Curvature and Combinatorics of Triangulations

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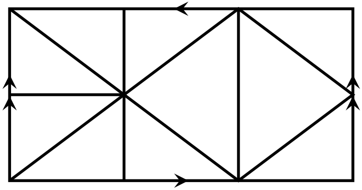
Workshop on Computational Geometry
INRIA, Sophia Antipolis, 2010 December 8



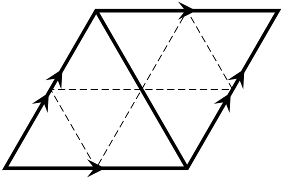
Berlin opportunities

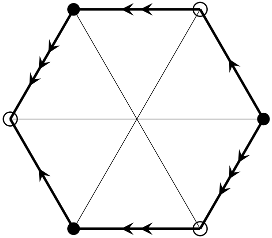
Berlin Mathematical School

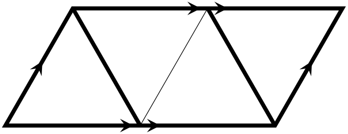
- New international math graduate school
- Courses in English at three universities
- `www.math-berlin.de`

Triangulations of the torus T^2

- Average vertex degree 6
- *Exceptional* vertices have $d \neq 6$
- *Regular* triangulations have $d \equiv 6$

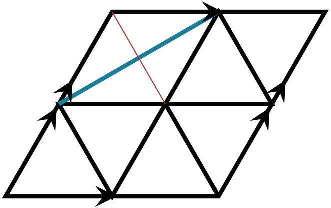


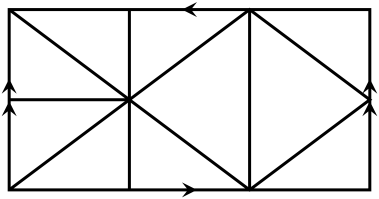




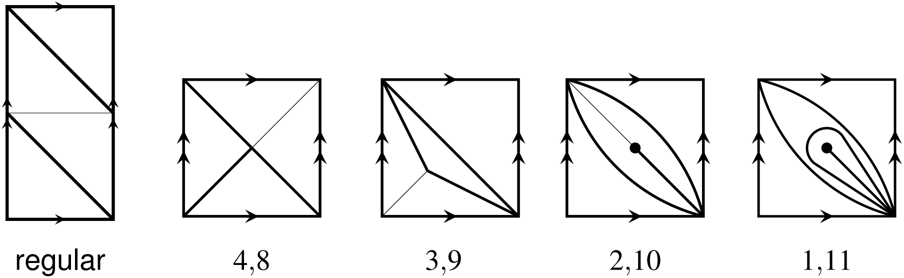
Edge flips give new triangulations

- Flip changes four vertex degrees
- Can produce $5^2 7^2$–triangulations
  (four exceptional vertices)
- Quotients of some such tori are
  $5,7$–triangulations of Klein bottle

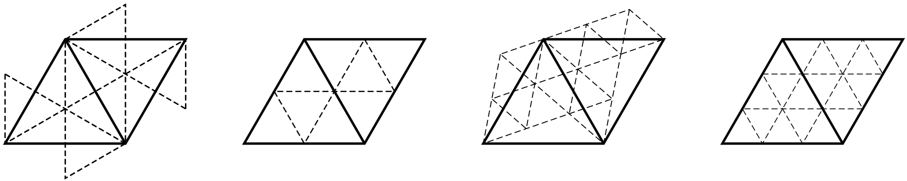




Two-vertex torus triangulations



Refinement or subdivision schemes



$\sqrt{3}$–fold 2–fold $\sqrt{7}$–fold 3–fold

Exceptional vertices preserved

- Old vertex degrees fixed
- New vertices regular

Lots more $4{,}8$–, $3{,}9$–, $2{,}10$– and $1{,}11$–triangulations

Is there a $5,7$–triangulation of the torus?

(any number of regular vertices allowed)

Is there a $5,7$–triangulation of the torus?

(any number of regular vertices allowed)

No!

