The Poincaré dodecahedral space

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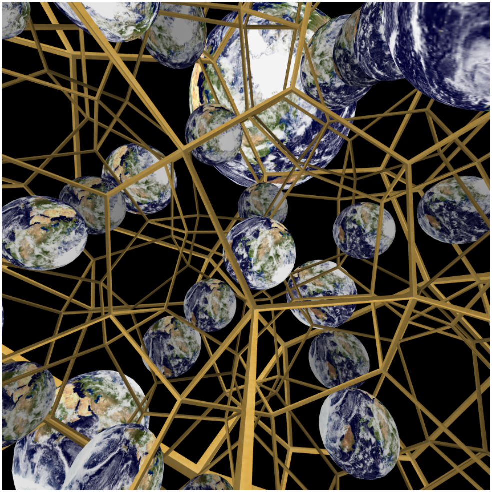
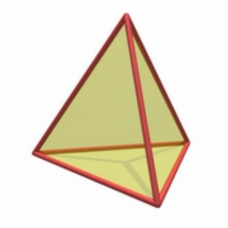
OrbiCG/Triangles Workshop
on
Computational Geometry

Poincaré dodecahedral space

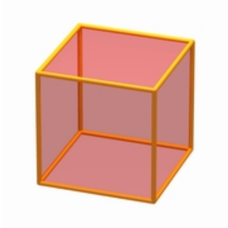


Poincaré dodecahedral space

Platonic solids

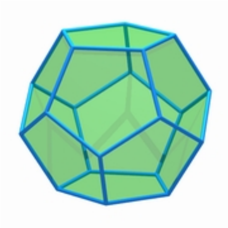


tetrahedron

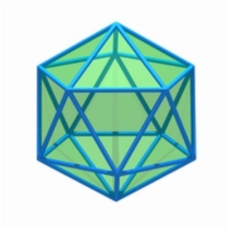


cube

octahedron



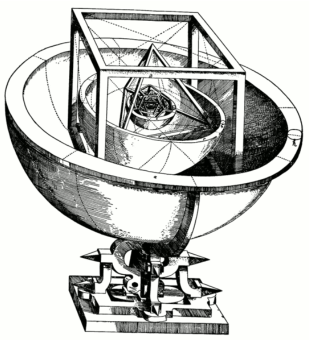
dodecahedron

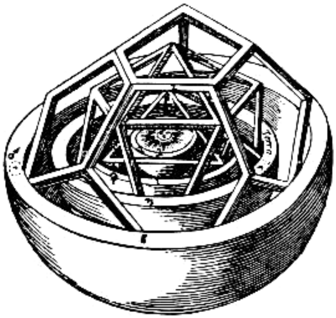


icosahedron

Kepler (1571–1630)

Kepler: Mysterium Cosmographicum (1596)





Mercury – *Octahedron* – Venus – *Icosahedron* – Earth – *Dodecahedron* – Mars – *Tetrahedron* – Jupiter – *Cube* – Saturn

"Van deze veelvlakken zijn er precies vijf en vijf zijn er nodig om de zes planeten uit elkaar te houden. Zo werkt God's denken!"

There are exactly five Platonic solids

Proof:

1. e_v: nr. edges/vertex $v_e = 2$: nr. vertices/edge
   e_f: nr. edges/face $f_e = 2$: nr. faces/edge
2. $$v\,e_v = e\,v_e = 2e \qquad f\,e_f = e\,f_e = 2e$$
3. Euler:
   $$2 = v - e + f = f\left(\frac{e_f}{e_v} - \frac{e_f}{2} + 1\right)$$
4. $$f = \frac{4e_v}{4 - (e_v - 2)(e_f - 2)}$$
5. So:
   $$(e_v - 2)(e_f - 2) < 4, \quad e_v, e_f \geq 3$$

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Five Platonic solids (cont'd)

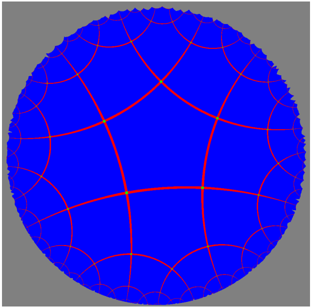
$$(e_f - 2)(e_v - 2) < 4, \quad \text{with} \quad e_v \geq 3 \quad \text{and} \quad e_f \geq 3.$$

e_v	e_f	f	Type
3	3	4	Tetrahedron
3	4	6	Kubus
3	5	12	Dodecahedron
4	3	8	Octahedron
5	3	20	Icosahedron

$$f = \frac{4e_v}{4 - (e_v - 2)(e_f - 2)}$$

Regular tesselations and (constant) curvature – 2D

$K > 0$ (spherical)

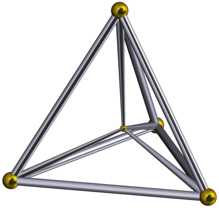


$K < 0$ (hyperbolic)

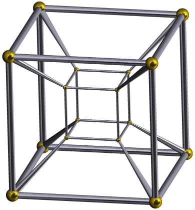
angle (Euclidean: 108°)

