Computing the Multiplicity Structure from Geometric Involutive Form

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Outline

- Compute an isolated primary component
  Bates etc.'06, Corless etc.'98, Dayton'07, Moller and Stetter'00

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- Construct differential operators
  Damiano etc.'07, Dayton and Zeng'05, Marinari etc.'95,'96, Mourrain'96
- Refine an approximate singular solution
  Lercerf'02, Leykin etc.'05,'07, Ojika etc.'83,'87

Notations

Consider a polynomial system $F \in \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_s]$

$$F : \begin{cases} f_1(x_1, \ldots, x_s) & = & 0, \\ f_2(x_1, \ldots, x_s) & = & 0, \\ & \vdots & \\ f_t(x_1, \ldots, x_s) & = & 0. \end{cases}$$

Let $I = (f_1, \ldots, f_t)$ be the ideal generated by $f_1, \ldots, f_t$.

- An ideal Q is *primary* if, for any $f, g \in \mathbb{C}[\mathbf{x}]$,

$$fg \in Q \Longrightarrow f \in Q \text{ or } \exists m \in \mathbb{N},\ g^m \in Q$$

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- Every ideal has an irredundant primary decomposition

$$I = \cap_{i=1}^{r} Q_i, \text{ and } Q_i \subsetneq \cap_{i \neq j} Q_j$$

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- The minimal nonnegative integer ρ s.t. $\sqrt{Q}^{\rho} \subset Q$ is called the index of Q.

Theorem 1. *[Van Der Waerden 1970] Suppose the polynomial ideal I has an isolated primary component Q whose associated prime P is maximal, and ρ is the index of Q, μ is the multiplicity.*

- *If* $\sigma < \rho$, *then*

$$\dim(\mathbb{C}[\mathbf{x}]/(I, P^{\sigma-1})) < \dim(\mathbb{C}[\mathbf{x}]/(I, P^{\sigma}))$$

- *If* $\sigma \geq \rho$, *then*

$$Q = (I, P^{\rho}) = (I, P^{\sigma})$$

Corollary 2. *The index is less than or equal to the multiplicity*

$$\rho \leq \mu = \dim(\mathbb{C}[\mathbf{x}]/Q)$$

Coefficient Matrix

F can be written in terms of its coefficient matrix $M_d^{(0)}$ as

$$M_d^{(0)} \cdot \begin{pmatrix} x_1^d \\ x_1^{d-1}x_2 \\ \vdots \\ x_s^2 \\ x_1 \\ \vdots \\ x_s \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Prolongation

- Successive prolongations yield

$$F^{(0)} = F, \;\; F^{(1)} = F \cup x_1 F \cup \cdots \cup x_s F, \ldots$$

$$M_d^{(0)} \cdot \mathbf{v_d} = \mathbf{0}, \;\; M_d^{(1)} \cdot \mathbf{v_{d+1}} = \mathbf{0}, \ldots$$

where $\mathbf{v_i} = \left[\mathbf{x}^i, \mathbf{x}^{i-1}, \ldots, \mathbf{x}, 1\right]^T$.

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where $\mathbf{v_i} = \left[\mathbf{x}^i, \mathbf{x}^{i-1}, \ldots, \mathbf{x}, 1\right]^T$.

- $\dim F^{(0)} = \dim \text{ Nullspace}(M_d^{(0)})$

Geometric Projection

- A single geometric projection is defined as

$$\pi(F) = \left\{ [\mathbf{x}^{d-1}, \ldots, 1] \in \mathbb{C}^{N_{d-1}} \mid \exists \mathbf{x}^{d}, M_d^{(0)} \cdot [\mathbf{x}^{d}, \ldots, 1]^T = \mathbf{0} \right\}$$

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- $\dim \pi(F^{(0)})$ is the dimension of a linear space spanned by the null vectors of $M_d^{(0)}$ corresponding to the monomials of the highest degree d being deleted.