First proved combinatorially by Jendrol' and Jucovič (1972)

We give geometric proofs

- using curvature and holonomy
- or complex function theory

Joint work with

- Ivan Izmestiev, Günter Rote, Boris Springborn (Berlin)
- Rob Kusner (Amherst)

Combinatorics and topology

Triangulation of any surface

Double-counting edges gives:

$$\tilde{d}V = 2E = 3F$$

$$\frac{\chi}{\tilde{d}V} = \frac{\chi}{2E} = \frac{\chi}{3F} = \frac{1}{\tilde{d}} - \frac{1}{2} + \frac{1}{3}$$

$$6\chi = \sum_d (6-d)v_d$$

Notation

- $\tilde{d}$:= average vertex degree
- v_d := number of vertices of degree d

Eberhard's theorem

Triangulation of $\mathbb{S}^2$

$$12 = \sum_d (6-d) v_d$$

Theorem (Eberhard, 1891)

Given any (v_d) satisfying this condition, there is a corresponding triangulation of $\mathbb{S}^2$, after perhaps modifying v_6.

Examples

- 5^{12}–triangulation exists for $v_6 \neq 1$
- 3^4–triangulation exists for v_6 even ($v_6 = 2$ only non-simplicial)

Torus triangulations

- The condition $0 = \sum(6-d)v_d$ is simply $\tilde{d} = 6$.
- Analog of Eberhard's Theorem would say
  $\exists \quad 5{,}7$–triangulation for some v_6
- Instead, this is the one exception
  (and there are no exceptions for higher genus [JJ'77])

Discrete Gauss curvature for polyhedral surface

Intrinsic Gauss curvature

- angle defect $= 2\pi - \sum \theta$ at a vertex
- Gauss/Bonnet holds $\int K\,dA = 2\pi - \int k_g\,ds$
- natural choice

Extrinsic Gauss curvature [BK82]

- $\frac{1}{2\pi}\int |K| =$ average $\#$ of critical points of height functions
- need different discretization
- some vertices have both $+$ and $-$ curvature

Euclidean cone metrics

Definition

Euclidean cone metric on M is locally euclidean
away from discrete set of cone points.

- Cone of angle $\omega > 0$ has *curvature* $\kappa := 2\pi - \omega$.

Definition

Triangulation on M induces *equilateral metric*:
each face an equilateral euclidean triangle.

- Exceptional vertices are cone points
- Vertex of degree d has curvature $(6 - d)\pi/3$

Regular triangulations on the torus

Theorem (cf. Alt73, Neg83, Tho91, DU05, BK06)

A triangulation of T^2 with no exceptional vertices is a quotient of the regular triangulation T_0 of the plane, or equivalently a finite cover of the 1-vertex triangulation.

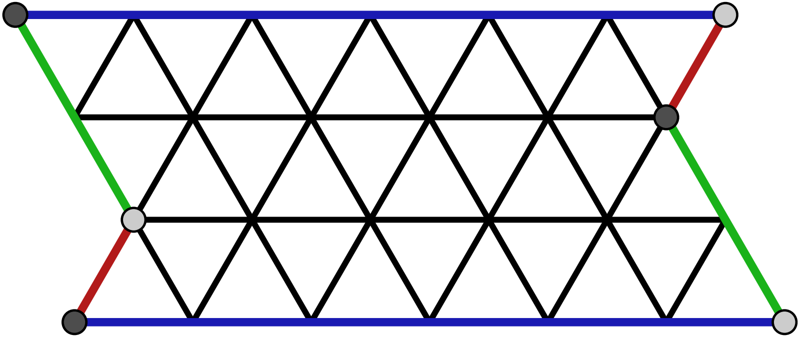
Proof:

Equilateral metric is flat torus $\mathbb{R}^2/\Lambda$. The triangulation lifts to the cover, giving T_0. Thus $\Lambda \subset \Lambda_0$, the triangular lattice. □

Regular triangulations on the torus

Corollary

Any degree-regular triangulation has vertex-transitive symmetry.



Holonomy of a cone metric

Definition

- $M^o := M \setminus$ cone points
- $h : \pi_1(M^o) \to SO_2$
- $H := h(\pi_1)$

Lemma

For a triangulation, H is a subgroup of $C_6 := \langle 2\pi/6 \rangle$.

