Convex hulls of spheres and convex hulls of convex polytopes lying on parallel hyperplanes

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Problem setup

- We are given a set Σ of n spheres in $\mathbb{E}^d$, such that n_i spheres have radius ρ_i, where $1 \leq i \leq m$ and $1 \leq m \leq n$.
- The radii are pairwise distinct, i.e., $\rho_i \neq \rho_j$ for $i \neq j$.
- The dimension d is considered fixed.

Problem

What is the worst-case combinatorial complexity of the convex hull $CH_d(\Sigma)$ of Σ, when m is fixed?

- Was posed as an open problem by [Boissonnat & K. 2003]
- The problem is interesting only for odd dimensions. Throughout the talk $d \geq 3$, and d odd.

Previous/Related work – Worst-case complexities

- For a point set P in $\mathbb{E}^d$ the worst-case complexity of $CH_d(P)$ is $\Theta(n^{\lfloor \frac{d}{2} \rfloor})$. Same for spheres with same radius.
- [Aurenhammer 1987]: The worst-case complexity of $CH_d(\Sigma)$ is $O(n^{\lceil \frac{d}{2} \rceil})$.
  - Reduction to power diagram in $\mathbb{E}^{d+1}$.
- [Boissonnat *et al.* 1996]: Showed that $CH_3(\Sigma) = \Omega(n^2)$.
- [Boissonnat & K. 2003]: Showed that $CH_d(\Sigma) = \Omega(n^{\lceil \frac{d}{2} \rceil})$ for all $d \geq 3$.
  - Correspondence between Möbius diagrams in $\mathbb{E}^{d-1}$ with additively weighted Voronoi cells in $\mathbb{E}^d$ and convex hulls of spheres in $\mathbb{E}^d$ (via inversions).
  - Worst-case bound for Möbius diagrams in $\mathbb{E}^{d-1}$ is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.

Previous/Related work – Worst-case optimal algorithms

- [Chazelle 1993]: The convex hull of n points in $\mathbb{E}^d$, $d \geq 2$, can be computed in worst-case optimal $O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n)$ time.
  - Worst-case optimal algorithms already existed for $d = 2, 3$.
- [Boissonnat *et al.* 1996]: Presented a $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ time algorithm: worst-case optimal only for even dimensions (at that point).
  - Lifting map from spheres in $\mathbb{E}^d$ to points in $\mathbb{E}^{d+1}$, using the radius as the last coordinate.
- [Boissonnat & K. 2003]: Due to the lower bound of $\Omega(n^{\lceil \frac{d}{2} \rceil})$ for $CH_d(\Sigma)$, the algorithm in [Boissonnat *et al.* 1996] is actually worst-case optimal for all $d \geq 2$.

Previous/Related work – Output-sensitive algorithms

- Many output-sensitive algorithms for the convex hull of points. We mention those related to this presentation:
  - [Seidel 1986]: Uses the notion of polytopal *shellings* to compute $CH_d(P)$ in $O(n^2 + f \log n)$ time.
  - [Matoušek & Schwarzkopf 1992]: Improved the n^2 part of Seidel's algorithm to get $O(n^{2-2/(\lfloor \frac{d}{2} \rfloor+1)+\epsilon} + f \log n)$.
  - [Chan, Snoeyink & Yap 1997]: Works in 4D and has running time $O((n + f) \log^2 f)$.
  - Many more algorithms for 2D and 3D, some of which are optimal (in the output-sensitive sense).
- Very few output-sensitive algorithms for spheres:
  - [Nielsen & Yvinec 1998]: Computes $CH_2(\Sigma)$ in optimal $O(n \log f)$ time.
  - [Boissonnat, Cérézo & Duquesne 1992]: Gift-wrapping algorithm for computing $CH_3(\Sigma)$; runs in $O(nf)$ time.

Our results – Some definitions

We have a set of spheres Σ consisting of n spheres, such that $n_i \leq n$ spheres have radius ρ_i, where $1 \leq i \leq m$ and $m \geq 2$ and m is fixed.

Definition

We say that ρ_λ *dominates* Σ if $n_\lambda = \Theta(n)$.

Definition

We say that Σ is *uniquely dominated* if, for some λ, $n_\lambda = \Theta(n)$, and $n_i = o(n)$ for all $i \neq \lambda$.

Definition

We say that Σ is *strongly dominated* if, for some λ, $n_\lambda = \Theta(n)$, and $n_i = O(1))$ for all $i \neq \lambda$.

Our results – Qualitative point-of-view

We use the term *generic worst-case complexity* to refer to the worst-case complexity of $CH_d(\Sigma)$ where there is no restriction on the number of distinct radii in Σ.

Result

If Σ is dominated by at least two radii, the worst-case complexity of $CH_d(\Sigma)$ matches the generic worst-case complexity.

Result

If Σ is uniquely dominated, the worst-case complexity of $CH_d(\Sigma)$ is asymptotically smaller than the generic worst-case complexity.

Result

If Σ is strongly dominated, the worst-case complexity of $CH_d(\Sigma)$ matches the worst-case complexity of convex hulls of points.

