Regeneration: A New Algorithm in Numerical Algebraic Geometry

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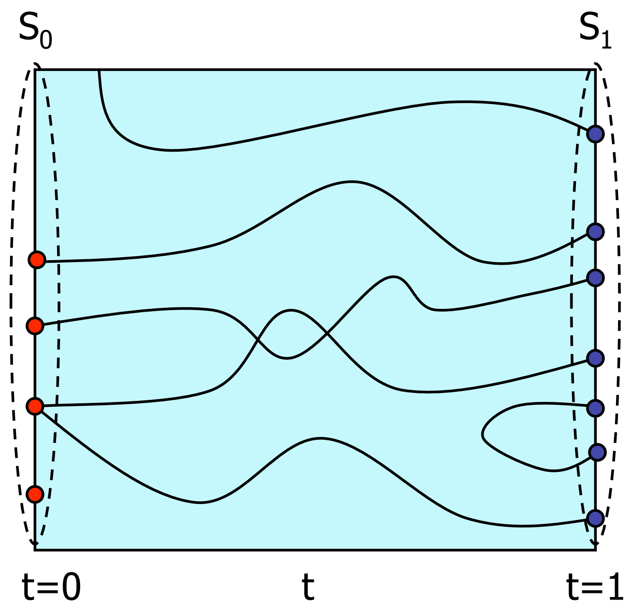
Outline

- Brief overview of Numerical Algebraic Geometry
- Building blocks for Regeneration
  - Parameter continuation
  - Polynomial-product decomposition
  - Deflation of multiplicity>1 components
- Description of Regeneration
  - A new equation-by-equation algorithm that can be used to find positive dimensional sets and/or isolated solutions
- Leading alternatives to regeneration
  - Polyhedral homotopy
    - For finding isolated roots of sparse systems
  - Diagonal homotopy
    - An existing equation-by-equation approach
- Comparison of regeneration to the alternatives

Introduction to Continuation

- Basic idea: to solve F(x)=0
  - (N equations, N unknowns)
  - Define a homotopy H(x,t)=0 such that
    - H(x,1) = G(x) = 0 has known isolated solutions, S_1
    - H(x,0) = F(x)
    - Example: $H(x,t) = (1-t)F(x) + \gamma t G(x)$
  - Track solution paths as t goes from 1 to 0
    - Paths satisfy the Davidenko o.d.e.
      - (dH/dx)(dx/dt) + dH/dt = 0
    - Endpoints of the paths are solutions of F(x)=0
    - Let S_0 be the set of path endpoints
    - A **good homotopy** guarantees that paths are nonsingular and S_0 includes all isolated points of V(F)
    - Many "good homotopies" have been invented



Basic Total-degree Homotopy

To find all isolated solutions to the polynomial system F: $C^N \rightarrow C^N$, i.e.,

$$\begin{bmatrix} f_1(x_1,\ldots,x_N) \\ \vdots \\ f_N(x_1,\ldots,x_N) \end{bmatrix} = 0, \quad \deg(f_i) = d_i$$