Proof:

As we parallel transport a vector, look at the angle it makes with each edge of the triangulation. □

Holonomy theorem

Theorem

A torus with two cone points $p_\pm$ *of curvature* $\kappa = \pm 2\pi/n$
has holonomy strictly bigger than C_n*.*

Corollary

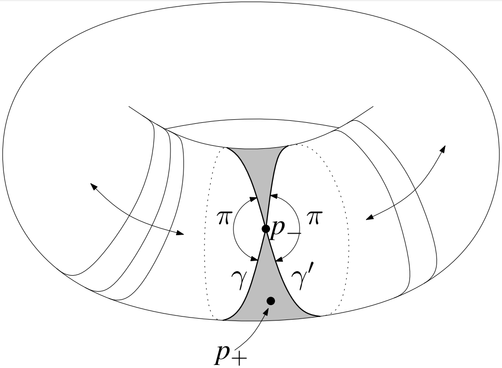
There is no $5{,}7$*–triangulation of the torus.*

Proof:

Lemma says H contained in C_6; theorem says H strictly bigger. □

Proof of Holonomy theorem:

Shortest nontrivial geodesic γ avoids p_+. If it hits p_- and splits excess angle $2\pi/n$ there, consider holonomy of a pertubation. Otherwise, γ avoids p_- or makes one angle π there, so slide it to foliate a euclidean cylinder. Complementary digon has two positive angles, so geodesic from p_- to p_- within the cylinder does split the excess $2\pi/n$. $\square$

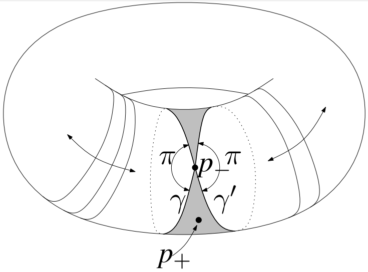


Meaning

Torus constructed from quadrilateral by gluing opposite sides.
But these are not parallel; more like cone than cylinder.

Berger's vector in crystallography

Finite piece with single exceptional vertex – disclination.
$5, 7$ or $4, 8$ piece – still a dislocation: can't fit in regular substrate.
Doesn't apply to torus – because not parallelogram.

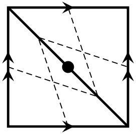


Quadrangulations and hexangulations

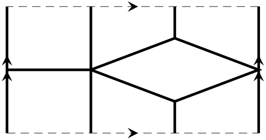
Theorem

The torus T^2 has

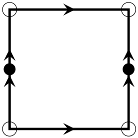
- *no $3,5$–quadrangulation*
- *no bipartite $2,4$–hexangulation*



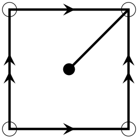
$2,6$–quad



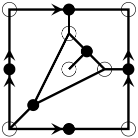
$3^2 5^2$–quad



$2,4$–hex



$1,5$–hex



bip $1,5$–hex

Generalizing the holonomy theorem

Question

Given $n > 0$ and a euclidean cone metric on T^2 whose curvatures are multiples of $2\pi/n$, when is its holonomy H contained in C_n?

Curvature as divisor

- Cone metric induces Riemann surface structure
- Cone point p_i has curvature $m_i 2\pi/n$
- Divisor $D = \sum m_i p_i$ has degree 0

Main theorem

Theorem

$$H < C_n \iff D \textit{ principal}$$

Proof:

Cone metric gives developing map from universal cover of M^o to $\mathbb{C}$. Consider the n^{th} power of the derivative of this developing map. This is well-defined on M iff $H < C_n$. If so, its divisor is D. Conversely, if D is principal, corresponding meromorphic function is this n^{th} power. □

Note: The case $n = 2$ is the classical correspondance between meromorphic quadratic differentials and “singular flat structrues”.

Combinatorial curvature in 3D

- Given a triangulation
- Put standard geometry on each simplex (euclidean regular)
- Measure discrete curvature around edges
  (or in higher dimensions, around codim-2 faces)
- Positive combinatorial curvature $\longleftrightarrow$ positive curvature operator

Forman's combinatorial Ricci curvature

- for surfaces it is different
- doesn't recover Gauss/Bonnet

Cubulations

- Edge of valence 4 is flat
- Edge valences $\leq 4 \iff CBB(0)$
- Edge valences $\geq 4 \iff CBA(0)$
- Works in any dimension