120° 90°

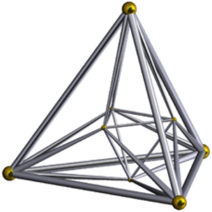
Polytopes in 4D (polychorons)



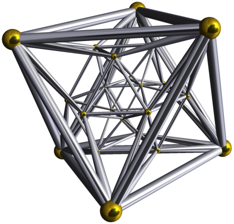
4-simplex



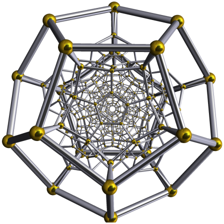
hypercube



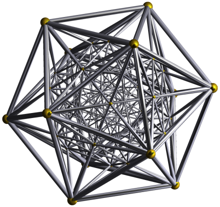
16-cell



24-cell



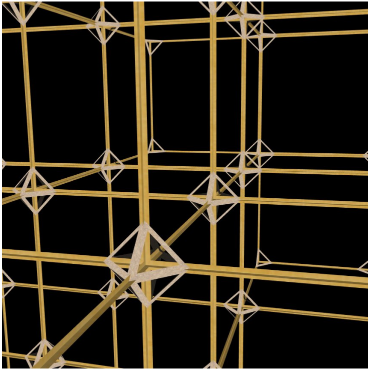
120-cell



600-cell

3D Regular Tesselations (by Platonic solids)

Vertex-figure: intersection of vertex-centered 2-sphere with tesselation



3D Regular Tesselations (by Platonic solids)

Vertex-figure: intersection of vertex-centered 2-sphere with tesselation

11 possible regular tesselations (of $\mathbb{S}^3$, $\mathbb{E}^3$ or $\mathbb{H}^3$):

1. By tetrahedra, cubes or dodecahedra,
   Vertex-figures: tetrahedra, octahedra or icosahedra
2. By octahedra
   Vertex-figure: cube
3. By icosahedra
   Vertex-figure: dodecahedron

3D Regular Tesselations (by Platonic solids)

Proof.

- Vertex-figure:
  Platonic solid, c_v faces, c_e faces/vertex.
- Euler for polyhedral 3-manifolds: $v - e + f - c = 0$.
- $$\left(\frac{4}{d-2} + c_v\right)\left(\frac{2d}{d-2} - c_e\right) = \frac{8d}{(d-2)^2}$$
  d: degree of vertex in (boundary of a) cell
- $d = 3$: $(c_v, c_e) \in \{(4,3),(8,4),(20,5)\}$
  $d = 4$: $(c_v, c_e) = (6,3)$
  $d = 5$: $(c_v, c_e) = (12,3)$

□

3D Regular Tesselations (by Platonic solids)

Cell	d	EDA	c_v	V-figure	c_e	DA	Space
Tetra	3	70.53^o	4	Tetra	3	120^o	$\mathbb{S}^3$
			8	Octa	4	90^o	$\mathbb{S}^3$ $(*)$
			20	Icosa	5	72^o	$\mathbb{S}^3$ $(*)$
Cube	3	90^o	4	Tetra	3	120^o	$\mathbb{S}^3$
			8	Octa	4	90^o	$\mathbb{E}^3$
			20	Icosa	5	72^o	$\mathbb{H}^3$ $(*)$
Dodeca	3	116.57^o	4	Tetra	3	120^o	$\mathbb{S}^3$
			8	Octa	4	90^o	$\mathbb{H}^3$ $(*)$
			20	Icosa	5	72^o	$\mathbb{H}^3$
Octa	4	109.47^o	6	Cube	3	120^o	$\mathbb{S}^3$
Icosa	5	138.19^o	12	Dodeca	3	120^o	$\mathbb{H}^3$

(E)DA: (Euclidean) Dihedral Angle

Group actions and quotient manifolds

- $I^* < \mathbb{S}^3$: binary icosahedral group (order: 120)
- 'Lift' of group $I <$ SO(3) of rotational symmetries of dodecahedron (order: 60) under universal covering map $\mathbb{S}^3 \rightarrow SO(3)$
- $\mathbb{S}^3/I^*$: Poincaré Dodecahedral Space (PDS), 3-manifold of constant positive curvature.
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Estimating the volume

The maximum number of cells for the different tesselations:

- Tetrahedron (V-figure: tetrahedron): $c < 12$
- Cube (V-figure: tetrahedron): $c < 13$
- Octahedron (V-figure: cube): $c < 30$
- Dodecahedron (V-figure: tetrahedron): $c < 127$

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Tetrahedron with V-figure octahedron or icosahedron:
not the orbit of a single cell!