Our results – Quantitative point-of-view

Theorem

The worst-case complexity of $CH_d(\Sigma)$, for m fixed, is $\Theta(\sum_{1 \leq i \neq j \leq m} n_i n_j^{\lfloor \frac{d}{2} \rfloor})$.

Result

If Σ is dominated by at least two radii, the complexity of $CH_d(\Sigma)$ is $\Theta(n^{\lceil \frac{d}{2} \rceil})$.

Result

If Σ is uniquely dominated, the complexity of $CH_d(\Sigma)$ is $o(n^{\lceil \frac{d}{2} \rceil})$.

Result

If Σ is strongly dominated, the complexity of $CH_d(\Sigma)$ is $\Theta(n^{\lfloor \frac{d}{2} \rfloor})$.

Our results – Methodology

- For the upper bound we reduce the sphere convex hull problem to the problem of computing the complexity of the convex hull of m d-polytopes lying on m parallel hyperplanes of $\mathbb{E}^{d+1}$.

Theorem

Let $\mathscr{P}$ be a set of m d-polytopes $\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ lying on m parallel hyperplanes of $\mathbb{E}^{d+1}$. The worst-case complexity of $CH_{d+1}(\mathscr{P})$ is $O(\sum_{1\leq i\neq j\leq m} n_i n_j^{\lfloor \frac{d}{2} \rfloor})$, where $n_i = f_0(\mathcal{P}_i)$, $1 \leq i \leq m$.

- For the lower bound we first construct a set Σ of $\Theta(n_1 + n_2)$ spheres in $\mathbb{E}^d$ such that the complexity of $CH_d(\Sigma)$ is $\Omega(n_1 n_2^{\lfloor \frac{d}{2} \rfloor} + n_2 n_1^{\lfloor \frac{d}{2} \rfloor})$. Then we generalize this construction for $m \geq 3$.
    - This construction also gives a matching lower bound for the parallel polytope convex hull problem.

Definitions

- Polytope $\mathcal{P}$: the convex hull of a set of points P
- Face of $\mathcal{P}$: intersection of $\mathcal{P}$ with at least one supporting hyperplane
- k-face: k-dimensional face
- trivial face: the unique face of dimension d ($\mathcal{P}$)
- proper faces: faces of dimension at most $d-1$
- d-polytope: a polytope whose trivial face is d-dimensional
- vertices: 0-faces; edges: 1-faces
- facets: $(d-1)$-faces; ridges: $(d-2)$-faces;
- simplicial polytope: all proper faces are simplices
- $f_k(\mathcal{P})$: number of k-faces of $\mathcal{P}$
- *f*-vector: $(f_{-1}(\mathcal{P}), f_0(\mathcal{P}), \ldots, f_{d-1}(\mathcal{P}))$
    - $f_{-1}(\mathcal{P}) = 1$ (empty set)

Definitions (cont.)

- For simplicial d-polytopes we can define the *h*-vector:

$$(h_0(\mathcal{P}), h_1(\mathcal{P}), \ldots, h_d(\mathcal{P})),$$

where

$$h_k(\mathcal{P}) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}(\mathcal{P}).$$

- $h_k(\mathcal{P})$: number of facets in a shelling of $\mathcal{P}$ whose restriction has size k.
- The elements of the *f*-vector determine the *h*-vector and vice versa.

Dehn-Sommerville equations

- The element of the *f*-vector are linearly dependent.
- They satisfy the so called *Dehn-Sommerville equations*:

$$h_k(\mathcal{P}) = h_{d-k}(\mathcal{P}), \quad 0 \leq k \leq d.$$

- Important implication: if we know the face numbers $f_k(\mathcal{P})$, $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - 1$, we can determine the remaining face numbers $f_k(\mathcal{P})$, $\lfloor \frac{d}{2} \rfloor \leq k \leq d-1$.

Simplicial vs. non-simplicial polytopes

For any non-simplicial d-polytope there exists a (nearby) simplicial polytope with the same number of vertices, and with at least as many faces as the non-simplicial polytope:

Theorem ([Klee 1964],[McMullen 1970])

Let $\mathcal{P}$ be a d-polytope.

1. *The d-polytope $\mathcal{P}'$ we obtain by pulling a vertex of $\mathcal{P}$ has the same number of vertices with $\mathcal{P}$, and $f_k(\mathcal{P}) \leq f_k(\mathcal{P}')$ for all $1 \leq k \leq d-1$.*
2. *The d-polytope $\mathcal{P}'$ we obtain by successively pulling each of the vertices of $\mathcal{P}$ is simplicial, has the same number of vertices with $\mathcal{P}$, and $f_k(\mathcal{P}) \leq f_k(\mathcal{P}')$ for all $1 \leq k \leq d-1$.*

Adaptation to parallel polytopes

Lemma

Let $\mathscr{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ be a set of $m \geq 2$ d-polytopes lying on m parallel hyperplanes $\Pi_1, \Pi_2, \ldots, \Pi_m$ of $\mathbb{E}^{d+1}$, respectively, where Π_j is above Π_i for all $j > i$. Let P_i be the vertex set of $\mathcal{P}_i$, $1 \leq i \leq m$, $P = P_1 \cup P_2 \ldots \cup P_m$, and $\mathcal{P} = CH_{d+1}(P)$. The points in P can be perturbed in such a way that:

1. *the points of P_i remain in Π_i, $1 \leq i \leq m$,*
2. *all the faces of $\mathcal{P}'$, except possibly the facets $\mathcal{P}_1'$ and $\mathcal{P}_m'$, are simplices, and,*
3. *$f_k(\mathcal{P}) \leq f_k(\mathcal{P}')$ for all $1 \leq k \leq d$,*

where $\mathcal{P}'$ is the polytope we obtain after having perturbed the vertices of $\mathcal{P}$ in P.

