Approximate Algebraic Geometry

for curves and surfaces

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1st October 2003

Symbolic-numeric methods

for curves and surfaces

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Certified/controlled algorithms for curves and surfaces

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$\overline{\mathbb{Q}}$: *Yes, but* $\sqrt{2}$ *is special*, says Galois.

Effective Geometric Objects

- **Effective real numbers** are such that we can ask:
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Numbers

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Represented as

- ☐ an arithmetic tree ($\sqrt{x+y+2\sqrt{x\,y}}-\sqrt{x}-\sqrt{y}$), and/or
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• Comparison of two numbers by refinement until a separating bound:

$$\alpha \neq 0 \Rightarrow |\alpha| > B(Symbolic\ Expression\ of\ \alpha).$$

Subdvision solver

□ **Bernstein basis**: $f(x) = \sum_{i=0}^{d} b_i \, B_d^i(x)$, where $B_d^i(x) = \binom{d}{i} x^i (1-x)^{d-i}$.

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$$b_i^0 = b_i, \;\; i = 0, \ldots, d,$$
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- For $t = \frac{1}{2}$, $b_i^r = \frac{1}{2}(b_i^{r-1} + b_{i+1}^{r-1})$.; use of adapted arithmetic.
- Number of arithmetic operations bounded by $\mathcal{O}(d^2)$, memory space $\mathcal{O}(d)$. Indeed, asymptotic complexity $\mathcal{O}(d \log(d))$.

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Algorithm: isolation of the roots of f on the interval $[a,b]$

INPUT: *A representation* $(\mathbf{b}, [a,b])$ *associate with* f *and* ϵ.
- *If* $V(\mathbf{b}) > 1$ *and* $|b-a| > \epsilon$, *subdivide;*
- *If* $V(\mathbf{b}) = 0$, *remove the interval.*
- *If* $V(\mathbf{b}) = 1$, *output interval containing one and only one root.*
- *If* $|b-a| \leq \epsilon$ *and* $V(\mathbf{b}) > 0$ *output the interval and the multiplicity.*

OUTPUT: *list of isolating intervals in* $[a,b]$ *for the real roots of* f *or the* ϵ*-multiple root.*

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OUTPUT: *list of isolating intervals in* $[a,b]$ *for the real roots of* f *or the* ϵ*-multiple root.*

- Multiple roots (and their multiplicity) computed within a precision ϵ.
- $x := t/(1-t)$: Uspensky method.
- Complexity: $\mathcal{O}(\frac{1}{2}d(d+1)\,r\,\left(\lceil \log_2\left(\frac{1+\sqrt{3}}{2s}\right)\rceil - \log_2(r) + 4\right))$ [MVY02+]
- Natural extension to B-splines.

Benchmarks

Pentium III 933Mhz.

The number of equations per s. (C++ with 64-bit floats; $\epsilon = 0.000001$):

d	8	9	12	16	18	20
	25 000	20-22 000	12-13 000	7.5-8 000	5.9-6.2 000	5.4 000

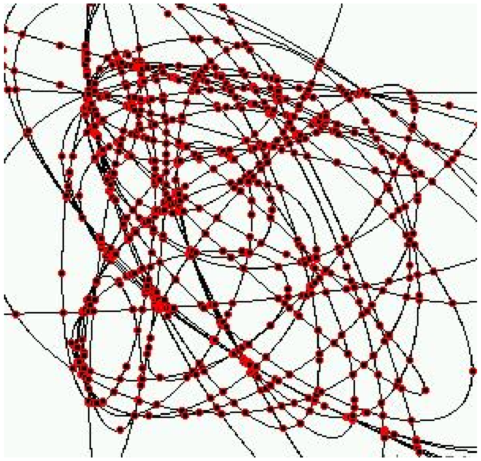
Equations per s. (precision bits vs. degree; $\epsilon = 0.000001$) using GMP library:

d	16	20	30	40	60	80	100
128-bit	96	62.5	25.4	12.5	–	–	–
192-bit	83.3	53.2	21.5	10.8	4.0	–	–
256-bit	73.5	47.2	18.9	9.5	3.6	1.8	–
384-bit	60.2	37.7	15.2	7.6	2.9	1.4	0.8
512-bit	51	31.2	12.2	6.1	2.3	1.2	0.7

Compare favorably with other efficient solvers (Aberth method, `mpsolve`).

[Demo]

Sweeping a set of (conic) circular arcs



[DFMT00]

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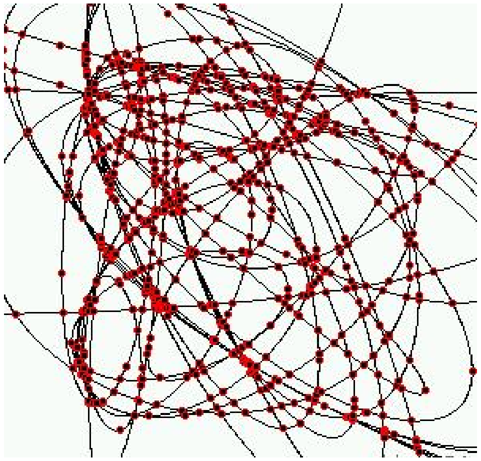
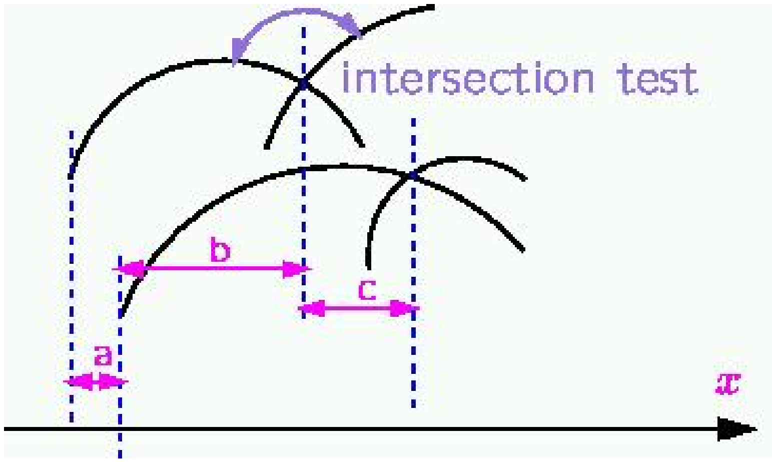




- Arcs represented by the equations of a circle and two lines.

