap-rounded extreme points

Experimental results

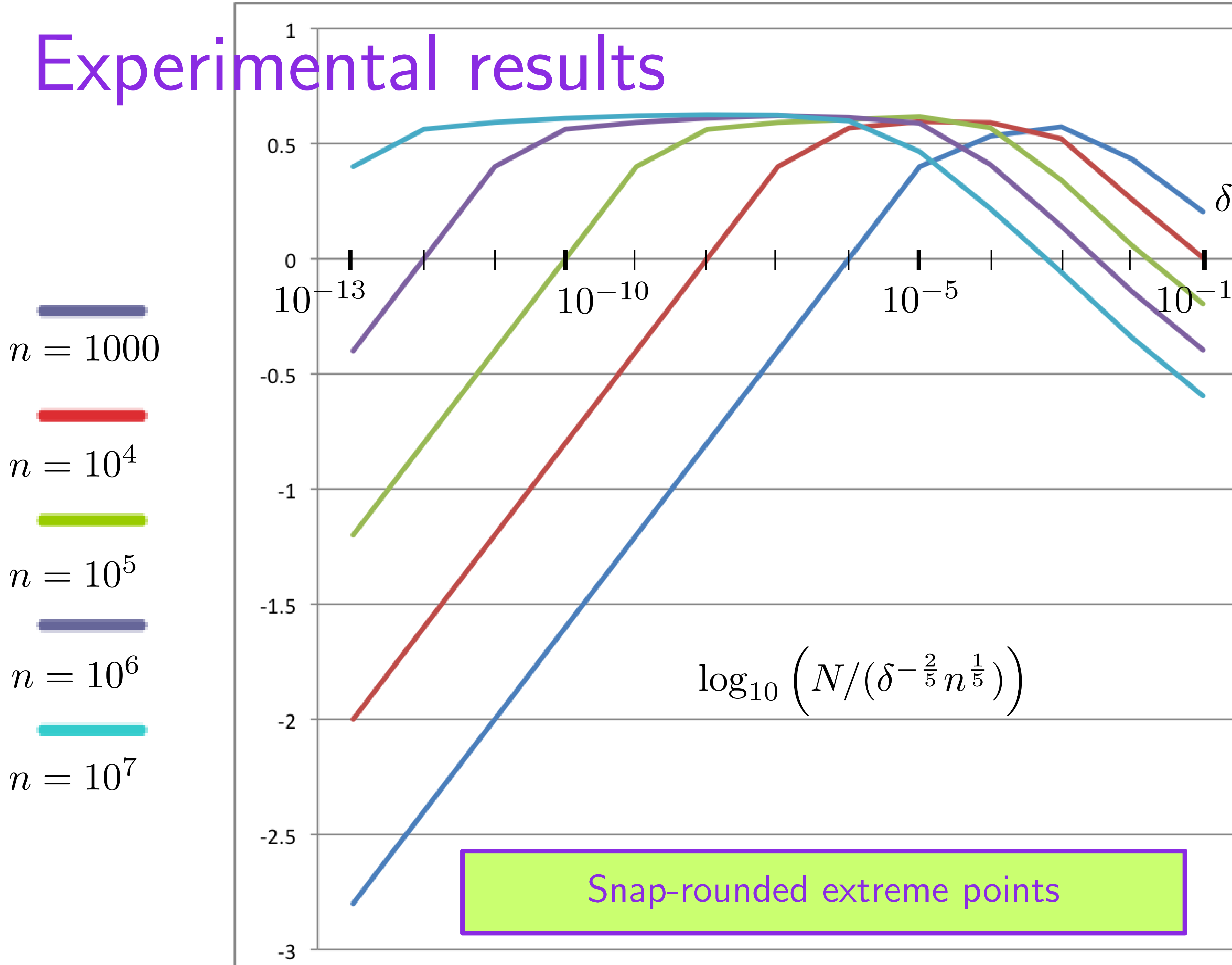
Snap-rounded extreme points



Experimental results

Snap-rounded extreme points

Experimental results



Snap-rounded extreme points

Open problems

Open problems

$\delta \in [1, \infty)$

Cubic noise in higher dimension

Noise shape (Gaussian noise)

Open problems

$\delta \in [1, \infty)$

Cubic noise in higher dimension

Noise shape (Gaussian noise)

Other problems

e.g. worst case for 3D Delaunay with noise δ ?