Criterion of Involution

Theorem 2. *[Zhi and Reid 2004] A zero dimensional polynomial system F is involutive at prolongation order m and projected order ℓ if and only if $\pi^{\ell}(F^{(m)})$ satisfies the projected elimination test:*

$$\dim \pi^{\ell}\left(F^{(m)}\right) = \dim \pi^{\ell+1}\left(F^{(m+1)}\right)$$

and the symbol involutive test:

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- For the tolerance τ, compute $\dim \hat{\pi}^{\ell}(F^{(m)})$ by SVD.

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- The number of solutions of polynomial system F is $d = \dim(\mathbb{C}[\mathbf{x}]/I) = \dim \hat{\pi}^{\ell}(F^{(m)})$.
- The multiplication matrices $M_{x_1}, \ldots, M_{x_s}$ are formed from the null vectors of $\hat{\pi}^{\ell}(F^{(m)})$ and $\hat{\pi}^{\ell+1}(F^{(m)})$.

Compute Primary Component I

- Form the prime ideal $P = (x_1 - \hat{x}_1, \ldots, x_s - \hat{x}_s)$.

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- Compute $d_k = \dim(\mathbb{C}[\mathbf{x}]/(I, P^k))$ by SNEPSolver for a given tolerance τ until $d_k = d_{k-1}$, set

$$\rho = k - 1, \ \mu = d_\rho, \ Q = (I, P^\rho).$$

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- Compute the multiplication matrices $M_{x_1}, \ldots, M_{x_s}$ of $\mathbb{C}[\mathbf{x}]/Q$ by SNEPSolver.

Example 1 [Ojika 1987]

$$I = (f_1 = x_1^2 + x_2 - 3,\ f_2 = x_1 + 0.125x_2^2 - 1.5)$$

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- $\dim F_2^{(1)} = \dim F_2^{(2)} = 2 \Longrightarrow \dim(\mathbb{C}[\mathbf{x}]/(I, P^2)) = 2$.
- $\dim F_3^{(1)} = \dim F_3^{(2)} = 3 \Longrightarrow \dim(\mathbb{C}[\mathbf{x}]/(I, P^3)) = 3$.
- $\dim F_4^{(1)} = \dim F_4^{(2)} = 3 \Longrightarrow \dim(\mathbb{C}[\mathbf{x}]/(I, P^4)) = 3$.

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Index $\rho = 3$, multiplicity $\mu = 3$.

Example 1 (continued)

The multiplication matrices(local ring) w.r.t. $\{x_1, x_2, 1\}$:

$$M_{x_1} = \begin{bmatrix} 0 & -1 & 3 \\ 6 & 3 & -10 \\ 1 & 0 & 0 \end{bmatrix}, \ M_{x_2} = \begin{bmatrix} 6 & 3 & -10 \\ -8 & 0 & 12 \\ 0 & 1 & 0 \end{bmatrix}$$

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The primary component of I associating to $(1,2)$ is

$$\{x_1^2 + x_2 - 3, x_2^2 + 8x_1 - 12, x_1x_2 - 6x_1 - 3x_2 + 10\}$$

Differential Operators

- Let $D(\alpha) = D(\alpha_1, \ldots, \alpha_s) : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]$ denote the differential operator defined by:

$$D(\alpha_1, \ldots, \alpha_s) = \frac{1}{\alpha_1! \cdots \alpha_s!} \partial x_1^{\alpha_1} \cdots \partial x_s^{\alpha_s},$$

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- Let $\mathfrak{D} = \{D(\alpha),\ |\alpha| \geq 0\}$, we define the space associated to I and $\hat{\mathbf{x}}$ as

$$\triangle_{\hat{\mathbf{x}}} := \{L \in Span_{\mathbb{C}}(\mathfrak{D}) | L(f)|_{\mathbf{x}=\hat{\mathbf{x}}} = 0,\ \forall f \in I\}$$

Construct Differential Operators I

- Write Taylor expansion of $h \in \mathbb{C}[x]$ at $\hat{x}$:

$$T_{\rho-1}h(x_1,\ldots,x_s) = \sum_{\alpha\in\mathbb{N}^s, |\alpha|<\rho} c_\alpha (x_1-\hat{x}_1)^{\alpha_1}\cdots(x_s-\hat{x}_s)^{\alpha_s}$$

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- Compute $\mathrm{NF}(h)$, and expand it at $\hat{\mathbf{x}}$

$$\mathrm{NF}(h(x)) = \sum_\beta d_\beta (\mathbf{x}-\hat{\mathbf{x}})^\beta$$

and find scalars $a_{\alpha\beta} \in \mathbb{C}$ such that $d_\beta = \sum_\alpha a_{\alpha\beta} c_\alpha$.