Adaptation to parallel polytopes – Sketch of Proof

- First we pull the vertices of $\mathcal{P}_1$ so that $\mathcal{P}_1$ becomes simplicial (we consider $\mathcal{P}_1$ embedded in $\mathbb{E}^d$).
- Secondly we pull the vertices of $\mathcal{P}_m$ so that $\mathcal{P}_m$ becomes simplicial (we consider $\mathcal{P}_m$ embedded in $\mathbb{E}^d$).
- Finally, for each i, $2 \leq i \leq m-1$, we pull the vertices of $\mathcal{P}_i$ in $\mathcal{P}$, so that all faces of $\mathcal{P}$, except $\mathcal{P}_1$ and $\mathcal{P}_m$, become simplicial.

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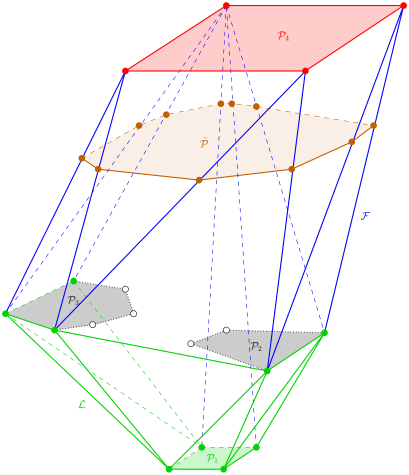
- Finally, for each i, $2 \leq i \leq m-1$, we pull the vertices of $\mathcal{P}_i$ in $\mathcal{P}$, so that all faces of $\mathcal{P}$, except $\mathcal{P}_1$ and $\mathcal{P}_m$, become simplicial.

✔ At each step above the number of vertices of the new polytope is the same as the old polytope.

✔ At each step above the number of faces of the new polytope is at least as big as the number of faces of the old polytope.

The setting



- By means of the lemma, we need only consider polytopes $\mathcal{P}$ such that $\mathcal{P}$ is simplicial, except possibly for its two facets $\mathcal{P}_1$ and $\mathcal{P}_m$.
- $\tilde{\Pi}$: hyperplane parallel and between Π_{m-1} and Π_m.
- $\mathcal{F}$: set of faces of $\mathcal{P}$ with non-empty intersection with $\tilde{\Pi}$.
- For each k-face $F \in \mathcal{F}$, at least one point comes from $\mathcal{P}_m$, whereas the remaining k points come from $\mathcal{P}_1, \ldots, \mathcal{P}_{m-1}$.

The *"easy"* upper bound

Let $A=(\alpha_1\ldots,\alpha_m)$, $B=(\beta_1,\ldots,\beta_m)\in\mathbb{N}^m$. Define: $|A|=\sum_{i=1}^m\alpha_i$, and $A\preccurlyeq B$, iff $\alpha_i\leq\beta_i$, $1\leq i\leq m$.

Theorem

The number of k-faces of $\mathcal{F}$ is bounded from above as follows:

$$f_k(\mathcal{F})\leq\sum_{\substack{(0,\ldots,0,1)\preccurlyeq A\preccurlyeq(k,\ldots,k)\\|A|=k+1}}\prod_{i=1}^m\binom{f_0(\mathcal{P}_i)}{\alpha_i},\qquad 1\leq k\leq d.$$

Proof.

A k-face $F\in\mathcal{F}$ is defined by $k+1$ vertices of $\mathcal{P}$, where at least one vertex comes from $\mathcal{P}_m$, whereas the remaining vertices are vertices of $\mathcal{P}_1,\ldots,\mathcal{P}_{m-1}$. Let α_i be the number of vertices of F from $\mathcal{P}_i$. We have: $0\leq\alpha_i\leq k$ for $1\leq i\leq m-1$, $1\leq\alpha_m\leq k$, and $\sum_{i=1}^m\alpha_i=k+1$. The maximum number of possible (α_i-1)-faces of $\mathcal{P}_i$ is $\binom{f_0(\mathcal{P}_i)}{\alpha_i}$. Hence, the maximum possible number of k-faces of $\mathcal{F}$ is $\prod_{i=1}^m\binom{f_0(\mathcal{P}_i)}{\alpha_i}$. Summing over all possible values for the α_i's we get the desired expression. □

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Corollary

Let $n_i=f_0(\mathcal{P}_i)$, $1\leq i\leq m$. The following asymptotic bounds hold:

$$f_k(\mathcal{F})=O(n_m^k\sum_{i=1}^{m-1}n_i+n_m\sum_{i=1}^{m-1}n_i^k),\quad 1\leq k\leq\lfloor\tfrac{d}{2}\rfloor.$$