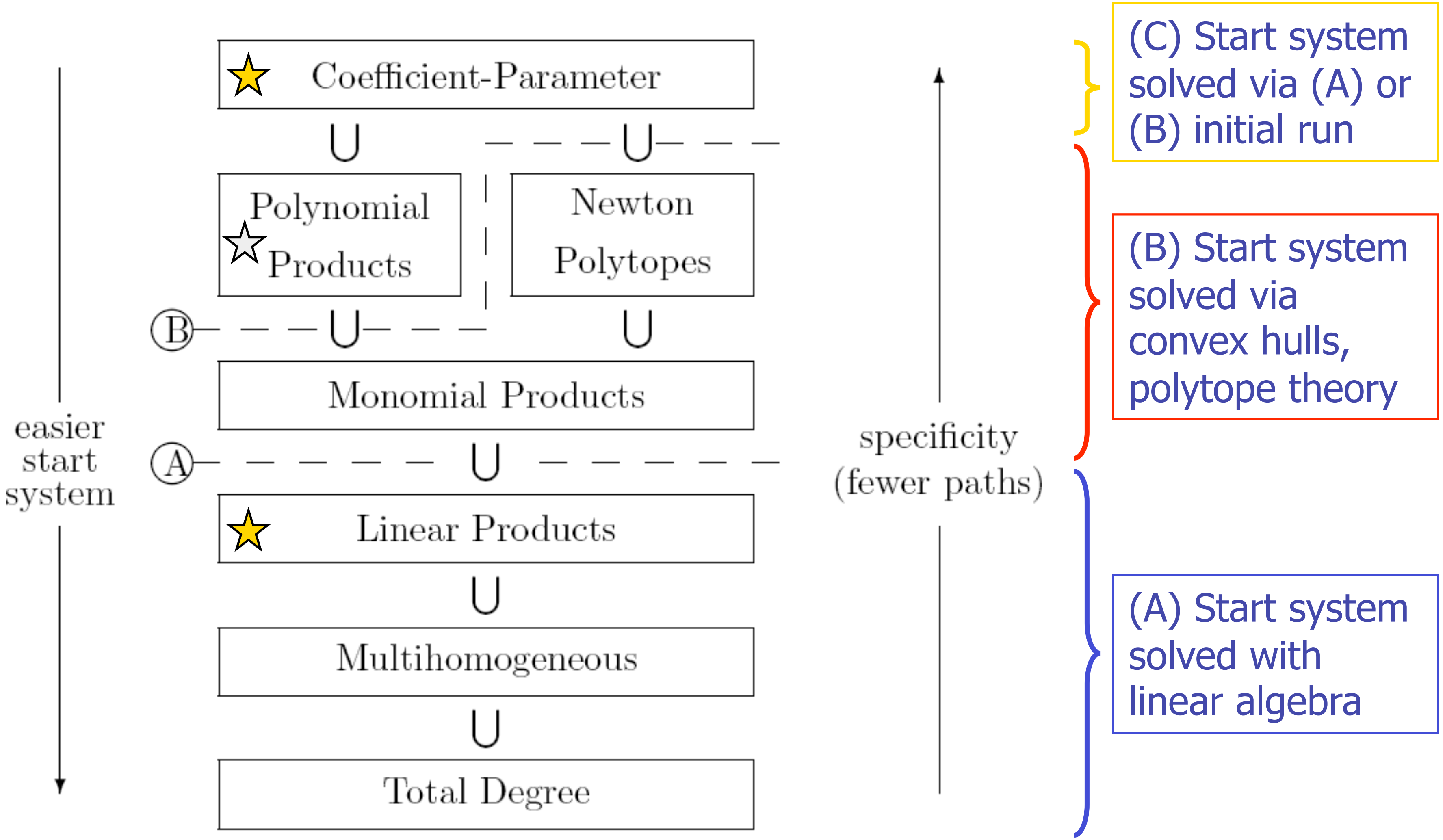
form the linear homotopy

H(x,t) = (1-t)F(x) + tG(x)=0,

where

$$g_i(x) = a_i x_i^{d_i} + b_i, \; a_i, b_i \text{ random, complex}$$

Polynomial Structures
- The basis of “good homotopies”

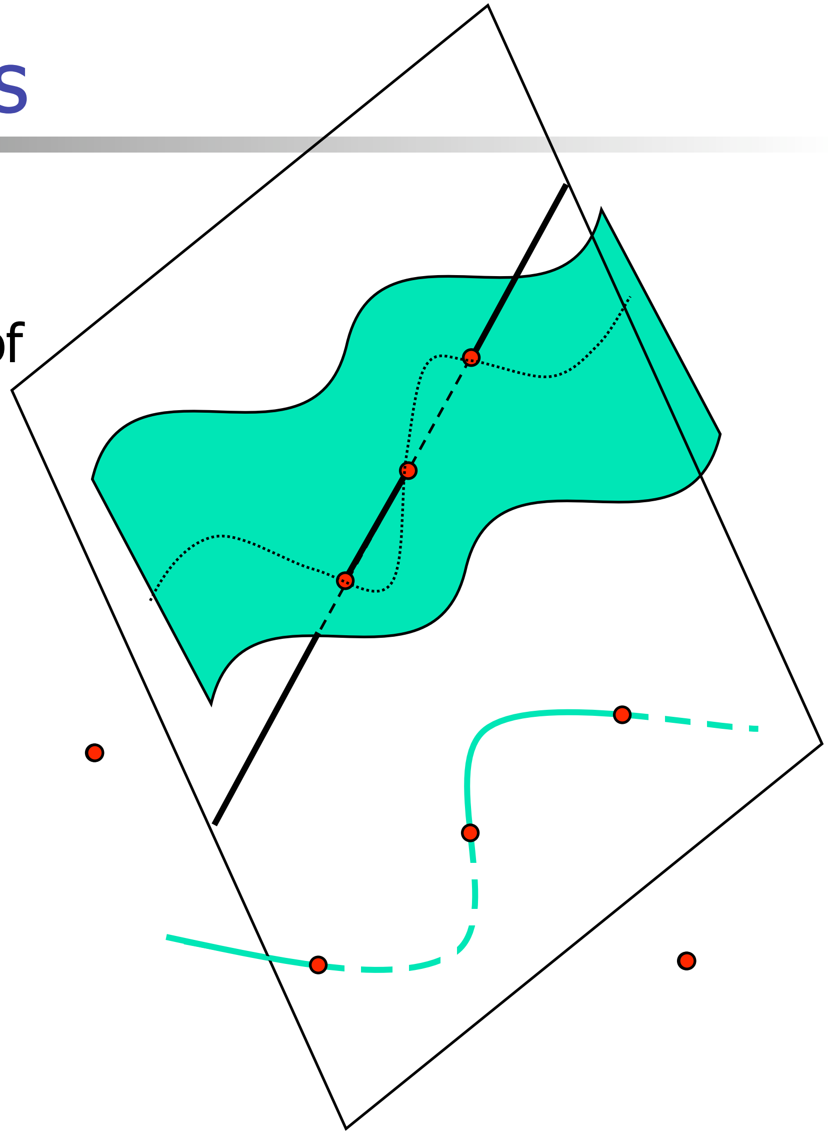


Numerical Algebraic Geometry

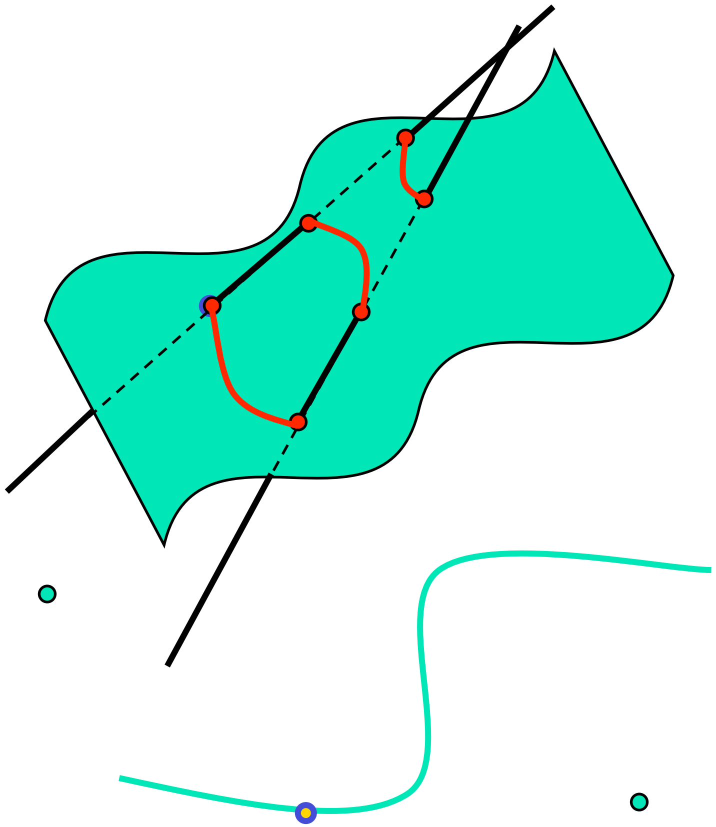
- Extension of polynomial continuation to include finding positive dimensional solution components and performing algebraic operations on them.
- First conception
  - Sommese & Wampler, FoCM 1995, Park City, UT
- Numerical irreducible decomposition and related algorithms
  - Sommese, Verschelde, & Wampler, 2000-2004
- Monograph covering to year 2005
  - Sommese & Wampler, World Scientific, 2005