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- For each β such that $d_\beta \neq 0$, return the operator

$$L_\beta = \sum_\alpha a_{\alpha\beta} \frac{1}{\alpha_1!\cdots\alpha_s!} \partial x_1^{\alpha_1}\cdots\partial x_s^{\alpha_s} = \sum_\alpha a_{\alpha\beta} D(\alpha).$$

  $L = \{L_1,\ldots,L_\mu\}$ is the set of differential operators.

Example 1 (continued)

Write Taylor expansion at $(1,2)$ up to degree $\rho - 1 = 2$,

$$\begin{aligned} h(\mathbf{x}) = c_{0,0} + c_{1,0}(x_1 - 1) + c_{0,1}(x_2 - 2) + c_{2,0}(x_1 - 1)^2 \\ + c_{1,1}(x_1 - 1)(x_2 - 2) + c_{0,2}(x_2 - 2)^2. \end{aligned}$$

Obtain the normal form of h by replacing $x_1^2, x_1 x_2, x_2^2$ with

$$x_1^2 = -x_2 + 3, x_1 x_2 = 6x_1 + 3x_2 - 10, x_2^2 = -8x_1 + 12.$$

The differential operators are:

$$\begin{cases} L_1 & = & D(0,0), \\ L_2 & = & D(0,1) - D(2,0) + 2D(1,1) - 4D(0,2), \\ L_3 & = & D(1,0) - 2D(2,0) + 4D(1,1) - 8D(0,2). \end{cases}$$

Refine an Approximate Singular Solution I

Example 1 (continued) Given an approximate singular solution:

$$\hat{\mathbf{x}} = (1 + 2.5428 \times 10^{-4} + 2.4352 \times 10^{-4} i,$$
$$2 + 8.4071 \times 10^{-4} + 3.6129 \times 10^{-4} i).$$

- Set $\tau = 10^{-4}$, the refined root:

$$(1 + 9.5829 \times 10^{-8} - 1.2762 \times 10^{-7} i,$$
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- Set $\tau = 10^{-6}$, use the refined root as initial, we obtain:

$$(1 - 1.0000 \times 10^{-15} + 2.5854 \times 10^{-14} i, 2 + 8.4457 \times 10^{-14}).$$

Criterion of Involution of F_k

The ideal $F_k = (I, P^k)$ is generated by

$$F_k = \{\mathrm{T}_k(f_1), \ldots, \mathrm{T}_k(f_t),\ (x_1 - \hat{x}_1)^{\alpha_1} \cdots (x_s - \hat{x}_s)^{\alpha_s},\ \sum_{i=1}^{s} \alpha_i = k\}.$$

where $\mathrm{T}_k(f_i) = \sum_{|\alpha|<k} f_{i,\alpha}(\mathbf{x} - \hat{\mathbf{x}})^{\alpha}$. The symbol matrix of F_k and its prolongations are of full column rank.

Let $M_k^{(j)}$ denote the coefficient matrices of $\mathrm{T}_k(F^{(j)})$ with $\binom{k+s-1}{s}$ columns. Let $d_k^{(j)} = \dim \mathrm{Nullspace}(M_k^{(j)})$.

Theorem 2. $F_k = (I, P^k)$ *is involutive at prolongation order* m *if and only if*

$$d_k^{(m)} = d_k^{(m+1)}$$

and $d_k = \dim(\mathbb{C}[\mathbf{x}]/(I, P^k)) = d_k^{(m)}$.

Compute Primary Component II

- Form the matrix $M_k^{(0)}$ by computing the truncated Taylor series expansions of $f_1, \ldots, f_t$ at $\hat{x}$ to order k. The prolonged matrix $M_k^{(j)}$ is computed by shifting $M_k^{(0)}$ accordingly.