Defining an auxilliary polytope

- Call $\mathcal{C}$ the polytopal complex with facets the facets of $\mathcal{F}$.
- Call $\mathcal{L}$ the faces of $\mathcal{P}$ not in $\mathcal{P}_m$ or $\mathcal{F}$. Call $\partial\mathcal{L} = \mathcal{C} \cap \mathcal{L}$.
- Call $\partial\mathcal{P}_m$ the boundary complex of $\mathcal{P}_m$.
- Define the point set Q so as to consist of the points of $\mathcal{P}_m$, $\partial\mathcal{L}$ and two additional points y and z.
    - y is below Π_1 and visible by the vertices of $\mathcal{P}_1$ only.
    - z is below Π_m and visible by the vertices of $\mathcal{P}_m$ only.
- Define $\mathcal{Q} = CH_{d+1}(Q)$; $\leadsto$ $\mathcal{Q}$ is simplicial $(d+1)$-polytope.
- $\partial\mathcal{L}$ is the *link* of y in $\mathcal{Q}$.
- $\partial\mathcal{P}_m$ is the *link* of z in $\mathcal{Q}$.

Relations on faces

1. For a vertex $v \in Q$, call $\mathcal{S}_v$ its star. Then:
$$f_k(\mathcal{Q}) = f_k(\mathcal{F}) + f_k(\mathcal{S}_y) + f_k(\mathcal{S}_z), \quad 0 \leq k \leq d,$$
where $f_0(\mathcal{F}) = 0$.
2. For the faces of $\mathcal{S}_z$ and $\partial\mathcal{P}_m$, we have for $0 \leq k \leq d$:
$$f_k(\mathcal{S}_z) = f_k(\partial\mathcal{P}_m) + f_{k-1}(\partial\mathcal{P}_m)$$
where $f_{-1}(\partial\mathcal{P}_m) = 1$ and $f_d(\partial\mathcal{P}_m) = 0$.
3. For the faces of $\mathcal{S}_y$ and $\partial\mathcal{L}$, we have for $0 \leq k \leq d$:
$$f_k(\mathcal{S}_y) = f_k(\partial\mathcal{L}) + f_{k-1}(\partial\mathcal{L})$$
where $f_{-1}(\partial\mathcal{L}) = 1$ and $f_d(\partial\mathcal{L}) = 0$.

Relations on faces

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2. For the faces of $\mathcal{S}_z$ and $\partial\mathcal{P}_m$, we have for $0 \leq k \leq d$:

$$f_k(\mathcal{S}_z) = f_k(\partial\mathcal{P}_m) + f_{k-1}(\partial\mathcal{P}_m) = O(n_m^{\lfloor\frac{d}{2}\rfloor})$$

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3. For the faces of $\mathcal{S}_y$ and $\partial\mathcal{L}$, we have for $0 \leq k \leq d$:

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2. For the faces of $\mathcal{S}_z$ and $\partial\mathcal{P}_m$, we have for $0 \leq k \leq d$:

$$f_k(\mathcal{S}_z) = f_k(\partial\mathcal{P}_m) + f_{k-1}(\partial\mathcal{P}_m) = O(n_m^{\lfloor \frac{d}{2} \rfloor})$$

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$$f_k(\mathcal{S}_y) = f_k(\partial\mathcal{L}) + f_{k-1}(\partial\mathcal{L}) = O((\sum_{i=1}^{m-1} n_i)^{\lfloor \frac{d}{2} \rfloor})$$

   where $f_{-1}(\partial\mathcal{L}) = 1$ and $f_d(\partial\mathcal{L}) = 0$.

Relations on faces

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where $f_{-1}(\partial\mathcal{L}) = 1$ and $f_d(\partial\mathcal{L}) = 0$.

Using the Dehn-Sommerville equations

- Combining the bounds for $f_k(\mathcal{F})$, $f_k(\mathcal{S}_y)$ and $f_k(\mathcal{S}_z)$, for $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$, we get:

$$f_k(\mathcal{Q}) = O(n_m^{\lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{m-1} n_i + n_m \sum_{i=1}^{m-1} n_i^{\lfloor \frac{d}{2} \rfloor}), \quad 0 \leq k \leq \lfloor \tfrac{d}{2} \rfloor.$$

- Hence for the elements of the *h*-vector we have:

$$h_k(\mathcal{Q}) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1}(\mathcal{Q}), \quad 0 \leq k \leq d+1$$

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$$= O(n_m^{\lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{m-1} n_i + n_m \sum_{i=1}^{m-1} n_i^{\lfloor \frac{d}{2} \rfloor}), \quad 0 \leq k \leq \lfloor \frac{d+1}{2} \rfloor$$

Using the Dehn-Sommerville equations (cont.)

- Using the Dehn-Sommerville equations we get for $\lfloor \frac{d+1}{2} \rfloor \leq k \leq d+1$:

$$h_k(\mathcal{Q}) = h_{d+1-k}(\mathcal{Q}) = O(n_m^{\lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{m-1} n_i + n_m \sum_{i=1}^{m-1} n_i^{\lfloor \frac{d}{2} \rfloor})$$

- Writing the *f*-vector in terms of the *h*-vector we have:

$$f_{k-1}(\mathcal{Q}) = \sum_{i=0}^{k} \binom{d+1-i}{k-i} h_i(\mathcal{Q}), \quad 0 \leq k \leq d$$

Using the Dehn-Sommerville equations (cont.)