Triangulations in 3D

- No “flat” case for euclidean regular tetrahedra
  every edge has nonzero angle defect
- Euler number $\chi := V - E + F - T$
- All 3-manifolds have $\chi = 0$
- For triangulation: $4T = 2F, \quad 3F = \bar{n}E, \quad 2E = \bar{z}V$
  $\bar{n} =$ average edge valence
  $\bar{z} =$ average vertex degree
- Implies $6 - \bar{n} = 12/\bar{z}$
- But no definite connection to topology of ambient space

Bounds in 3D

- Any value of $4.5 < \bar{n} < 6$ (corresponding to $8 < \bar{z} < \infty$) can be achieved for any ambient space
- $\bar{n} < 4.5$ ($\bar{z} < 8$) only for S^3 [Luo/Stong]
- So foam/triangulation with periodic boundary conditions (3-torus) must have $\bar{n} > 4.5$ ($\bar{z} > 8$)
- Implies some face has $n \geq 5$, some bubble has $z \geq 9$ faces

Combinatorics $\longrightarrow$ geometry in three dimensions

- Triangulated 3-manifold $\longrightarrow$ each tetrahedron regular euclidean
- Edge valence $\leq 5 \quad \Longleftrightarrow \quad$ curvature bounded below by 0

Enumeration (with Frank Lutz)

- All simplicial 3-manifolds with edge valence ≤ 5
- Exactly 4761 three-spheres plus 26 finite quotients

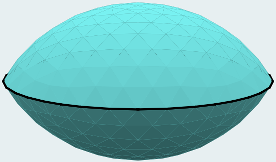
- Surely true that Ricci flow immediately gives positive curvature
- [Matveev, Shevchishin]: Can smooth to get positive curvature
- Can start with spherical geometry on each tetrahedron

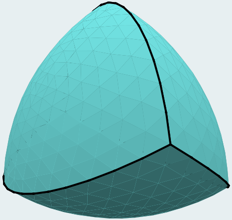
Enumeration interpreted for dual bubble clusters

“Sanity” conditions

Dual to simplicial complex means:

- never have multiple faces between the same two bubbles
- never have multiple edges between the same three bubbles
- in particular, no faces with $n = 1$ or $n = 2$