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- Compute $d_k^{(j)} = \text{dim Nullspace}(M_k^{(j)})$ for the given τ, until $d_k^{(m)} = d_k^{(m+1)} = d_k$.

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- If $d_k = d_{k-1}$, then set $\rho = k - 1$ and $\mu = d_\rho$.

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- Compute the multiplication matrices $M_{x_1}, \ldots, M_{x_s}$ from the null vectors of $M_{\rho+1}^{(m)}$.

Example 2 [Leykin etc 2006]

$$\{ f_1 = x_1^3 + x_2^2 + x_3^2 - 1,\ f_2 = x_1^2 + x_2^3 + x_3^2 - 1,\ f_3 = x_1^2 + x_2^2 + x_3^3 - 1 \}$$

has a 4-fold solution $\hat{\mathbf{x}} = (1,0,0)$. Transform it to the origin:

$$\begin{cases} g_1 &=& y_1^3 + 3y_1^2 + 3y_1 + y_2^2 + y_3^2, \\ g_2 &=& y_1^2 + 2y_1 + y_2^3 + y_3^2, \\ g_3 &=& y_1^2 + 2y_1 + y_2^2 + y_3^3. \end{cases}$$

has the 4-fold solution $\hat{\mathbf{y}} = (0,0,0)$. Let $I = (g_1, g_2, g_3)$, $P = (y_1, y_2, y_3)$.

$$[\mathrm{T}_3(g_1), \mathrm{T}_3(g_2), \mathrm{T}_3(g_3)]^T = M_3^{(0)} \cdot \left[y_1^2, \ldots, y_3, 1\right]^T,$$

$$M_3^{(0)} = \begin{bmatrix} 3 & 0 & 0 & 1 & 0 & 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}$$

$$M_3^{(1)} = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 1 & 3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

Example 2 (continued)

- $d_3^{(0)} = 7$, $d_3^{(1)} = d_3^{(2)} = 4 \Longrightarrow d_3 = \dim(\mathbb{C}[\mathbf{y}]/(I, P^3)) = 4$.

- $d_4^{(0)} = 17$, $d_4^{(1)} = 8$, $d_4^{(2)} = d_4^{(3)} = 4$,
  $\Longrightarrow d_4 = \dim(\mathbb{C}[\mathbf{y}]/(I, P^4)) = 4$.

- $d_3 = d_4 = 4$, then index $\rho = 3$, multiplicity $\mu = 4$.

Example 2 (continued)

The multiplication matrices(local ring) w.r.t. $\{y_2y_3, y_2, y_3, 1\}$):

$$M_{y_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, M_{y_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, M_{y_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

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The primary component of I associating to $(0,0,0)$ is

$$\{y_1, y_2^2, y_3^2\}.$$

Construct Differential Operators II

Theorem 2. *Let* $Q = (I, P^{\rho})$ *be an isolated primary component of* I *at* $\hat{\mathbf{x}}$ *and* $\mu \geq 1$. *Suppose* $F_{\rho} = \mathrm{T}_{\rho}(F) \cup P^{\rho}$ *is involutive after* m *prolongations, the null space of the matrix* $M_{\rho}^{(m)}$ *is generated by* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{\mu}$. *Then differential operators are:*

$$L_j = \mathbf{L} \cdot \mathbf{v}_j, \quad \textit{for } 1 \leq j \leq \mu,$$

$$\mathbf{L} = [D(\rho - 1, 0, \ldots, 0), D(\rho - 2, 1, 0, \ldots, 0), \ldots, D(0, \ldots, 0)].$$

Example 2 (continued)

The index $\rho = 3$, multiplicity $\mu = 4$, $d_3^{(0)} = 7$, $d_3^{(1)} = d_3^{(2)} = 4$, the null space of the matrix $M_3^{(1)}$ is:

$$N_3^{(1)} = [e_{10}, e_9, e_8, e_5],$$

Multiplying the diff. operators of order less than 3:

$$\{D(0,0,0),\ D(0,0,1),\ D(0,1,0),\ D(0,1,1)\}.$$

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- $\hat{\mathbf{y}} = -\hat{\mathbf{x}}_{\text{error}} = (-\hat{x}_{1,\text{error}}, \ldots, -\hat{x}_{s,\text{error}})$ is an exact solution of the system G.