Slicing & Witness Sets

- Slicing theorem
  - An degree *d* reduced algebraic set hits a general linear space of complementary dimension in *d* isolated points
- Witness generation
  - Slice at every dimension
  - Use continuation to get sets that contain all isolated solutions at each dimension
    - "Witness supersets"
- Irreducible decomposition
  - Remove "junk"
  - Monodromy on slices finds irreducible components
  - Linear traces verify completeness

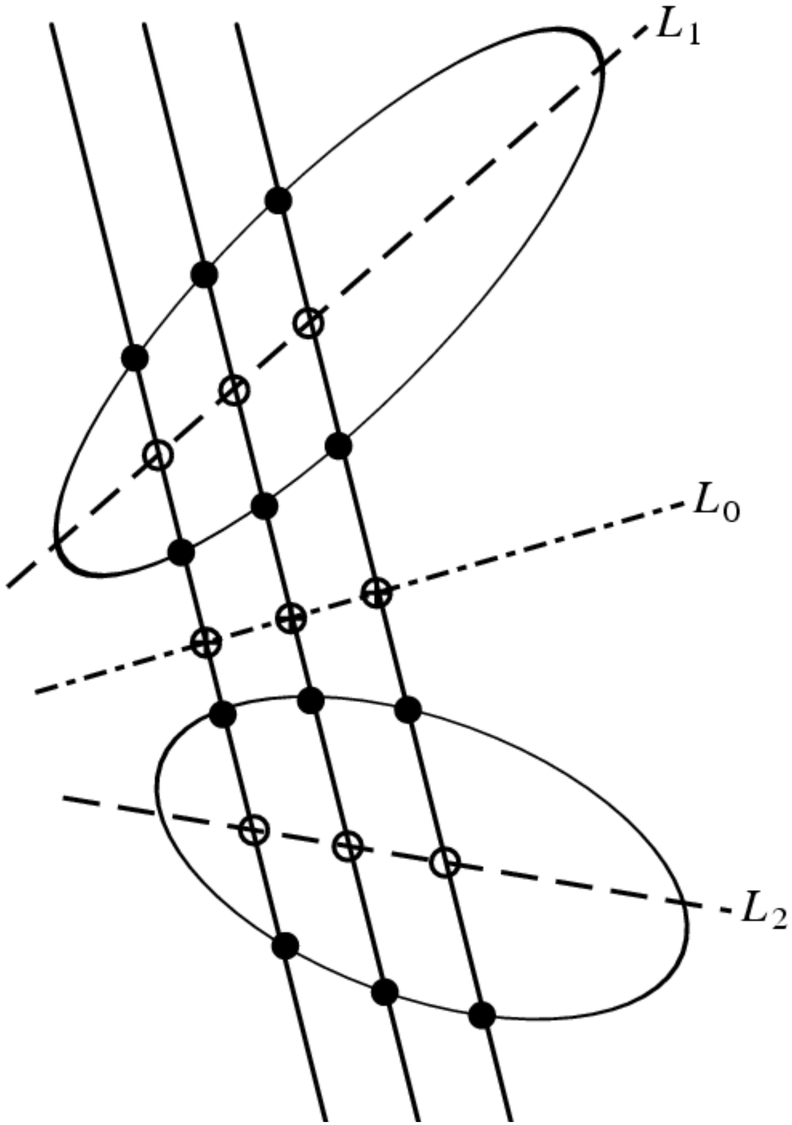


Membership Test



Linear Traces

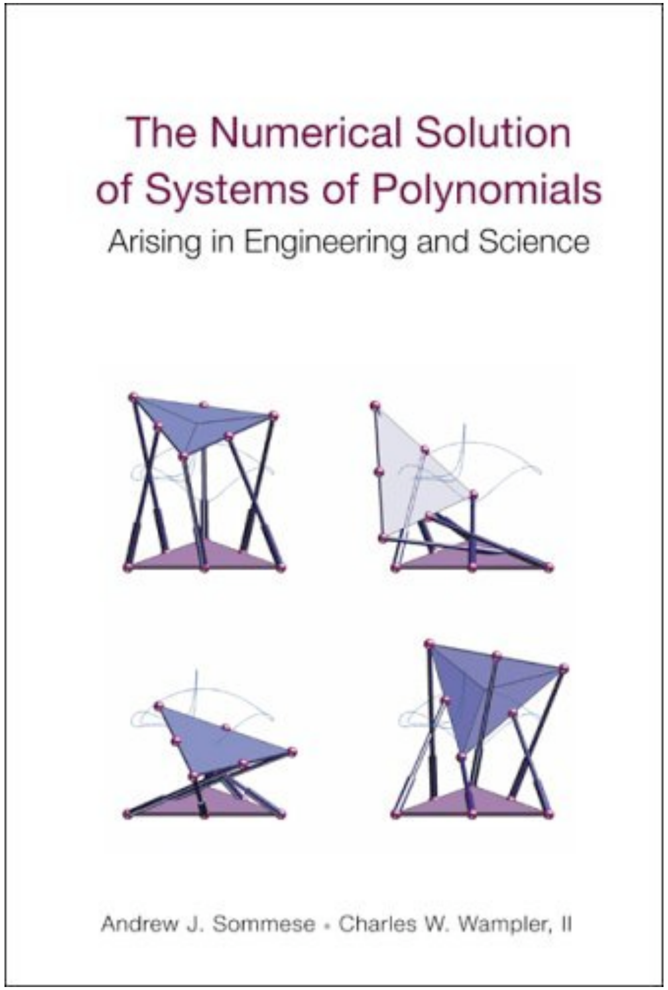
- Track witness paths as slice translates parallel to itself.
- Centroid of witness points for an algebraic set must move on a line.