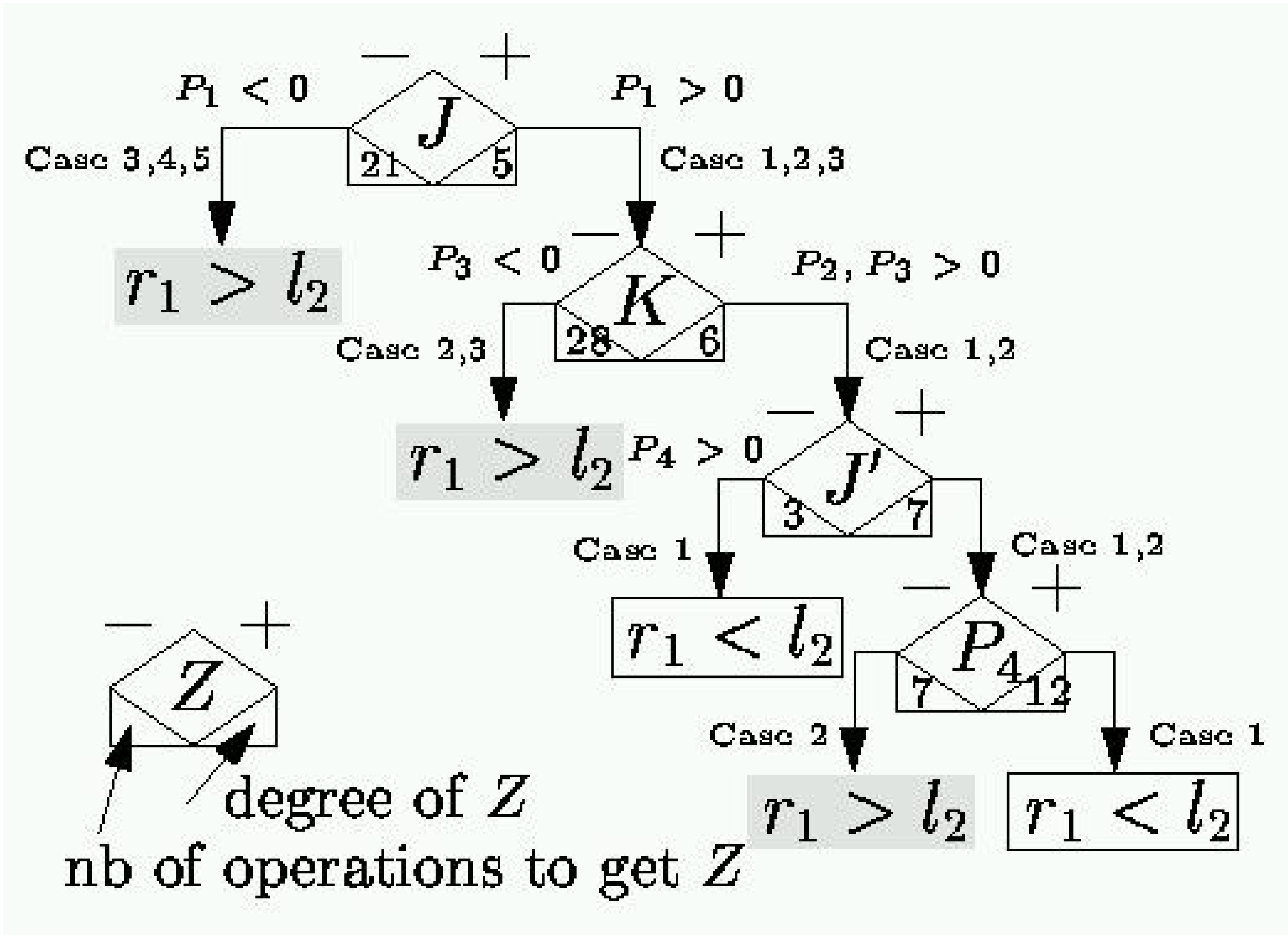
- By resultant computation, it reduces to sign evaluations of algebraic polynomials P_i (degree ≤ 12):

$$P_4 t^4 + P_3 t^3 + P_2 t^2 + P_1 t + P_0 = 0$$

- Inputs parameters in a bounded grid; static error bounds.

[DFMT00]

- Filtering technics can be used easily.
- Efficiency:



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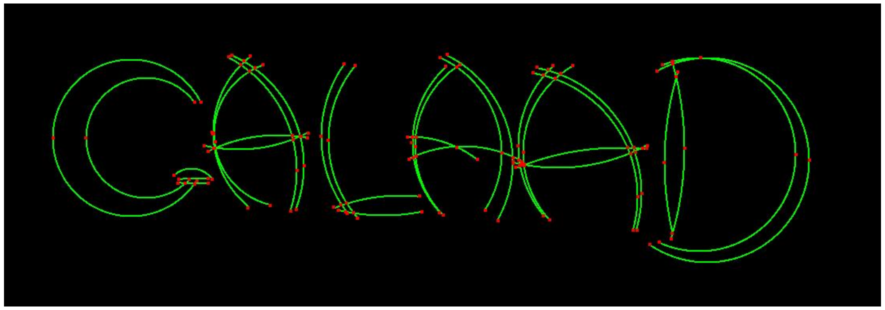
Preliminary benchmarks



A: 100, S:23123, V: 1916, E: 6878	Sweep (s)	Incr. (s)
SplCart, ExDFMT, double	0.6	0.7
RefCart, ExDFMT, double	0.8	0.2
SplCart, ExDFMT, ZZraw	9.3	11.0
SCN, ExDFMT, ZZraw	6.0	7.0
SCN, ExDFMT, Gmpz	9.0	6.0
RefCart, ExDFMT, ZZraw	8.2	9.3
SplCart, Naive, Core	23.2	34.7
RefCart, Naive, Core	24.9	36.3
SplCart, Naive, Leda	30.7	21.6
RefCart, Naive, Leda	30.8	21.9
SplCart, ExDFMT, Gmpq	51.7	60.5
RefCart, ExDFMT, Gmpq	57.5	61.2

A:Arcs, S:Seed, V:Vertices, E:Edges. Random data.

