

# Understanding hyperbolic surfaces

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# Tiling the hyperbolic plane



## From tilings to hyperbolic surfaces

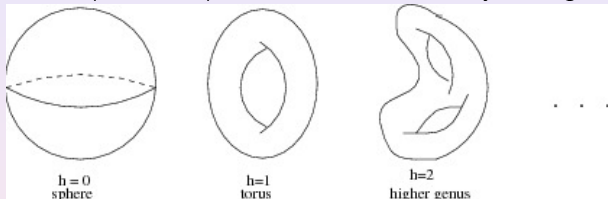
A tiling determines an identification by isometries between different areas of  $\mathbb{H}^2$  corresponding to the same tile. So a group action on  $\mathbb{H}^2$ . The quotient is a *hyperbolic surface*, possibly with *orbifold points* (points where total angle  $2\pi/k$ ).

Some remarks on hyperbolic surfaces.

- There are many! (Much “more” than Euclidean surfaces).
- They stand at a “center” of maths : relations to complex analysis, algebraic geometry, representation theory, number theory, etc.
- Key point : understand the geometry of the space of hyperbolic metrics on a surface (Teichmüller space).

# Hyperbolic surfaces

Closed (orientable) surfaces are classified by their genus.



- The sphere has a spherical metric ( $K = 1$ ),
- The torus has (several) Euclidean metrics (the space of those metrics is... the hyperbolic plane),
- All other surfaces have (many) hyperbolic metrics.

Often useful to allow for point singularities : *cone singularities*, where the angle is  $\theta < 2\pi$ .

In particular, *orbifold points* where angle  $2\pi/k$  (cf Porti's talk).

# Riemann surfaces

A *Riemann surface* is a surface with a complex structure. Modeled on  $\mathbb{C}$ , considered up to isotopies (=deformations).

A complex structure is the same (for *orientable* surfaces) as a conformal structure : enough to measure angles (not lengths).

**Poincaré Uniformization Thm.** Any conformal class of metrics on a surface of genus  $\geq 2$  contains a unique hyperbolic metric.

Extension to surfaces with cone singularities : choose where the cone points are, and their “angle”.

**Riemann Mapping Thm.** Given  $\Omega, \Omega' \subset \mathbb{C}$  simply connected domains,  $\exists!$  conformal diffeo  $\Omega \rightarrow \Omega'$ , with given images for  $x, y, z \in \partial\Omega$ .

Discrete version based on circle packings (Koebe, Thurston), converges well to Riemann map (Rodin-Sullivan, Zhe,...).

## Motivations and applications

Conformal maps can be used for texture mapping. Picture : Kharevych, Springborn, Schröder. Conformality seems to be visually relevant. Discrete version  $\implies$  circle patterns  $\implies$  hyperbolic geometry.



Cone singularities are necessary to keep length distortion within reasonable bounds. Higher genus surfaces happen  $\implies$  hyperbolic geometry.

## Teichmüller space

**Def.** The space of all hyperbolic metrics on a given surface (of genus  $g \geq 2$ ), up to isotopy (deformation).  $\mathcal{T}_g$ .

**Thm.**  $\mathcal{T}_g$  is topologically a ball, of dimension  $6g - 6$ .

**Def.** With  $n$  marked points :  $\mathcal{T}_{g,n}$ , topologically a ball of dimension  $6g - 6 + 2n$ . One can fix the cone angles,  $\theta_i (< \pi)$ .

Rich geometric structure on  $\mathcal{T}_g$  : Kähler (Weil-Petersson) metric + Finsler metric. Action on  $\mathcal{T}_g$  of the *mapping-class group*.

Recent development : discrete/combinatorial models of  $\mathcal{T}_g$ , e.g. the “curve complex” (proof of Ending Lamination Conjecture, Minski).

## Fenchel-Nielsen coordinates

Consider a surface  $S$  of genus  $\geq 2$ , with a *pant decomposition* : a family of  $n$  disjoint simple closed curves with complement pairs of pants.

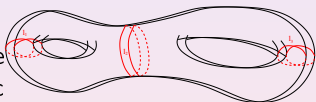
Let  $g$  be a hyperbolic metric on  $S$ . The lengths of the curves define  $n$  numbers  $l_1, \dots, l_n$ .

Moreover, the *gluing angles* define  $\theta_1, \dots, \theta_n$ . The  $l_i, \theta_i$  describe the metric up to diffeo.

Thm : the  $\theta_i \in \mathbb{R}$  are well-defined on  $\mathcal{T}_S$ .

The  $(l_i, \theta_i)$  are global coordinates on  $\mathcal{T}_S$ .

Cor :  $\mathcal{T}_S$  is homeomorphic to a ball of dimension  $6g - 6$ .



The construction strongly depends on the pant decomposition.



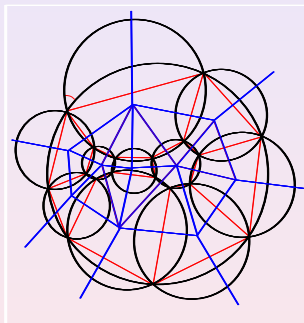
## Angles of Delaunay tessellations

Let  $x_1, \dots, x_n \in S^2/\mathbb{R}^2/H^2$ , the corresponding Delaunay tessellation defines an “ideal” circle pattern.

Consider the angles between the circles, and the incidence graph. Two conditions :

- the sum on the boundary of a face is  $2\pi$ ,
- the sum on any other simple closed curve is  $> 2\pi$ .

The Delaunay tessellation is then unique..



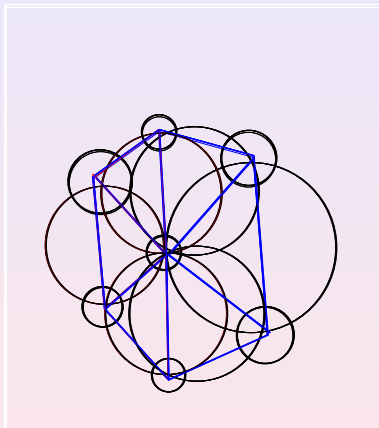
Result (Andreev 1971, Rivin 1992) based on 3d hyperbolic geometry. Extended to higher degree surfaces (Leibon, Thurston, Rivin).

Parameterization of Teichmüller space with marked points by the angles (in polyhedron), depends on graph. Cone sing possible (Bobenko-Springborn, ...)

## Weighted Delaunay

Delaunay tessellations are defined for a family of disjoint disks.  $\exists!$  tessellation with vertices at centers of disks, faces corresponding to circles orthogonal to the disks/adjacent vertices, with interior angle  $< \pi/2$  with other disks.

They form an “hyperideal” circle pattern.



The angle satisfy polyhedral relations. Other parameterization of Teichmüller space w/ marked points, no linear constraint on angles (S. 05,08).