Real Points on a Complex Curve

- Go to Griffis-Duffy movie…

Further Reading



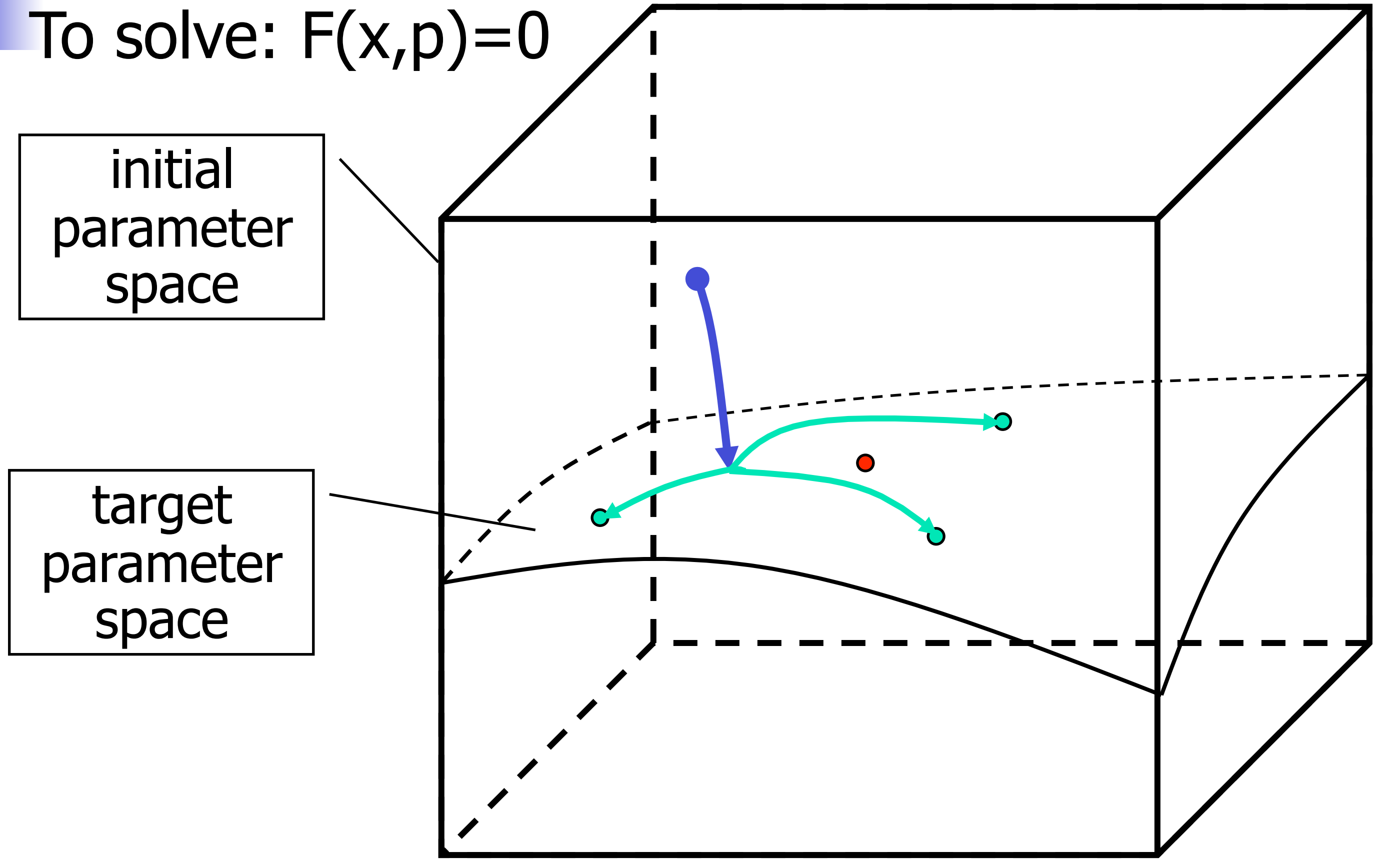
World Scientific
2005

Regeneration

- Building blocks
- Regeneration algorithm
- Comparison to pre-existing numerical continuation alternatives