A: 400, S: 43123, V: 33653, E: 131402	Sweep (sec)	Incr. (sec)
SplCart, ExDFMT, double	3.5	0.6
RefCart, ExDFMT, double	5.0	1.3
SplCart, ExDFMT, ZZraw	179.5	174.9
RefCart, ExDFMT, ZZraw	156.9	147.2
SCN, ExDFMT, ZZraw	100.0	95.0
SCN, ExDFMT, Gmpz	140.0	25.0
SplCart, Naive, Core	Abort	Abort
RefCart, Naive, Core	Abort	Abort
SplCart, Naive, Leda	520.6	337.7
RefCart, Naive, Leda	523.6	345.6
SplCart, ExDFMT, Gmpq,	989.9	960.3
RefCart, ExDFMT, Gmpq,	1104.0	1009.3

Benchmarks

Times							Exactness	
Data		Naive		Polynomial			Naive	Pol.
		`double`	Interval + real	`double`	GMP	static + semi-static + Naive Int. + GMP	`double`	`double`
		μs	μs	μs	μs	μs	%	%
$l_1 \leq r_2?$	rnd22	0.60	2.48	0.25	82	0.36	100	100
	rnd16	0.60	2.48	0.25	78	0.41	100	100
	almost	0.60	67.	0.25	115	6.80	99	78
	degenerate	0.60	2170.	0.25	115	128.	8	17
$l_1 \leq l_2?$	rnd22	0.60	2.45	0.28	107	0.44	100	100
	rnd16	0.60	2.45	0.28	102	0.56	100	100
	almost	0.60	38.1	0.28	130	5.64	99	99
	degenerate	0.60	2180.	0.28	130	144.	7	5

- PC-Linux Pentium-III 500MHz, `g++` 2.95.1-02 `-mcpu=pentiumpro -march=pentiumpro`
- **rnda:** pick at random in $[-M, M]^2$, $M = 2^a$, the centers of two cercles and a common point. Arc defined by radical axis and first cercle.
- **degenerate:** same construction with ending points of same abcissae.
- **almost:** same construction with one square radius slightly incremented.

Points

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Multiplication operators

We assume that $\mathcal{Z}(I) = \{\zeta_1, \ldots, \zeta_d\} \Leftrightarrow \mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ of finite dimension D over $\mathbb{K}$.

$$\begin{array}{rcl} M_a : \mathcal{A} & \longrightarrow & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} M_a^{\mathsf{t}} : \widehat{\mathcal{A}} & \longrightarrow & \widehat{\mathcal{A}} \\ \Lambda & \mapsto & a \cdot \Lambda = \Lambda \circ M_a \end{array}$$

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Corollary: (Chow form)

$\Delta(\mathbf{u}) = \det(u_0 + u_1\,\mathtt{M}_{x_1} + \cdots + u_n\,\mathtt{M}_{x_n}) = \prod_{\zeta\in\mathcal{Z}(I)}(u_0 + u_1\zeta_1 + \cdots + u_n\zeta_n)^{\mu_\zeta}.$

Resultant in one variable

Let $f_0 = c_{0,0} + \cdots + c_{0,d_0}\, x^{d_0}$, $\; f_1 = c_{1,0} + \ldots + c_{1,d_1}\, x^{d_1}$ (with $d_0 \leq d_1$).

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Sylvester (1840)

$$\begin{array}{c} \overbrace{\begin{array}{cccccc} f_0 & \cdots & x^{d_1-1}f_0 & f_1 & \cdots & x^{d_0-1}f_1 \end{array}}^{d_0+d_1} \\ \left[\begin{array}{ccc|ccc} c_{0,0} & & \mathbf{0} & c_{1,0} & & \mathbf{0} \\ \vdots & \ddots & & \vdots & \ddots & \\ \vdots & & c_{0,0} & \vdots & & c_{1,0} \\ c_{0,d_0} & & \vdots & c_{1,d_1} & & \vdots \\ & \ddots & \vdots & & \ddots & \vdots \\ \mathbf{0} & & c_{0,d_0} & \mathbf{0} & & c_{1,d_1} \end{array}\right] \end{array} \qquad \left.\begin{array}{l} 1 \\ x \\ \cdot \\ x^{d_1-1} \\ \cdot \\ x^{d_0+d_1-1} \end{array}\right\} d_0+d_1$$

Resultant in one variable

Let $f_0 = c_{0,0} + \cdots + c_{0,d_0}\, x^{d_0}$, $\;f_1 = c_{1,0} + \ldots + c_{1,d_1}\, x^{d_1}$ (with $d_0 \le d_1$).

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\overbrace{\begin{array}{cccc|cccc} f_0 & \cdots & & x^{d_1-1} f_0 & f_1 & \cdots & & x^{d_0-1} f_1 \end{array}}^{d_0+d_1}
$$

$$
\left[\begin{array}{cccc|cccc}
c_{0,0} & & & \mathbf{0} & c_{1,0} & & & \mathbf{0} \\
\vdots & \ddots & & & \vdots & \ddots & & \\
\vdots & & & c_{0,0} & \vdots & & & c_{1,0} \\
c_{0,d_0} & & & \vdots & c_{1,d_1} & & & \vdots \\
 & \ddots & & \vdots & & \ddots & & \vdots \\
\mathbf{0} & & & c_{0,d_0} & \mathbf{0} & & & c_{1,d_1}
\end{array}\right]
\qquad
\left.\begin{array}{c} 1 \\ x \\ \cdot \\ x^{d_1-1} \\ \cdot \\ x^{d_0+d_1-1} \end{array}\right\} d_0 + d_1
$$

Bézout (1779)

$$
\Theta_{f_0,f_1}(x,y) := \frac{f_1(x)\, f_0(y) - f_1(y)\, f_0(x)}{y - x} = \sum_{i=0}^{d_1-1} \theta_{f_0,f_1,i}(x)\, y^i = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_1-1} \theta_{i,j}\, x^i\, y^j .
$$

The Bézout matrix is $\mathsf{B}_{f_0,f_1} = (\theta_{i,j})_{0 \le i,i \le d_1}$.

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\begin{array}{c}
\overbrace{\begin{array}{cccccc} f_0 & \cdots & x^{d_1-1}f_0 & f_1 & \cdots & x^{d_0-1}f_1 \end{array}}^{d_0+d_1} \\
\left[\begin{array}{ccc|ccc}
c_{0,0} & & \mathbf{0} & c_{1,0} & & \mathbf{0} \\
\vdots & \ddots & & \vdots & \ddots & \\
\vdots & & c_{0,0} & \vdots & & c_{1,0} \\
c_{0,d_0} & & \vdots & c_{1,d_1} & & \vdots \\
& \ddots & \vdots & & \ddots & \vdots \\
\mathbf{0} & & c_{0,d_0} & \mathbf{0} & & c_{1,d_1}
\end{array}\right]
\end{array}
\qquad
\left.\begin{array}{l} 1 \\ x \\ . \\ x^{d_1-1} \\ . \\ x^{d_0+d_1-1} \end{array}\right\} d_0+d_1
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$$

The Bézout matrix is $\mathsf{B}_{f_0,f_1} = (\theta_{i,j})_{0 \leq i,i \leq d_1}$.