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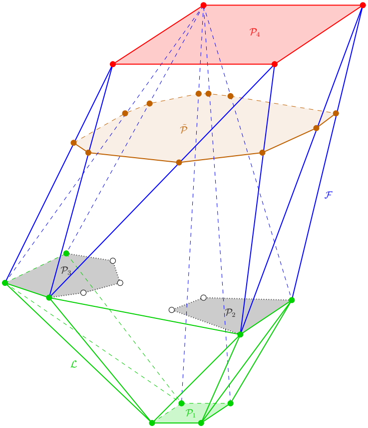
$$= O(n_m^{\lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{m-1} n_i + n_m \sum_{i=1}^{m-1} n_i^{\lfloor \frac{d}{2} \rfloor}), \quad 0 \leq k \leq d.$$

The complexity of $CH_{d+1}(\mathscr{P})$



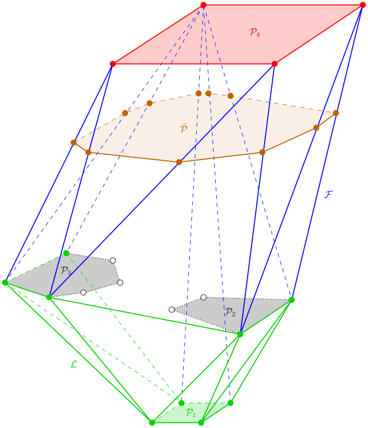
Theorem

Let $\mathscr{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ be a set of a fixed number of $m \geq 2$ parallel d-polytopes, where $d \geq 3$ and d is odd. The worst-case complexity of $CH_{d+1}(\mathscr{P})$ is

$$O\Big(\sum_{1 \leq i \neq j \leq m} n_i n_j^{\lfloor \frac{d}{2} \rfloor}\Big),$$

where $n_i = f_0(\mathcal{P}_i)$, $1 \leq i \leq m$.

The complexity of $CH_{d+1}(\mathscr{P})$ – Inductive proof



Proof.

Let $T(m)$ be the worst-case complexity of $CH_{d+1}(\mathscr{P})$.
The set of faces of $\mathcal{P}$ is the disjoint union of $\mathcal{L}$, $\mathcal{F}$ and the set of faces of $\mathcal{P}_m$. The faces in $\mathcal{L}$ are also faces of $CH_{d+1}(\mathscr{P} \setminus \{\mathcal{P}_m\})$, which implies that the complexity of $\mathcal{L}$ is at most $T(m-1)$. Therefore, we have:

$$T(m) \leq T(m-1) + O(n_m^{\lfloor \frac{d}{2} \rfloor})$$
$$+ O(n_m^{\lfloor \frac{d}{2} \rfloor} \sum_{i=1}^{m-1} n_i + n_m \sum_{i=1}^{m-1} n_i^{\lfloor \frac{d}{2} \rfloor})$$
$$\rightsquigarrow \quad T(m) \leq c(\sum_{1 \leq i \neq j \leq m} n_i n_j^{\lfloor \frac{d}{2} \rfloor} + \sum_{i=1}^{m} n_i^{\lfloor \frac{d}{2} \rfloor})$$

□

Context

- Σ: set of n spheres in $\mathbb{E}^d$
- The radii of the spheres in Σ take m distinct values: $\rho_1 < \rho_2 < \ldots < \rho_m$
- $CH_d(\Sigma)$: convex hull of Σ
- Face of circularity ℓ of $CH_d(\Sigma)$: maximal connected portion of the boundary of $CH_d(\Sigma)$, where the supporting hyperplanes are tangent to a given set of $(d-\ell)$ spheres
- n_i: number of spheres in Σ with radius ρ_i

Lifting map

- Let H_0 be the hyperplane $\{x_{d+1} = 0\}$ of $\mathbb{E}^{d+1}$.
- Map the sphere $\sigma_k = (c_k, \rho_k)$ to the point $p_k = (c_k, \rho_k) \in \mathbb{E}^{d+1}$.
- Let $P = \{p_1, p_2, \ldots, p_n\}$. Call $\mathcal{P} = CH_{d+1}(P)$.
- Let λ_0 be the lower halfcone in $\mathbb{E}^{d+1}$ with arbitrary apex, vertical axis, and apex angle equal to $\frac{\pi}{4}$.
- Let $\lambda(p)$ be the translated copy of λ_0 with apex at p.
- Define the set Λ to be the set

$$\Lambda = \{\lambda(p_1), \lambda(p_2), \ldots, \lambda(p_n)\}$$

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Fact

$\lambda(p_k) \cap H_0 = \sigma_k, \quad 1 \leq k \leq n$

Fact

$CH_{d+1}(\Lambda) \cap H_0 = CH_d(\Sigma)$

Relation between faces of $CH_{d+1}(\Lambda)$ and $CH_d(\Sigma)$

Let O' be any point inside $\mathcal{P}$.