Building Block 1: Parameter Continuation

To solve: F(x,p)=0

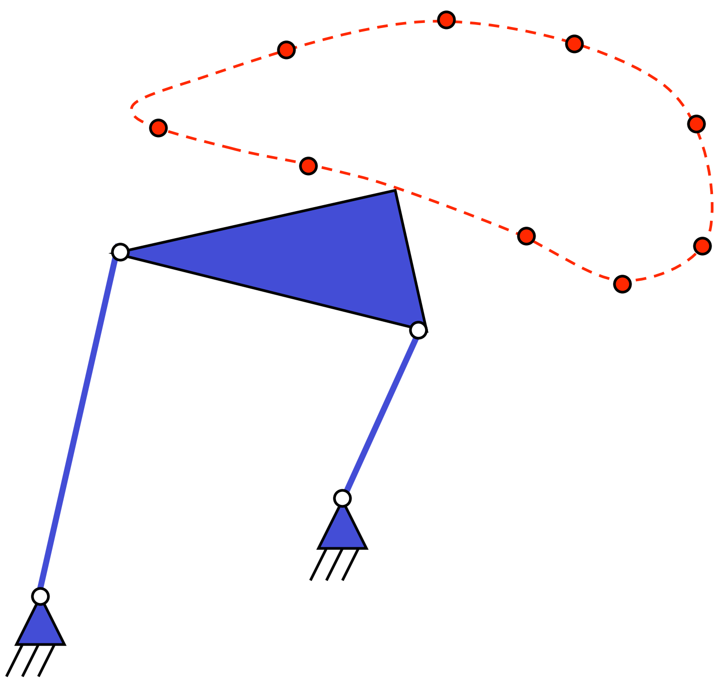


Morgan & Sommese, 1989

- Start system easy in initial parameter space
- Root count may be much lower in target parameter space
- Initial run is 1-time investment for cheaper target runs

Kinematic Milestone

- 9-Point Path Generation for Four-bars
  - Problem statement
    - Alt, 1923
  - Bootstrap partial solution
    - Roth, 1962
  - Complete solution
    - Wampler, Morgan & Sommese, 1992
    - m-homogeneous continuation
    - 1442 Robert cognate triples



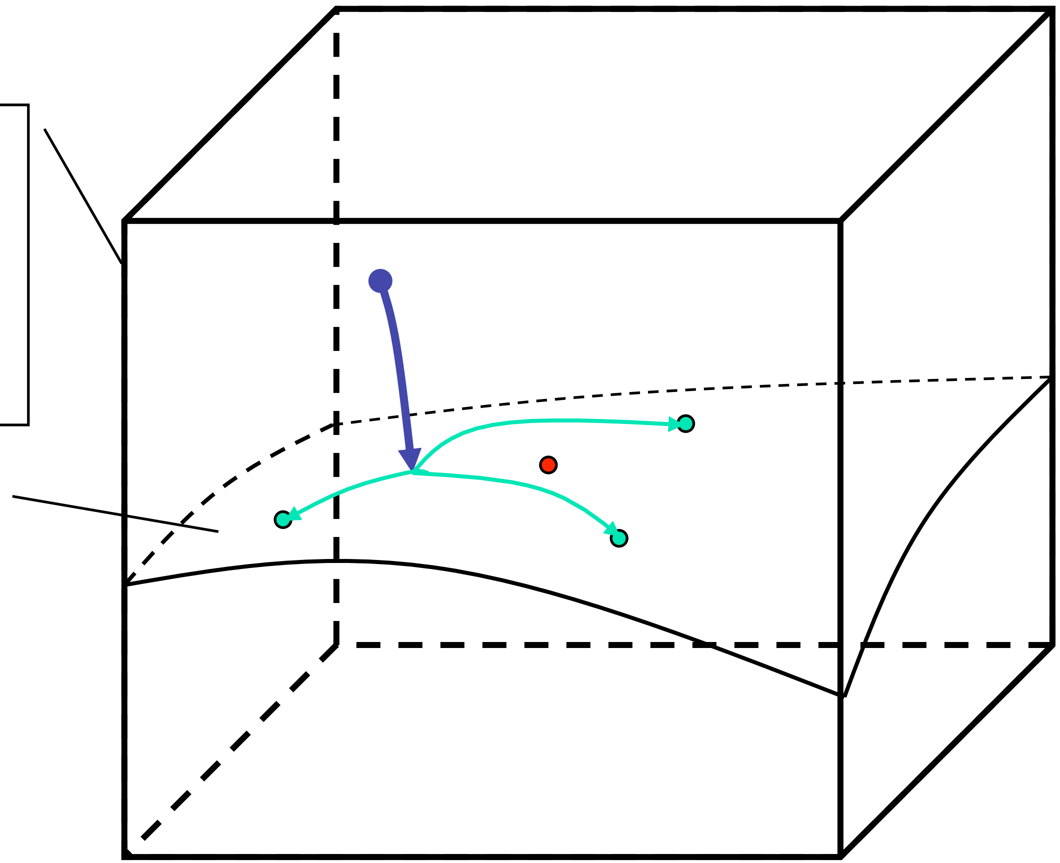
Nine-point Four-bar summary

- Symbolic reduction
  - Initial total degree $\approx 10^{10}$
  - Roth & Freudenstein, tot.deg.=5,764,801
  - Our reformulation, tot.deg.=1,048,576
  - Multihomogenization 286,720
  - 2-way symmetry 143,360
- Numerical reduction (Parameter continuation)
  - Nondegenerate solutions 4326
  - Roberts cognate 3-way symmetry 1442
- Synthesis program follows 1442 paths

Parameter Continuation: 9-point problem

2-homogeneous systems with symmetry:
143,360 solution pairs

9-point problems*:
1442 groups of 2x6 solutions



*Parameter space of 9-point problems is 18 dimensional (complex)

Building Block 2: Product Decomposition

- To find: isolated roots of system F(x)=0
- Suppose i-th equation, f(x), has the form:

$$f(x) \in \left\langle \{p_{11}, \ldots, p_{1k_1}\} \otimes \cdots \otimes \{p_{j1}, \ldots, p_{jk_j}\} \right\rangle$$

where $p_{jk} = p_{jk}(x)$ are all polynomials.