Theorem : $R(c_{i,j}) := det(S)$ **vanishes iff** $f_0 = 0, f_1 = 0$ **has a common root.**

Solving $f_1 = \cdots = f_n = 0$ by hiding a variable

1. Construct the resultant matrix $\mathtt{S}(x_n)$ of $f_1, \ldots, f_n$ as polynomials in $x_1, \ldots, x_{n-1}$ with coefficients in $\mathbb{K}[x_n]$.

2. Solve $\mathtt{S}(x_n)^t \mathbf{w} = 0$.

   - Either by solving $\det(\mathtt{S}(x_n)) = 0$ and by deducing the corresponding $\mathbf{w}$.
   - or by reducing it to an eigenproblem:

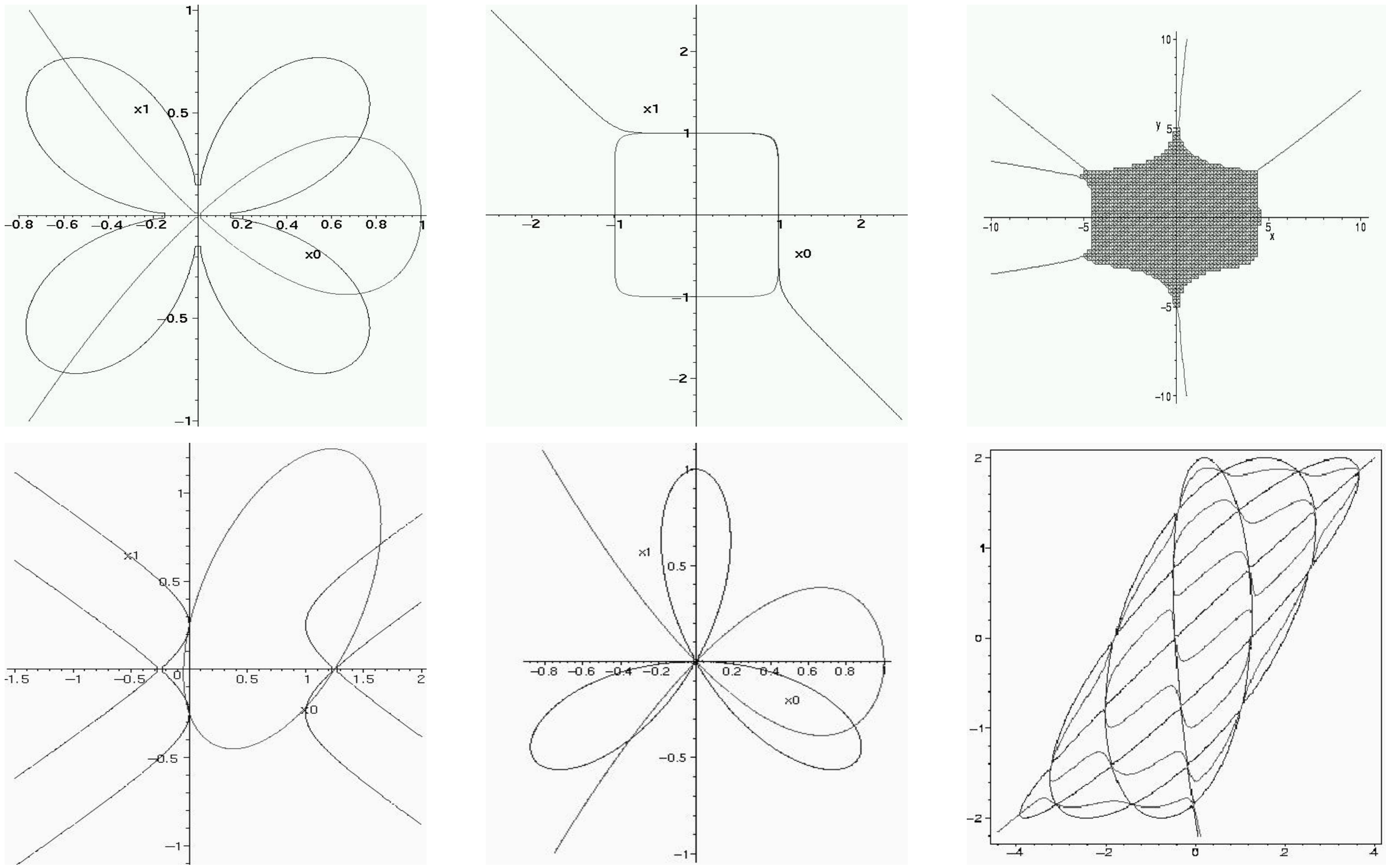
$$(\mathtt{S}_d^t\, x_n^d + \mathtt{S}_{d-1}^t x_n^{d-1} + \cdots + \mathtt{S}_0^t)\, \mathbf{w} = 0$$

   or

$$\left(\begin{bmatrix} 0 & \mathbb{I} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mathbf{0} & \mathbb{I} \\ \mathtt{S}_0^t & \cdots & \mathtt{S}_{d-2}^t & \mathtt{S}_{d-1}^t \end{bmatrix} - x_n \begin{bmatrix} \mathtt{S}_d^t & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathtt{S}_d^t & 0 \\ 0 & \cdots & 0 & \mathtt{S}_d^t \end{bmatrix} \right) \underline{\mathbf{w}} = 0,$$

3. Deduce the other coordinates of the roots from $\mathbf{w}$.

Comparison



Alg. mth.	# r-roots	deg.	# x-roots	Time
1	2	4	2	0.02 s
2	6	3	6	0.02 s
3	6	3	6	0.02 s
4	4	2	4	0.02 s
5	16	4	8	0.02 s
6	4	2	4	0.02 s
7	8	4	8	0.02 s
8	6	3	0	–
9	8	4	8	0.03 s
10	12	6	12	0.03 s
11	10	10	10	0.12 s
12	16	6	0	–
13	6	5	6	0.02 s
14	42	6	42	0.12 s