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- Set $\hat{\mathbf{x}} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$ and run the first two steps for the refined solution and smaller τ.

Refining Approximate Singular Solution II

- For approximate $\hat{\mathbf{x}}$ and tolerance τ, estimate μ and ρ.

- $G_{\rho+1} = \mathrm{T}_{\rho+1}(G) \cup P^{\rho+1}$ is involutive at m, form $M_{x_1}, \ldots, M_{x_s}$ from null vectors of $M_{\rho+1}^{(m)}$ and compute $\hat{\mathbf{y}}$.

- Set $\hat{\mathbf{x}} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$ and run the first two steps for the refined solution and smaller τ.

- If $\hat{\mathbf{y}}$ converges to the origin, we get $\hat{\mathbf{x}}$ with high accuracy.

Example 2 (continued)

Given an approximate solution $\hat{\mathbf{x}} = (1.001, -0.002, -0.001\,i)$.
Use tolerance $\tau = 10^{-2}$, the computed solution is

$$\begin{aligned} \hat{\mathbf{y}} = (&-0.0009994 - 7.5315 \times 10^{-10}\,i, \\ &0.002001 + 2.8002 \times 10^{-8}\,i, \\ &-1.4949 \times 10^{-6} + 0.0010000\,i). \end{aligned}$$

Apply twice for $\tau = 10^{-5}, 10^{-8}$ respectively, we get:

$$\begin{aligned} \hat{\mathbf{x}} = (&1 + 7.0405 \times 10^{-18} - 7.8146 \times 10^{-19}\,i, \\ &1.0307 \times 10^{-16} - 1.9293 \times 10^{-17}\,i, \\ &1.5694 \times 10^{-16} + 7.9336 \times 10^{-17}\,i). \end{aligned}$$

Example 2 (continued)

Given an approximate solution $\hat{\mathbf{x}} = (1.001, -0.002, -0.001\,i)$.
Use tolerance $\tau = 10^{-2}$, the computed solution is

$$\hat{\mathbf{y}} = (-0.0009994 - 7.5315 \times 10^{-10}\,i,\quad 0.002001 + 2.8002 \times 10^{-8}\,i,\quad -1.4949 \times 10^{-6} + 0.0010000\,i).$$

Apply twice for $\tau = 10^{-5}, 10^{-8}$ respectively, we get:

$$\hat{\mathbf{x}} = (1 + 7.0405 \times 10^{-18} - 7.8146 \times 10^{-19}\,i,\quad 1.0307 \times 10^{-16} - 1.9293 \times 10^{-17}\,i,\quad 1.5694 \times 10^{-16} + 7.9336 \times 10^{-17}\,i).$$

Algorithm Performance

System	Zero	ρ	μ	MRRI	MRRII
cmbs1	$(0,0,0)$	5	11	$3 \to 8 \to 16$	$3 \to 11 \to 15$
cmbs2	$(0,0,0)$	4	8	$3 \to 9 \to 16$	$3 \to 13 \to 15$
mth191	$(0,1,0)$	3	4	$4 \to 8 \to 16$	$4 \to 9 \to 15$
LVZ	$(0,0,-1)$	7	18		$5 \to 10 \to 14$
KSS	$(1,1,1,1,1,1)$	5	16		$5 \to 11 \to 14$
Caprasse	$(2,-i\sqrt{3},2,i\sqrt{3})$	3	4	$4 \to 8 \to 14$	$4 \to 12 \to 15$
DZ1	$(0,0,0,0)$	11	131	$5 \to 14$	$5 \to 14$
DZ2	$(0,0,-1)$	8	16		$4 \to 7 \to 14$
D2	$(0,0,0)$	5	5	$5 \to 10 \to 15$	$5 \to 10 \to 15$
Ojika1	$(1,2)$	3	3	$3 \to 6 \to 14$	$3 \to 6 \to 18$
Ojika2	$(0,1,0)$	2	2	$5 \to 10 \to 15$	$5 \to 10 \to 14$

Thank you !