- Then, a generic g of the form

$$g(x) \in \langle p_{11}, \ldots, p_{1k_1} \rangle \otimes \cdots \otimes \langle p_{j1}, \ldots, p_{jk_j} \rangle$$

is a good start function for a linear homotopy.
- Linear product decomposition = all p_{jk} are linear.

Linear products: Verschelde & Cools 1994

Polynomial products: Morgan, Sommese & W. 1995

Product decomposition

- For a product decomposition homotopy:
  - Original articles assert:
    - Paths from all *nonsingular* start roots lead to all *nonsingular* roots of the target system.
  - New result extends this:
    - Paths from all *isolated* start roots lead to all *isolated* roots of the target system.

Building Block 3: Deflation

- Let X be an irreducible component of V(F) with multiplicity > 1.
- Deflation produces an augmented system G(x,y) such that there is a component Y in V(G) of multiplicity 1 that projects generically 1-to-1 onto X.
  - Multiplicity=1 means Newton’s method can be used to get quadratic convergence

Isolated points: Leykin, Verschelde & Zhao 2006, Lecerf 2002

Positive dimensional components: Sommese & Wampler 2005

Related work: Dayton & Zeng ’05; Bates, Sommese & Peterson ’06; LVZ, L preprints

Regeneration

- Suppose we have the isolated roots of
  - $\{F(x), g(x)\}=0$

  where F(x) is a system and
  - $g(x)=L_1(x)L_2(x)\ldots L_d(x)$

  is a linear product decomposition of f(x).
- Then, by *product decomposition*,
  - $H(x,t)=\{F(x), \gamma t\, g(x)+(1-t)f(x)\}=0$

  is a good homotopy for solving
  - $\{F(x), f(x)\}=0$
- How can we get the roots of $\{F(x), g(x)\}=0$?

Regeneration

- Suppose we have the isolated solutions of
  - $\{F(x),L(x)\}=0$

  where L(x) is a linear function.
- Then, by *parameter continuation* on the coefficients of L(x) we can get the isolated solutions of
  - $\{F(x),L'(x)\}=0$.

  for any other linear function L'(x).
  - Homotopy is $H(x,t)=\{F,\gamma tL(x)+(1-t)L'(x)\}=0$.
- Doing this d times, we find all isolated solutions of
  - $\{F(x), L_1(x)L_2(x)\ldots L_d(x)\} = \{F(x),g(x)\} = 0$.
- We call this the "regeneration" of {F,g}.

Tracking multiplicity > 1 paths

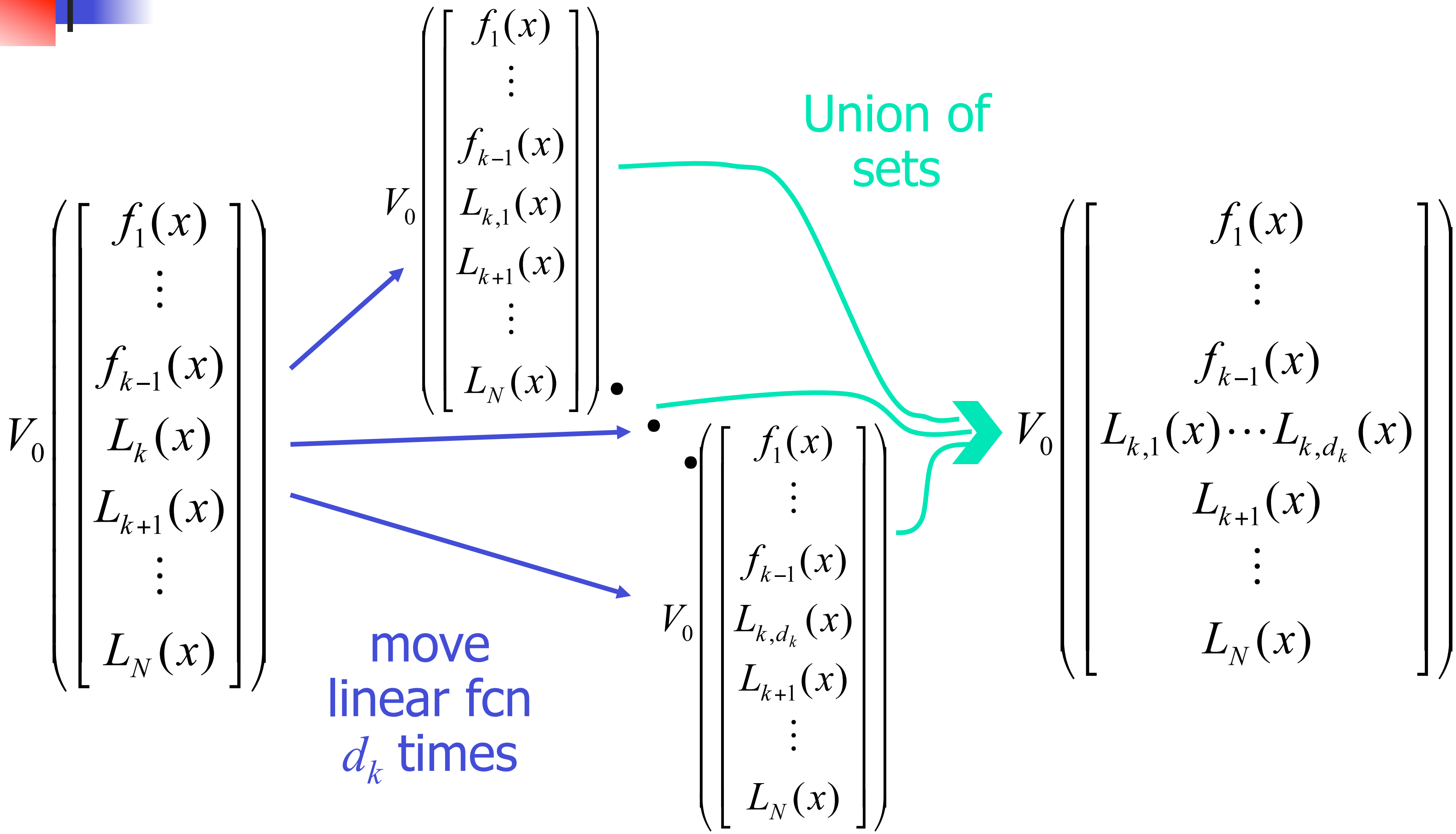
- For both regenerating {F,g} and tracking to {F,f}, we want to track all isolated solutions.
  - Some of these may be multiplicity > 1.
- In each case, there is a homotopy H(x,t)=0
- The paths we want to track are curves in V(H)
  - Each curve has a *deflation*.
  - We track the deflated curves.