Sylv. SVD	# r-roots	deg.	# x-roots	Time
1	2	4	2	0.00 s
2	6	3	4	0.00 s
3	6	3	1	0.00 s
4	4	2	2	0.00 s
5	16	4	0	–
6	4	2	4	0.00 s
7	8	4	3	0.00 s
8	6	3	2	0.00 s
9	8	4	Np	–
10	12	6	Np	–
11	10	10	Np	–
12	16	6	10	0.07 s
13	6	5	2	0.00 s
14	42	6		

Sylv. Eig.	# r-roots	deg.	# x-roots	Time
1	2	4	Np	–
2	6	3	6	0.00 s
3	6	3	4	0.00 s
4	4	2	4	0.00 s
5	16	4	16	0.01 s
6	4	2	Np	–
7	8	4	2	0.00 s
8	6	3	0	0.00 s
9	8	4	Np	–
10	12	6	Np	–
11	10	10	3	0.82 s
12	16	6	Np	–
13	6	5	1	0.00 s
14	42	6	Np	–

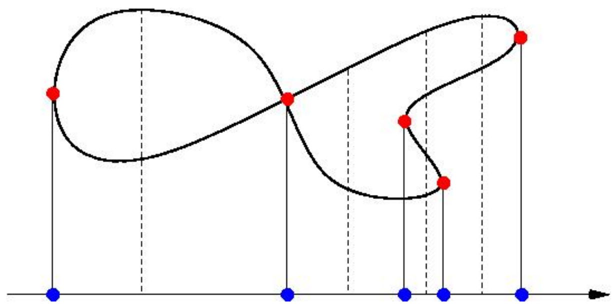
Bez. Eig.	# r-roots	deg.	# x-roots	Time
1	2	4	Np	–
2	6	3	Uc	–
3	6	3	3	0.00 s
4	4	2	Np	–
5	16	4	Np	–
6	4	2	Np	–
7	8	4	1	0.01 s
8	6	3	2	0.01 s
9	8	4	6	0.00 s
10	12	6	11	0.00 s
11	10	10	6	0.19 s
12	16	6	Np	–
13	6	5	2	0.00 s
14	42	6	Np	–

- **Subdvision**

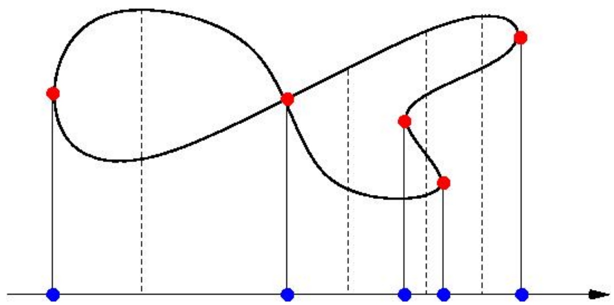
Example	ε	Evaluation	Number of intervals	Time	Number of real roots
10	10^{-5}	10^{-5}	5	0.030	5
11	10^{-5}	10^{-4}	770	79.188	2
15	10^{-5}	10^{-4}	4	0.016	4

Curves

Curves



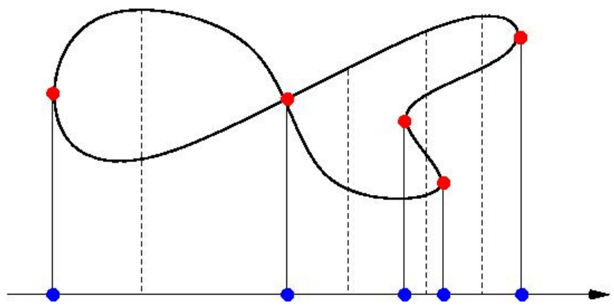
Curves



Algorithm: Topology of an implicit curve

- *Compute the critical value for the projection along the y-abcisses.*
- *Above each point, compute the y-value, with their multiplicity.*
- *Between two critical points, compute the number of branches.*
- *Connect the points between two consecutive levels by y-order, the multi-branches beeing at* **the** *multiple point.*

Curves



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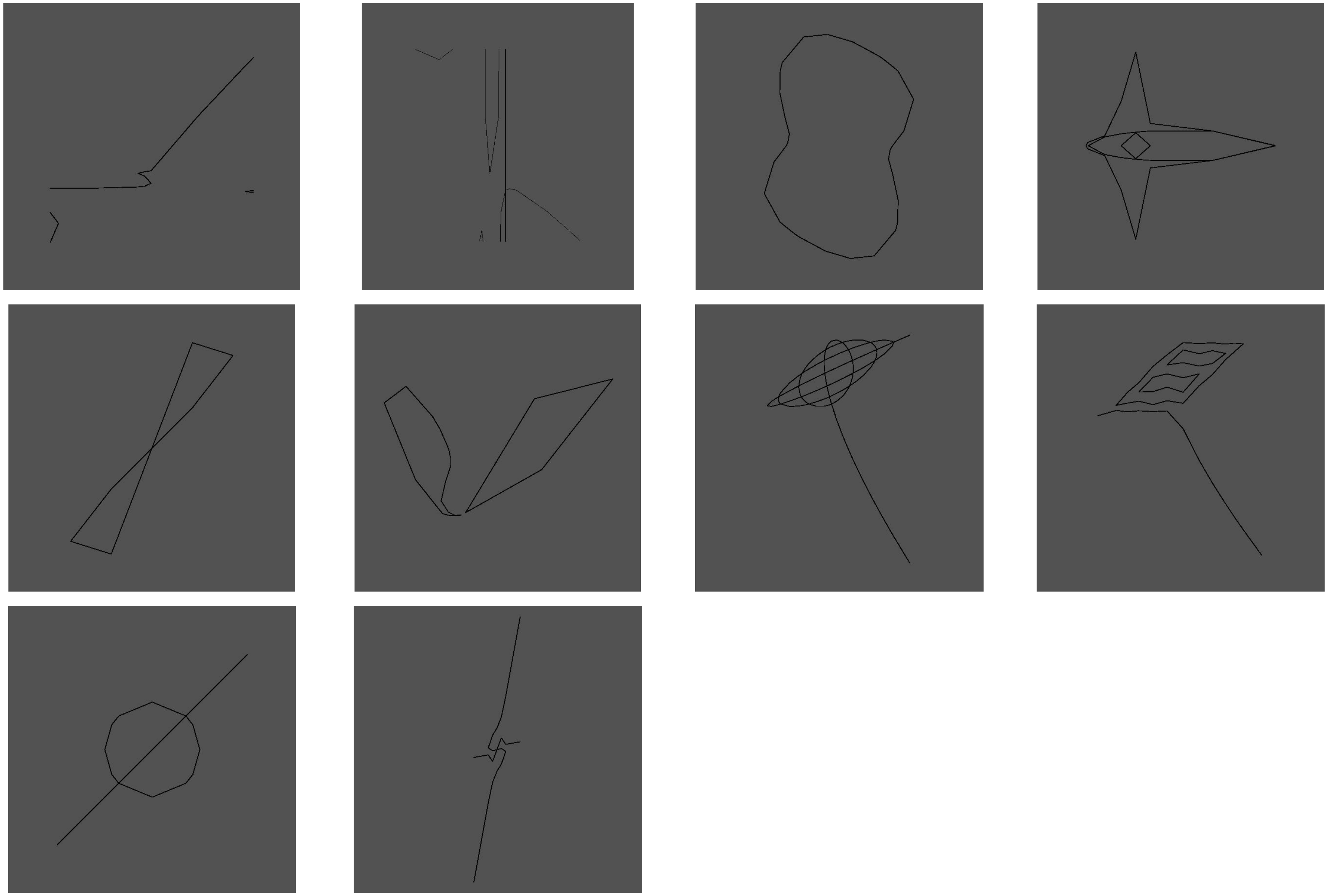
- *Connect the points between two consecutive levels by y-order, the multi-branches beeing at* **the** *multiple point.*

