The total order of reducibility: counting multiplicities

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Several authors have studied the problem of the total order of reducibility of a pencil of algebraic plane curves. This problem can be view as follows:
Consider a rational function $r(X,Y) = f(X,Y)/g(X,Y) \in \mathbb{K}(X,Y)$, where $\mathbb{K}$ is a field and suppose that r is non-composite, meaning that it is not possible to write $r = u \circ h$ with $u \in \mathbb{K}(T)$ with $h(X,Y) \in \mathbb{K}(X,Y)$ such that $\deg u \geq 2$. We want to study the fibers of r. The question is when $r^{-1}(\alpha)$ is reducible? That is to say when $f + \alpha g$ is reducible in $\overline{\mathbb{K}}[X,Y]$ (where $\overline{\mathbb{K}}$ is an algebraic closure of $\mathbb{K}$)?
Here α belongs to $\mathbb{P}_1(\overline{\mathbb{K}})$. Thus $f + \infty g = g$. So we can restrict our problem to a question about the pencil $\mu f + \lambda g$, where $(\mu : \lambda) \in \mathbb{P}_1(\overline{\mathbb{K}})$.
The set $\sigma(f,g) = \{(\mu : \lambda) \in \mathbb{P}_1(\overline{\mathbb{K}}) \mid \mu f + \lambda g \text{ is reducible in } \overline{\mathbb{K}}[X,Y]\}$ is called the spectrum. We recall that a polynomial is called absolutely irreducible when it is irreducible over $\overline{\mathbb{K}}$.
A classical theorem of Bertini and Krull says that r is non-composite implies that the spectrum is finite. If $(\mu : \lambda) \in \sigma(f,g)$ we have:

$$\mu f + \lambda g = \prod_{i=1}^{n(\mu:\lambda)} P_{(\mu:\lambda),i}^{e_{(\mu:\lambda),i}}, \text{ where } P_{(\mu:\lambda),i} \text{ is irreducible in } \overline{\mathbb{K}}[X,Y].$$

The total order of reducibility of a non-composite rational function is

$$\rho(f,g) = \sum_{(\mu:\lambda)\in\mathbb{P}_1(\overline{\mathbb{K}})} \big(n(\mu:\lambda) - 1\big).$$

This sum is finite because $n(\mu : \lambda) \neq 0$ if and only if $(\mu : \lambda) \in \sigma(f,g)$. This sum is bounded by $d^2 - 1$ where d is the maximum of the degree of f and g.
Denote by $m(\mu : \lambda)$ the sum $\sum_{i=1}^{n(\mu:\lambda)} e(\mu : \lambda)$. This number is the number of factors of $\mu f + \lambda g$ when we count the multiplicities of the absolute factors. In this talk, we will show that we have:

$$m(f,g) = \sum_{(\mu:\lambda)\in\mathbb{P}_1(\overline{\mathbb{K}})} \big(m(\mu:\lambda) - 1\big) \leq d^2 - 1.$$

That is to say we count the multiplicities of the absolute irreducible factors and we get the same inequality than the one known for $\rho(f,g)$. Then we generalize this result for polynomials f and g with a given Newton polygon. Finally, thanks to Bertini's theorem we show that the above bound works also for polynomials with n variables.
This is a joint work with Laurent Busé.