Working Equation-by-Equation

- Basic step

$$V_0\left(\begin{bmatrix} f_1(x) \\ \vdots \\ f_{k-1}(x) \\ L_k(x) \\ L_{k+1}(x) \\ \vdots \\ L_N(x) \end{bmatrix}\right) \Rightarrow V_0\left(\begin{bmatrix} f_1(x) \\ \vdots \\ f_{k-1}(x) \\ f_k(x) \\ L_{k+1}(x) \\ \vdots \\ L_N(x) \end{bmatrix}\right)$$

Regeneration: Step 1

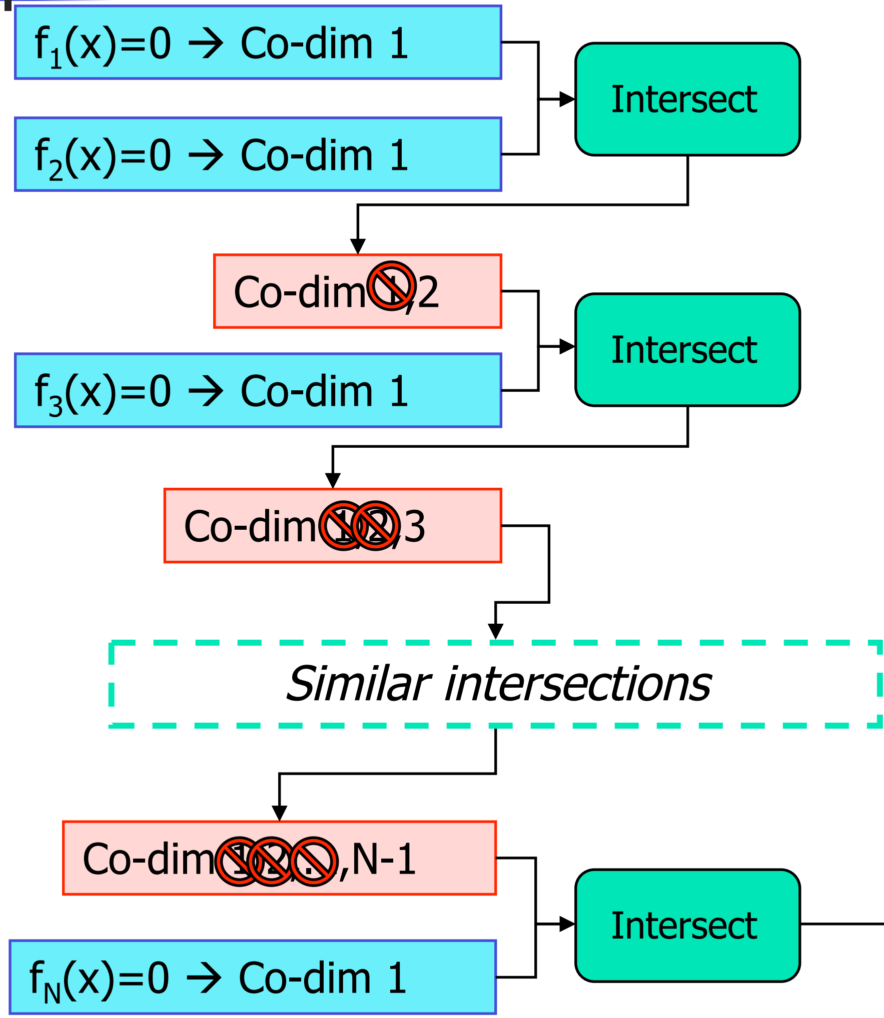


Regeneration: Step 2

$$V_0\left(\begin{bmatrix} f_1(x) \\ \vdots \\ f_{k-1}(x) \\ L_{k,1}(x)\cdots L_{k,d_k}(x) \\ L_{k+1}(x) \\ \vdots \\ L_N(x) \end{bmatrix}\right) \xrightarrow{\text{Linear homotopy}} V_0\left(\begin{bmatrix} f_1(x) \\ \vdots \\ f_{k-1}(x) \\ f_k(x) \\ L_{k+1}(x) \\ \vdots \\ L_N(x) \end{bmatrix}\right)$$

Repeat for k+1,k+2,…,N

Equation-by-Equation Solving



N equations, n variables

- Special case:
  - N=n
  - nonsingular solutions only
  - results are very promising

Theory is in place for $\mu>1$ isolated and for full witness set generation.