⇨ Rationnal representation of the singular y in terms of the x.

⇨ Descartes rule to detect the multiple point among the regular ones.

Examples



topology	time execution running
1	0.03 s
2	0.04 s
3	0.04 s
4	0.10 s
5	0.05 s
6	0.05 s
7	0.62 s
8	0.17 s
9	0.02 s
10	0.09 s

Solving by subdivision methods

Rectangular patches: $f(x,y) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} b_{j,i} B_{d_1}^i(x) B_{d_2}^j(y)$ associated with the box $[0,1] \times [0,1]$.

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- **Subdivision** by row or by column, similar to the univariate case.

- Arithmetic **complexity** of a subdivision bounded by $\mathcal{O}(d^3)$ ($d = max(d_1, d_2)$), memory space $\mathcal{O}(d^2)$.

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Triangular patches: $f(x,y) = \sum_{i+j+k=d} b_{i,j,k} \frac{d!}{i!j!k!} x^i \, y^j \, (1-x-y)^k$ associated with the representation on the 2d **simplex**.

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• Subdivision at a **new point**. Arithmetic complexity $\mathcal{O}(d^3)$, memory space $\mathcal{O}(d^2)$.

• Combined with **Delaunay triangulations**.

• Extension to A-patches.

Approximating an implicit curve

Algorithm: Representation of the implicit curve $f(x, y) = 0$

INPUT: *A triangular representation of f $L := ((A, B, C), \mathbf{b})$ and a precision ϵ.*

• *If at least one of the triangle edges is bigger that ϵ, split the triangle and insert the new triangles in L:*
- when the number of sign changes of some row (column or diagonal) is ≥ 2,
- or when the coefficients of f'_x (or f'_y, f'_z) have not the same sign.

• *Remove the triangle from L if the coefficients of f have the same sign.*

• *Save it*
- when all the edges of the triangle are smaller than ϵ,
- or when the total number of sign changes on the border sides is 2 and f'_x or f'_y, f'_z, has a constant sign. Isolate the roots.

OUTPUT: *A list of segments approximating the curve $f(x, y) = 0$.*

- Insertion of the circumcenter (barycenter), in order to break the *bad triangle*.

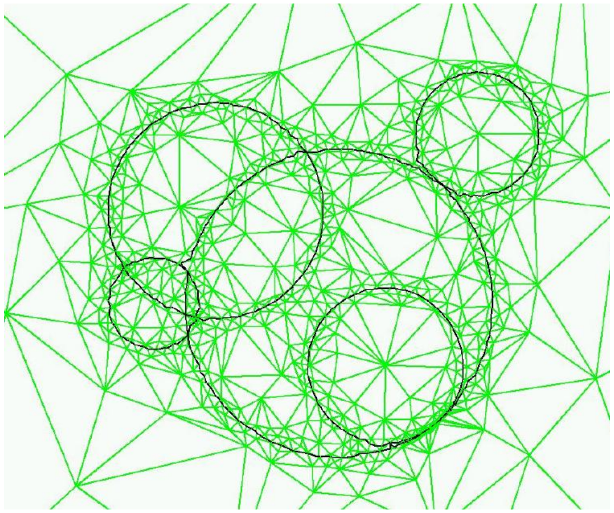
- No specific directions/axes used.

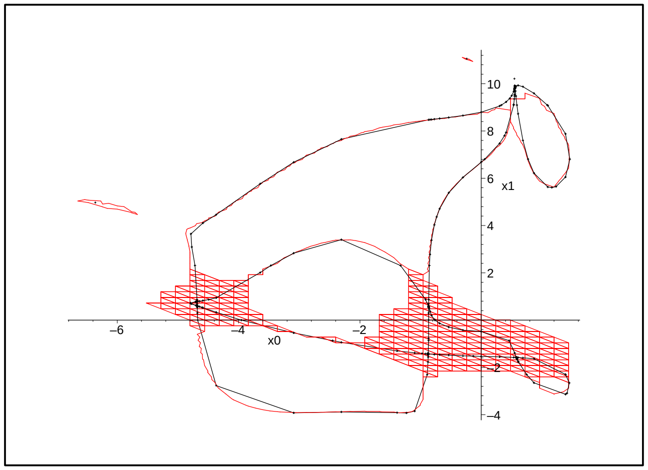
- New edges are constructed, no tangency problem.

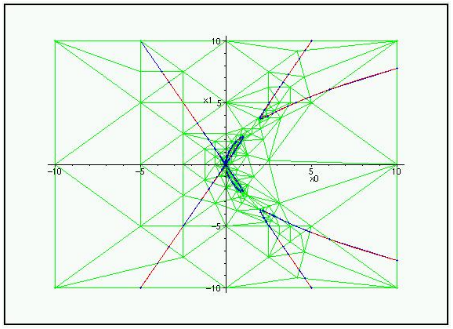
- Number of triangles related to the complexe local feature size.