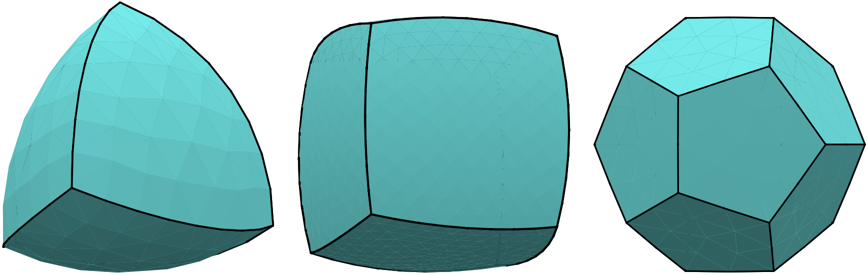




$n \leq 5$

Foam structures (bubble clusters) with $n \leq 5$ for all faces

- 11 types of foam cells (tetrahedron to dodecahedron) allowed

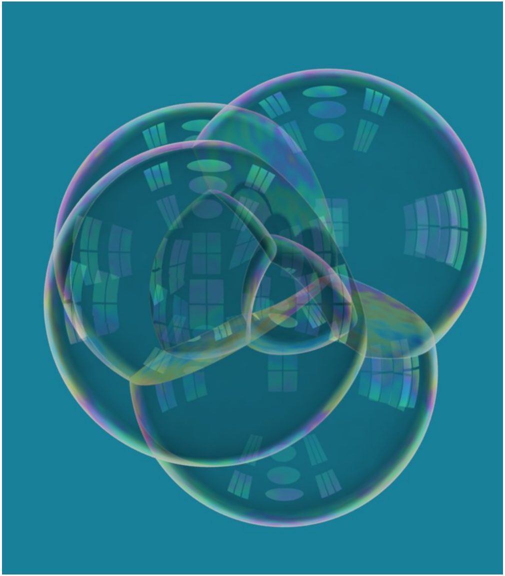


$n \leq 5$

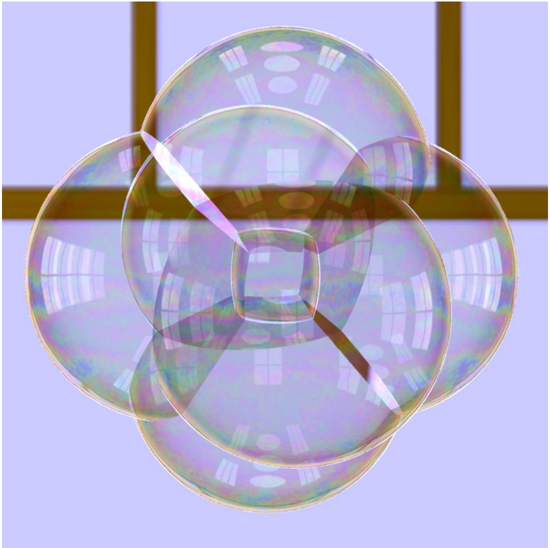
Foam structures (bubble clusters) with $n \leq 5$ for all faces

- Enumerated by [Lutz/Sullivan 2005]
- All are finite clusters
  best thought of as foams in $\mathbb{S}^3$
- Exactly 4761 combinatorial types
  (in $\mathbb{R}^3$ also have to choose which bubble infinite)

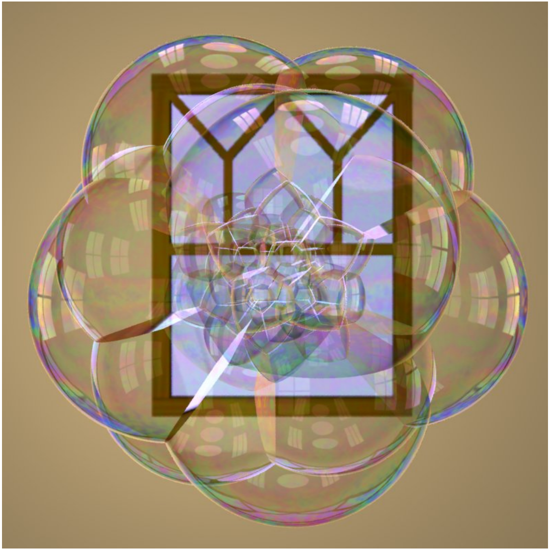
Example $n \leq 5$



Example $n \equiv 4$



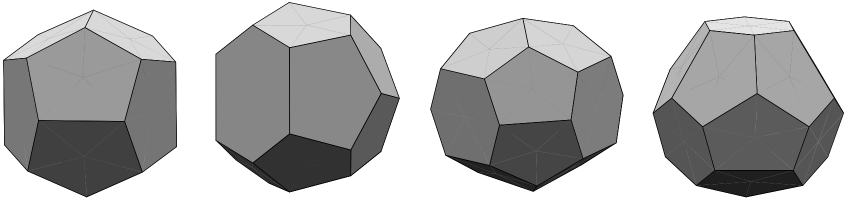
Example $n \equiv 5$



TCP foams

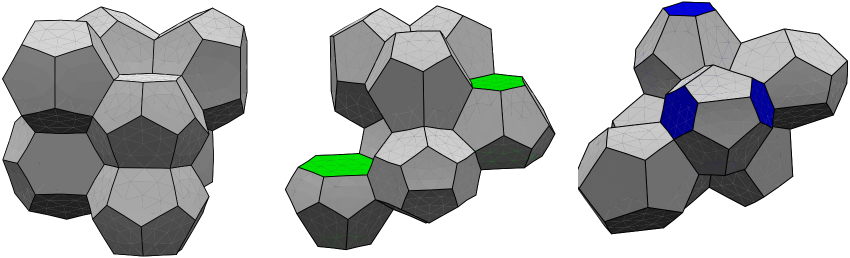
TCP structures from transition metal alloy chemistry

- large atoms pack at vertices of nearly regular tetrahedra
- Voronoi cells (Dirichlet domains) have faces
  with $n = 5$ or $n = 6$, no adjacent 6s
- Allows four cell types in foam, $z = 12, 14, 15, 16$



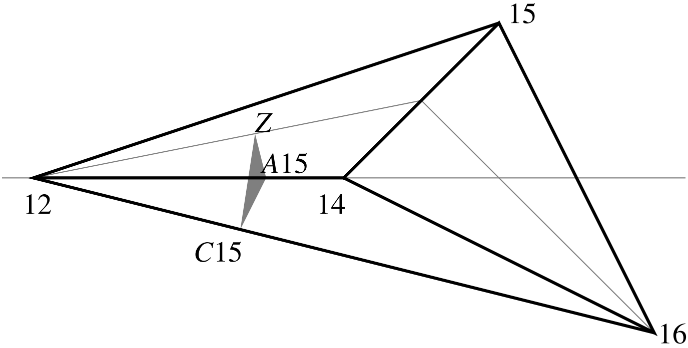
Why TCP?