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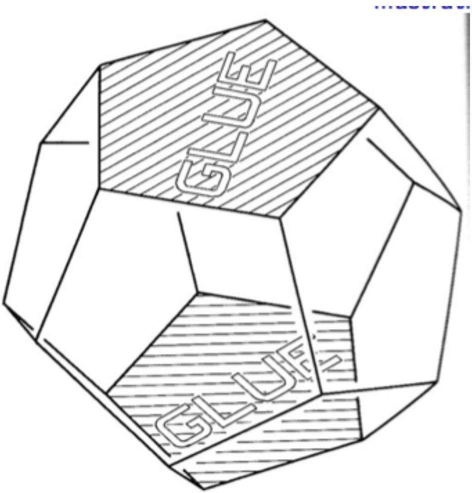
$c = 120$, so $\mathbb{S}^3/I^*$ must be obtained by gluing dodecahedra (identifying faces), such that

- four dodecahedra incident to each vertex ($c_v = 4$)
- three tetrahedra incident to each edge ($c_e = 3$)

PDS: Identify opposite faces with twist



Spherical PDS: $c_e = 3$

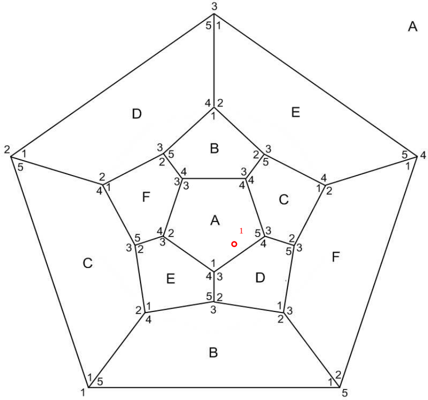


Figure: Schlegel diagram of dodecahedron. Opposite faces identified with minimal twist $\pi/5$

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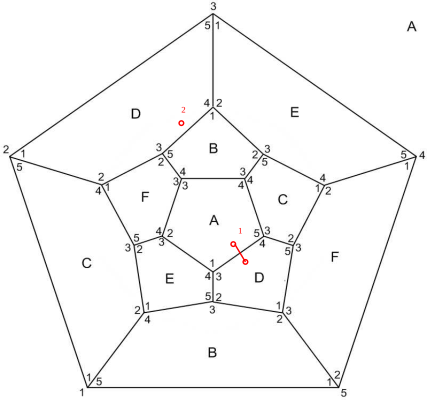

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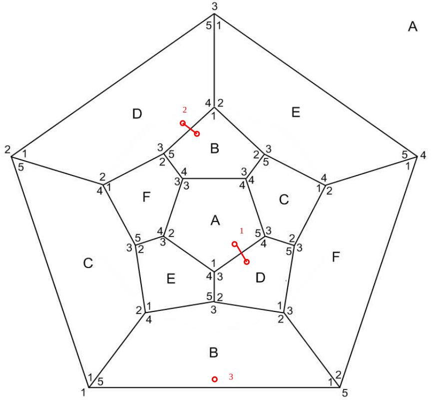

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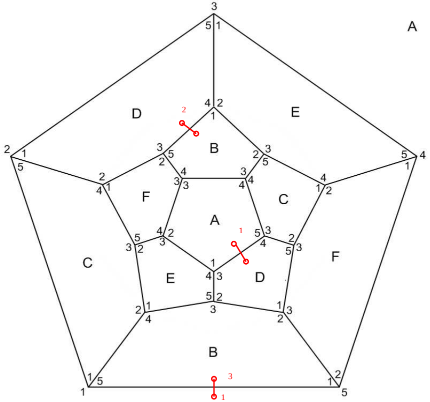


Figure: Schlegel diagram of dodecahedron. Opposite faces identified with minimal twist $\pi/5$

Hyperbolic PDS: $c_e = 5$

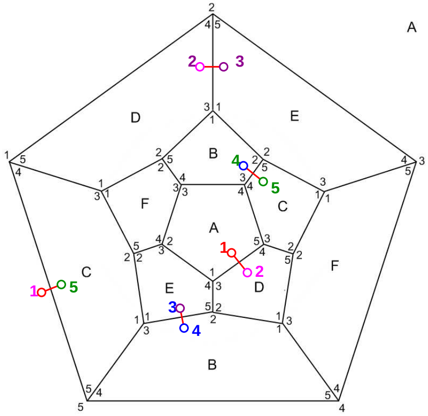
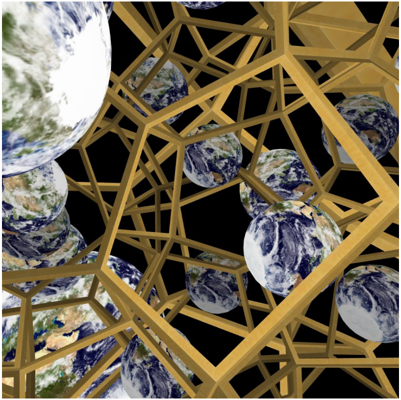


Figure: Schlegel diagram of dodecahedron. Opposite faces identified with twist $3\pi/5$

Dodecahedral tesselation of $\mathbb{S}^3$ $(c_v = 4, c_e = 3)$



Poincaré Dodecahedral Space

Dodecahedral tesselation of $\mathbb{H}^3$ $(c_v = 8, c_e = 4)$