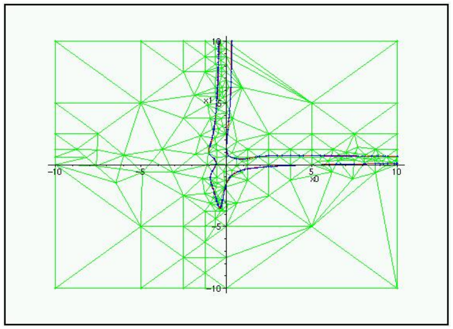
- Application to the intersection of curves, surfaces.

Examples



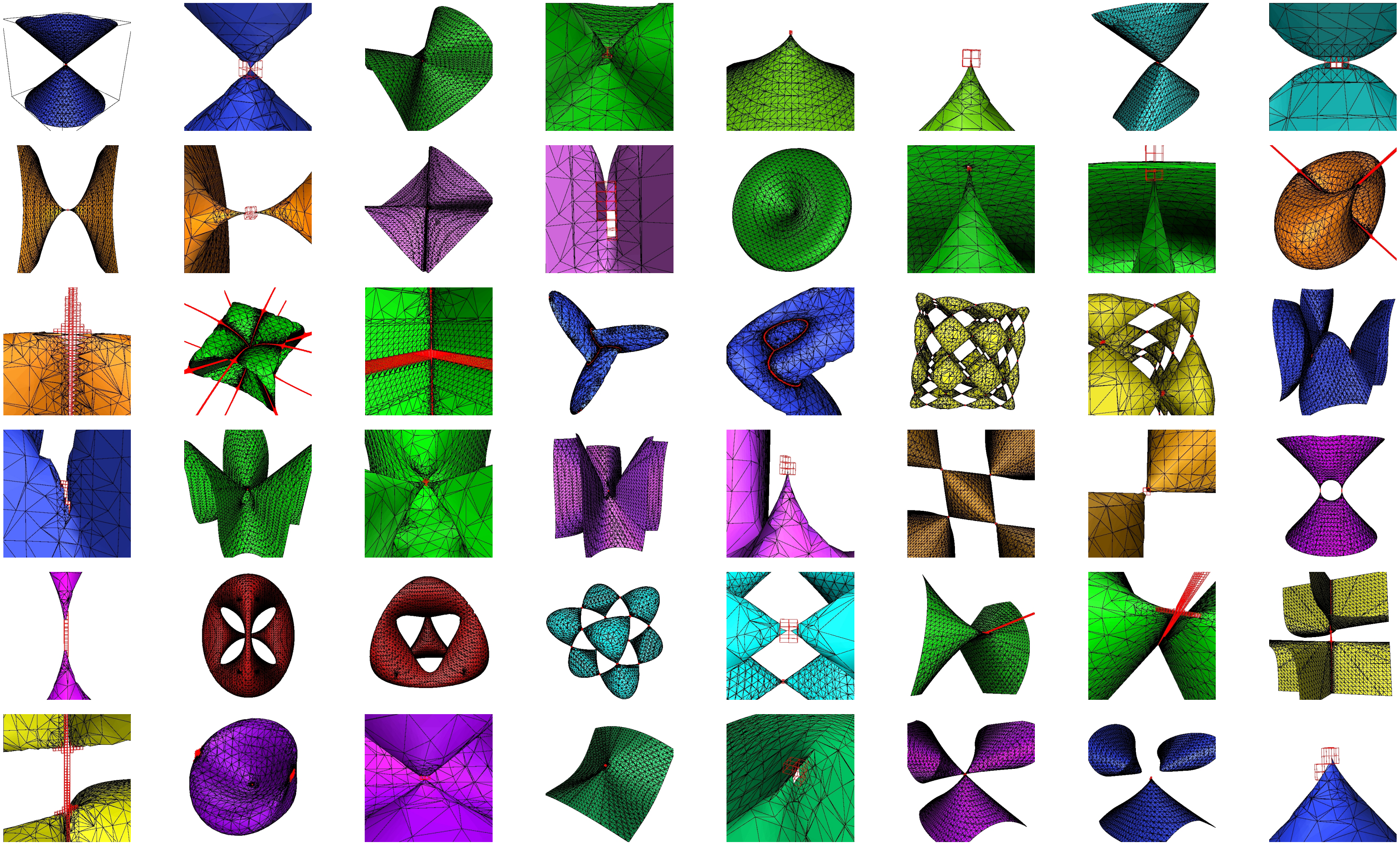


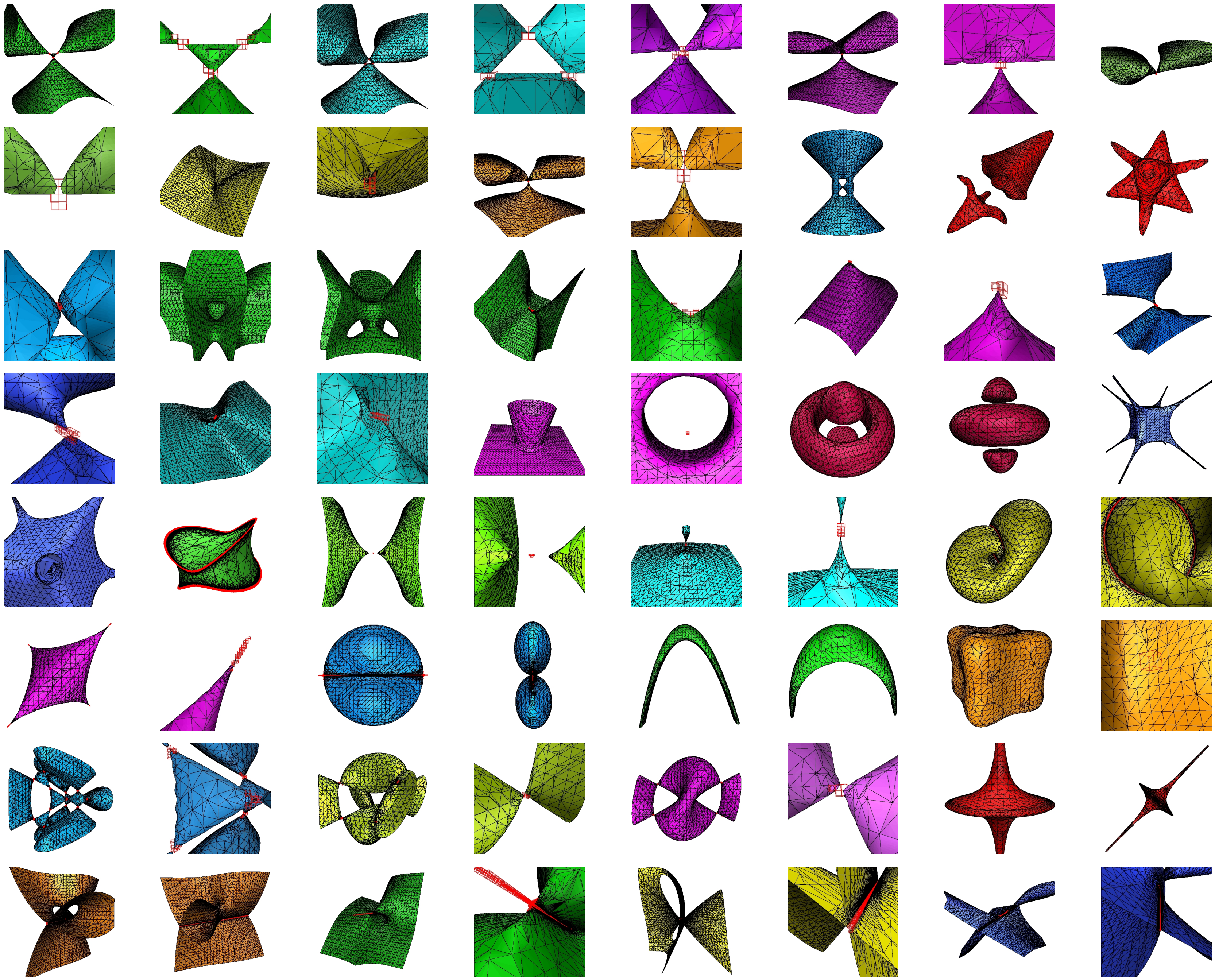


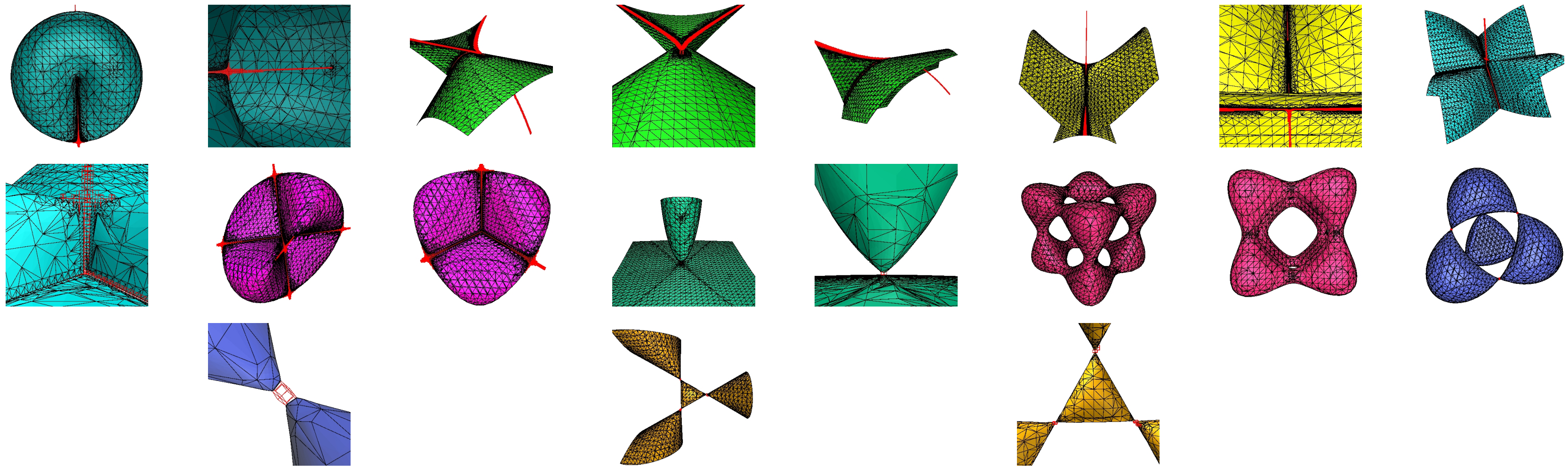


Surfaces

Examples







	nb triangles	nb cellules	temps d'execution
A1 - -	8093	3962 + 6 nt	3.37818 s
A2 + -	6395	3134 + 6 nt	2.90356 s
A2 - -	3598	1768+ 7 nt	1.9934 s
A3 + -	6179	3009 + 10 nt	3.01165 s
A3 - -	6290	6290 + 6 nt	3.14654 s
A4 + -	5859	2825 + 13 nt	3.14721 s
D4 + -	4035	1975 + 6 nt	1.84261 s
D4 - -	8937	4340 + 12 nt	4.29918 s
D4 - - unfold 1	7291	3577 + 10 nt	3.53391 s
D4 - - unfold 2	8269	4019 + 31 nt	3.86696 s
D4 – unfold 3	9633	4615 + 14 nt	4.35244 s

	nb triangles	nb cellules	temps d'execution
D5 + -	7407	3596 + 6 nt	3.61756 s
D5 - -	4254	2014 + 9 nt	2.23405 s
D6 + -	3925	1882 +12 nt	2.1853 s