Theorem ([Boissonnat *et al.* 1996])

Any hyperplane of $\mathbb{E}^d$ supporting $CH_d(\Sigma)$ is the intersection with H_0 of a unique hyperplane H of $\mathbb{E}^{d+1}$ satisfying the following three properties:

1. *H supports $\mathcal{P}$,*
2. *H is the translated copy of a hyperplane tangent to λ_0 along one of its generatrices,*
3. *H is above O'.*

Conversely, let H be a hyperplane of $\mathbb{E}^{d+1}$ satisfying the above three properties. Its intersection with H_0 is a hyperplane of $\mathbb{E}^d$ supporting $CH_d(\Sigma)$.

Parallel polytopes

- Π_i: the hyperplane in $\mathbb{E}^{d+1}$ with eq. $\{x_{d+1} = \rho_i\}$
- P_i: the points of P in Π_i
- n_i: the cardinality of P_i
- $\hat{P}_i$: the vertex set of $\mathcal{P}_i$, i.e., $CH_d(\mathcal{P}_i) = CH_d(\hat{\mathcal{P}}_i)$
- $\hat{n}_i$: the cardinality of $\hat{P}_i \rightsquigarrow \hat{n}_i \leq n_i$
- $\hat{P} = \hat{P}_1 \cup \hat{P}_2 \cup \ldots \cup \hat{P}_m$
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- $\hat{\mathcal{P}} = CH_{d+1}(\hat{P})$

It is possible that $\mathcal{P} \neq \hat{\mathcal{P}}$

- This can happen if $P_1 \neq \hat{P}_1$ or $P_m \neq \hat{P}_m$

Upper bound

- ✔ There is an injection $\varphi : CH_d(\Sigma) \to \mathcal{P}$ mapping a face of circularity $(d-\ell-1)$ of $CH_d(\Sigma)$ to a unique ℓ-face of $\mathcal{P}$.
  - Points in $P_i \setminus \hat{P}_i$ can never be points on a supporting hyperplane H of $\mathcal{P}$ of the theorem.
- ✔ The injection φ is in fact an injection from $CH_d(\Sigma)$ to $\hat{\mathcal{P}}$
  - $\hat{\mathcal{P}}$ is the convex hull of m parallel polytopes.

Therefore the complexity of $CH_d(\Sigma)$ is

$$O\Big(\sum_{1\leq i\neq j\leq m} \hat{n}_i \hat{n}_j^{\lfloor \frac{d}{2} \rfloor}\Big) = O\Big(\sum_{1\leq i\neq j\leq m} n_i n_j^{\lfloor \frac{d}{2} \rfloor}\Big)$$

The trigonometric moment curve

For any even dimension 2δ we can define the so called *trigonometric moment curve*:

$$\gamma_{2\delta}^{tr}(t) = (\cos t, \sin t, \cos 2t, \sin 2t, \ldots, \cos \delta t, \sin \delta t), \quad t \in [0, \pi)$$

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Fact

For any set P of n points on $\gamma^{tr}_{2\delta}(t)$, the convex hull $CH_{2\delta}(P)$ is a polytope $\mathcal{Q}$ combinatorially equivalent to the cyclic polytope $C_{2\delta}(n)$

Fact

$f_{2\delta-1}(\mathcal{Q}) = \Theta(n^{\delta})$

Fact

Points on $\gamma^{tr}_{\delta}(t)$ lie on the sphere of $\mathbb{E}^{2\delta}$ centered at the origin with radius $\sqrt{\delta}$.

First step: the prism

We assume that the ambient space is $\mathbb{E}^d$, where $d \geq 3$ and d odd.

1. Define $H_1 = \{x_d = z_1\}$ and $H_2 = \{x_d = z_2\}$, $z_2 > z_1 + 2(n_2 + 2)\sqrt{\delta}$
2. Σ_1: set of $n_1 + 1$ points on $\gamma_{d-1}^{tr}(t) \in H_1$, where for n_1 points $t \in (0, \frac{\pi}{2})$, whereas for the last point $t \in (\frac{\pi}{2}, \pi)$
3. Σ_2: *"vertical"* projection of Σ_1 on H_2.
4. $Q_i = CH_{d-1}(\Sigma_i)$, and $\Delta = CH_d(\Sigma_1 \cup \Sigma_2)$

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Some facts:

1. $f_{d-2}(Q_i) = \Theta(n_1^{\lfloor \frac{d-1}{2} \rfloor}) = \Theta(n_1^{\lfloor \frac{d}{2} \rfloor})$ (# of facets of Q_i)
2. Δ consists of the *bottom facet* Q_1, the *top facet* Q_2, and $\Theta(n_1^{\lfloor \frac{d}{2} \rfloor})$ vertical facets

More facts

- For a vertical facet F denote by $\vec{\nu}_F$ the unit normal vector and by F^+ and F^- the two open halfspaces bounded by F.
- Call *vertical ridges*, the ridges that are intersections of vertical facets of Δ
- Call Y the hyperplane with unit normal vector $\vec{\nu} = (1, 0, \ldots, 0)$, and Y^+, Y^- the two oriented halfspaces delimited by Y.