- Plateau rules say foam dual to triangulation
  and suggest tetrahedra close to regular
- Best known equal-volume foams are TCP duals
- All known (Euclidean) TCP foams are combinations of:

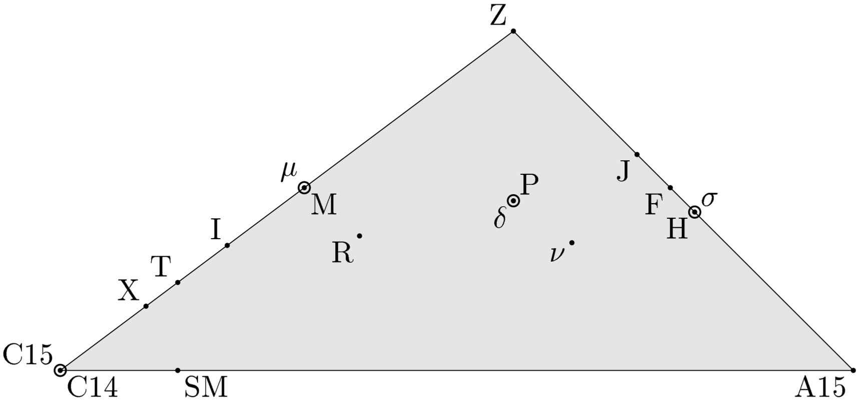


TCP ratios

- TCP triangulations by definition have
  $5 \leq \bar{n} \leq 5\frac{1}{4} \quad (12 \leq \bar{z} \leq 16)$
- Why do all known Euclidean ones have
  $5\frac{1}{10} \leq n \leq 5\frac{1}{9} \quad \left(13\frac{1}{3} \leq \bar{z} \leq 13\frac{1}{2}\right)$?



Known Euclidean TCP foams

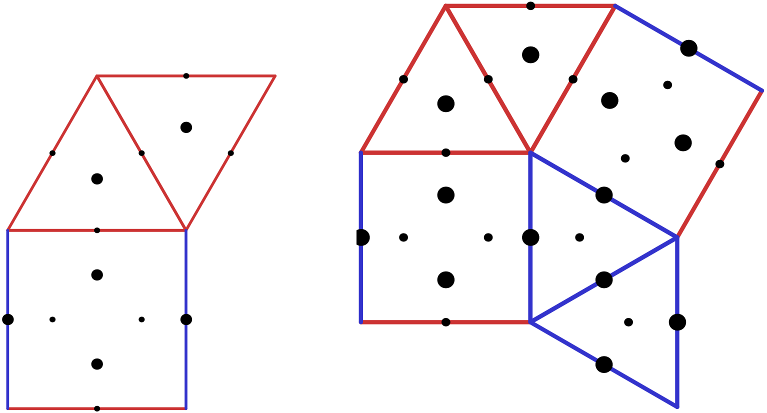


New TCP foams

- TCP foams constructed [Sullivan 2002 Delft]
  in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$
- These lie to the expected sides of the plane
  $6X - 2P - 7Q - 12R = 0$
  of the known Euclidean TCPs

New TCP foams

One Euclidean family gives arbitrary blend of A15, Z



Generalize by allowing green edges with no vertex

Newer TCP foams

New results [Lutz/Sulanke/Sullivan 2006]

- No TCP foam with only 16s (in any ambient space)
- Look for TCP foams with just 12s and 14s
  Examples found with $12 \leq \bar{z} \leq 13$ tile $\mathbb{S}^3$
  Examples found with just 14s have Heisenberg geometry
  (not hyperbolic)

Open questions

With restrictions can we relate combinatorics to topology?

- Any 3-manifold can be tiled with $n = 4, 5, 6$
- Conj: can be tiled with TCP foam
- For such restricted classes of foams
  are there connections between $\bar{z}$ and the ambient geometry?