D6 - -	6983	3364 + 12 nt	3.83701 s
E6 + -	6017	2920 + 18 nt	3.53177 s
E6 - -	3638	1751 + 10 nt	2.11355 s
E7	6019	2927 + 24 nt	3.20513 s
E8	4524	2182 + 14 nt	2.78838 s
Q1	11634	4760 + 185 nt	9.04296 s
Q2	8833	3821 + 84 nt	7.30251 s
Q3	6661	3074 + 24 nt	6.55459 s
S20	8016	3498 + 258 nt	2.9955 s
S21	13542	4848 + 1962 nt	4.16841 s
S22	16722	6234 + 792 nt	5.3028 s
S23	17757	6551 + 1263 nt	5.28855 s
S26	8699	3310 + 1178 nt	4.02468 s

	nb triangles	nb cellules	temps d'execution
D5 + -	7407	3596 + 6 nt	3.61756 s
D5 - -	4254	2014 + 9 nt	2.23405 s
D6 + -	3925	1882 +12 nt	2.1853 s
D6 - -	6983	3364 + 12 nt	3.83701 s
E6 + -	6017	2920 + 18 nt	3.53177 s
E6 - -	3638	1751 + 10 nt	2.11355 s
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S22	16722	6234 + 792 nt	5.3028 s
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Thanks for your attention.