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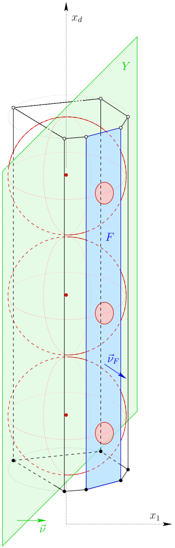
3. n_1 points of Σ_1 (or Σ_2) belong to Y^+.
4. 1 point of Σ_1 (or Σ_2) belongs to Y^-.
5. The $(d-2)$-polytope $\tilde{\mathcal{Q}}_i = \mathcal{Q}_i \cap Y$ has at most n_1 vertices.
6. The faces $\mathcal{F}_i$ of $\mathcal{Q}_i$ intersected by Y is at most $O(n_1^{\lfloor \frac{d-2}{2} \rfloor}) = O(n_1^{\lfloor \frac{d}{2} \rfloor - 1})$.
7. The number of vertices facets of Δ in Y^+ is $\Theta(n_1^{\lfloor \frac{d}{2} \rfloor})$.

Second step: spheres with non-zero radius



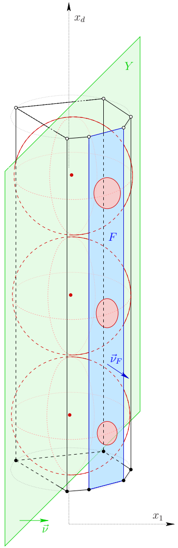
- Define a new set of $n_2 + 2$ spheres Σ_3, where

  $$\sigma_k = ((0, 0, \ldots, 0, (2k+1)\sqrt{\delta}), \rho)$$

- Choose ρ so that
  1. each σ_k does not intersect any vertical ridge of $\Delta \rightsquigarrow \rho < \sqrt{\delta}$
  2. each σ_k intersects all vertical facets of Δ

Third step: perturd the spheres of Σ_3

View from the top



- Define the spheres σ'_k with radius ρ and center:

$$c'_k = c_k + \left(\sum_{\ell=0}^{k} \frac{\varepsilon}{2^\ell} \right) \vec{\nu} = c_k + \varepsilon \left(2 - \frac{1}{2^k} \right) \vec{\nu}$$

- Choose ε so that
  1. each σ'_k does not intersect any vertical ridge of $\Delta \rightsquigarrow \rho < \sqrt{\delta}$
  2. each σ'_k intersects all vertical facets of Δ in Y^+
  3. the $(d-2)$-sphere $\sigma_k \cap \sigma'_k$ is contained in F^- for all vertical facets F of Δ in Y^+
- Call Σ'_3 the set of perturbed spheres
- Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma'_3$

Faces of circularity $(d-1)$

- For each pair (σ'_k, F), where $1 \leq k \leq n_2$ and F a vertical facet of Δ in Y^+, we have a unique face of circularity $(d-1)$ in $CH_d(\Sigma)$

$\rightsquigarrow$ $CH_d(\Sigma)$ has $n_2\Theta(n_1^{\lfloor \frac{d}{2} \rfloor})$ faces of circularity $(d-1)$

$\rightsquigarrow$ The complexity of $CH_d(\Sigma)$ is $\Omega(n_2 n_1^{\lfloor \frac{d}{2} \rfloor})$

- WLOG $n_2 \leq n_1$, which implies that the complexity of $CH_d(\Sigma)$ is $\Omega(n_2 n_1^{\lfloor \frac{d}{2} \rfloor} + n_1 n_2^{\lfloor \frac{d}{2} \rfloor})$

Construction with $m \geq 3$ radii

- Let $N_1 = \sum_{i=2}^{m} n_i$, $N_2 = n_1$.
- Construct the sets Σ_1, Σ_2 and Σ_3' as in the case $m = 2$, where Σ_1 and Σ_2 contain each $N_1 + 1$ points and Σ_3' contains $N_2 + 2$ spheres.
- Replace n_i among the N_1 points of Σ_1, Σ_2 in Y^+ by a sphere of radius r^i.
- Replace the point of Σ_1, Σ_2 in Y^- by a sphere of radius r^2.
- Choose r, where $0 < r \ll 1$, such that the following two conditions are satisfied:
  1. $\Delta_r = CH_d(\Sigma_1 \cup \Sigma_2)$ is combinatorially equivalent to Δ_0
  2. The two requirements for the spheres of Σ_3' should still be satisfied.

$\rightsquigarrow$ $CH_d(\Sigma)$ has $N_2\Theta(N_1^{\lfloor \frac{d}{2} \rfloor})$ faces of circularity $(d-1)$

Results

- The complexity of $CH_d(\Sigma)$ is $\Omega(\sum_{1\leq i\neq j\leq m} n_i n_j^{\lfloor \frac{d}{2} \rfloor})$
- WLOG, assume that $n_2 \geq n_1 \geq n_i$, $3 \leq i \leq m$. Then:

$$n_1(\sum_{i=2}^{m} n_i)^{\lfloor \frac{d}{2} \rfloor} \geq n_1 n_2^{\lfloor \frac{d}{2} \rfloor} \geq \frac{1}{m(m-1)}(\sum_{1\leq i\neq j\leq m} n_i n_j^{\lfloor \frac{d}{2} \rfloor})$$

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Theorem

Fix some odd $d \geq 3$. There exists a set Σ of spheres in $\mathbb{E}^d$, consisting of n_i spheres of radius ρ_i, with $\rho_1 < \rho_2 < \ldots < \rho_m$ and $m \geq 3$ fixed, such that the complexity of $CH_d(\Sigma)$ is $\Omega(\sum_{1\leq i\neq j\leq m} n_i n_j^{\lfloor \frac{d}{2} \rfloor})$.

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Corollary

Let $\mathscr{P} = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_m\}$ be a set of m d-polytopes, lying on m parallel hyperplanes of $\mathbb{E}^{d+1}$, with $d \geq 3$ odd, and both d, m fixed. The worst-case complexity of $CH_{d+1}(\mathscr{P})$ is $\Omega(\sum_{1\leq i\neq j\leq m} n_i n_j^{\lfloor\frac{d}{2}\rfloor})$, where $n_i = f_0(\mathcal{P}_i)$, $1 \leq i \leq m$.

Minkowski sum of two convex d-polytopes

- $\mathcal{P}$: n-vertex convex d-polytope
- $\mathcal{Q}$: m-vertex convex d-polytope
- Embed $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{E}^{d+1}$ and on the hyperplanes $\{x_{d+1} = 0\}$ and $\{x_{d+1} = 1\}$, respectively.
- The Minkowski sum $(1-\lambda)\mathcal{P} \oplus \lambda\mathcal{Q}$, $\lambda \in (0,1)$ is combinatorially equivalent to the intersection of $CH_{d+1}(\{\mathcal{P},\mathcal{Q}\})$ with the hyperplane $\{x_{d+1} = \lambda\}$.

Corollary

Let $\mathcal{P}$ and $\mathcal{Q}$ be two convex d-polytopes in $\mathbb{E}^d$, with n and m vertices, respectively, where $d \geq 3$ and d odd. The complexity of the weighted Minkowski sum $(1-\lambda)\mathcal{P} \oplus \lambda\mathcal{Q}$, $\lambda \in (0,1)$, as well as the complexity of the Minkowski sum $\mathcal{P} \oplus \mathcal{Q}$, is $\Theta(mn^{\lfloor \frac{d}{2} \rfloor} + nm^{\lfloor \frac{d}{2} \rfloor})$.

Computing convex hulls of parallel polytopes

We can use output-sensitive algorithms for computing $CH_{d+1}(\mathscr{P})$.

- For $d \geq 5$ the algorithm in [Seidel 1986] and the algorithm in [Matoušek & Schwarzkopf 1992] yield the same running time:

$$O\Big(\Big(\sum_{1\leq i\neq j\leq m} n_i n_j^{\lfloor \frac{d}{2} \rfloor}\Big) \log n\Big), \quad n = \sum_{i=1}^{m} n_i.$$

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- For $d = 3$ the best two choices are the algorithm in [Matoušek & Schwarzkopf 1992], and that in [Chan, Snoeyink & Yap 1997]. The running time is in:

$$O\Big(\min\{n^{4/3+\epsilon} + \big(\sum_{1\leq i\neq j\leq m} n_i n_j\big)\log n, \big(\sum_{1\leq i\neq j\leq m} n_i n_j\big)\log^2 n\}\Big).$$

Computing convex hulls of spheres

- We can slightly modify the algorithm in [Boissonnat *et al.* 1996] to get a new algorithm adapted to the fact that the spheres have a constant number of radii.
  - The algorithm first computes the convex hull for each subset of spheres having the same radius, and keeps only the spheres defining each convex hull. These spheres are then used to compute the convex hull of the entire sphere set.

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- Time complexity:

$$O(n^{\lfloor \frac{d}{2} \rfloor} + n \log n + T_{d+1}(n_1, n_2, \ldots, n_m))$$

where n_i is the number of spheres of radius ρ_i, n the total number of spheres, and $T_{d+1}(n_1, n_2, \ldots, n_m)$ stands for the time to compute the convex hull of m parallel convex d-polytopes in $\mathbb{E}^{d+1}$, where the i-th polytope has n_i vertices (cf. previous slide).

Open problems

- Worst-case optimal algorithms for all odd dimensions
- Refinement of worst-case complexity of additively weighted Voronoi cells, when the number of radii is fixed.
- Tight bound on the complexity of the Minkowski sum when we have at least three summands.
- Exact complexity for the Minkowski sum of two (or more) polytopes.

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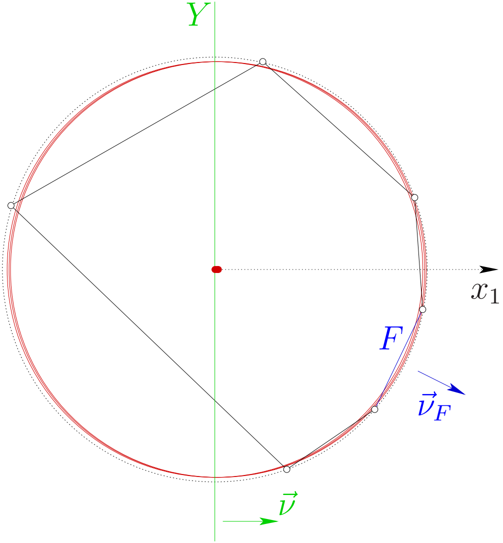
THANK YOU FOR YOUR ATTENTION

View from the top



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