Final Result

Co-dim 1,2,..,min(n,N)

Alternatives 1

- Polyhedral homotopies (a.k.a., BKK)
  - Finds all isolated solutions
  - Parameter space = coefficients of all monomials
    - Root count = mixed volume (Bernstein's Theorem)
    - Always ≤ root count for best linear product
      - Especially suited to sparse polynomials
  - Homotopies
    - Verschelde, Verlinden & Cools, '94; Huber & Sturmfels, '95
    - T.Y. Li with various co-authors, 1997-present
- Advantage:
  - Reduction in # of paths
- Disadvantage:
  - Mixed volume calculation is combinatorial

Alternatives 2: Diagonal homotopy

- Given:
  - W_X = Witness set for irreducible X in V(F)
  - W_Y = Witness set for irreducible Y in V(G)
- Find:
  - Intersection of X and Y
- Method:
  - $X \times Y$ is an irreducible component of V(F(x),G(y))
  - $W_X \times W_Y$ is its witness set
  - Compute irreducible decomposition of the diagonal, $x - y = 0$ restricted to $X \times Y$
- Can be used to work equation-by-equation
  - Let F be the first k equations & G be the $(k+1)^{st}$ one
- Sommese, Verschelde, & Wampler 2004, 2008.

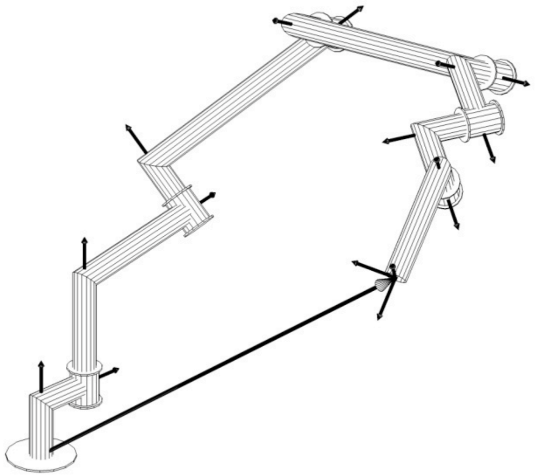
Other alternatives

- Numerical
  - Exclusion methods (e.g., interval arithmetic)
- Symbolic
  - Grobner bases
  - Border bases
  - Resultants
  - Geometric resolution
- Here, we will compare only to the alternatives using numerical homotopy. A more complete comparison is a topic for future work.

Software for polynomial continuation

- PHC (first release 1997)
  - J. Verschelde
  - First publicly available implementation of polyhedral method
  - Used in SVW series of papers
  - Isolated points
    - Multihomogeneous & polyhedral method
  - Positive dimensional sets
    - Basics, diagonal homotopy
- Hom4PS-2.0 (released 2008)
  - T.Y. Li
  - Isolated points:
    - Multihomogeneous & polyhedral method
    - Fastest polyhedral code available
- Bertini (ver1.0 released Apr.20, 2008)
  - D. Bates, J. Hauenstein, A. Sommese, C. Wampler
  - Isolated points
    - Multihomogeneous, regeneration
  - Positive dimensional sets
    - Basics, diagonal homotopy
  - Automatically adjusts precision: adaptive multiprecision

Test Run 1: 6R Robot Inverse Kinematics

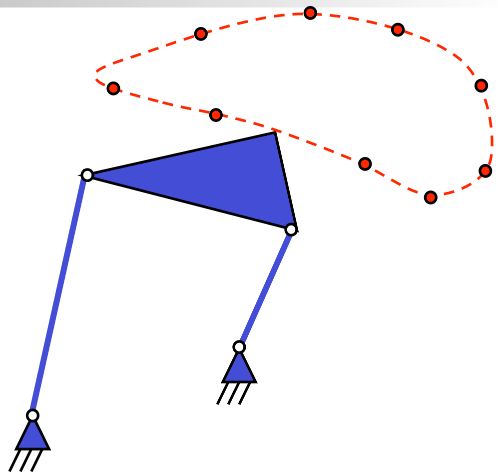


Method*	Work	Time
Total-degree traditional	1024 paths	54 s
Diagonal eqn-by-eqn	649 paths	23 s
Regeneration eqn-by-eqn	628 paths 313 linear moves	9 s

*All runs in Bertini

Test Run 2: 9-point Four-bar Problem

Method	Work	Time
Polyhedral (Hom4PS-2.0)	Mixed volume 87,639 paths	11.7 hrs
Regeneration (Bertini)	136,296 paths 66,888 linear moves	8.1 hrs



1442 Roberts cognates

Test Run 3: Lotka-Volterra Systems

- Discretized (finite differences) population model
  - Order n system has 8n sparse bilinear equations
  - Only 6 monomials in each equation

Work Summary

	total degree	2-homogeneous	polyhedral	regeneration	
n	paths	paths	paths	paths	slices moved
1	256	70	16	60	42
2	65,536	12,870	256	1020	762
3	16,777,216	2,704,156	4096	16,380	12,282
4	4,294,967,296	601,080,390	65,536	262,140	196,602
5	1,099,511,627,776	137,846,528,820	1,048,576	4,194,300	3,145,722

+ mixed volume

Total degree = 2^{8n}

Mixed volume = 2^{4n} is exact

Lotka-Volterra Systems (cont.)

- Time Summary

n	PHC polyhedral	HOM4PS-2.0 polyhedral	Bertini regeneration
1	0.56s	0.06s	0.34s
2	4m57s	7.33s	17.30s
3	18d10h18m56s	9m32s	10m3s
4	xx	3d8h28m30s	5h5m50s
5	xx	xx	9d23h32m40s

xx = did not finish

All runs on a single processor

Summary

- Continuation methods for isolated solutions
  - Highly developed in 1980's, 1990's
- Numerical algebraic geometry
  - Builds on the methods for isolated roots
  - Treats positive-dimensional sets
  - Witness sets (slices) are the key construct
- Regeneration: equation-by-equation approach
  - Uses moves of linear fcns to regenerate each new equation
    - Based on
      - parameter continuation, product decomposition, & deflation
  - Captures much of the same structure as polytope methods, *without a mixed volume computation*
  - Most efficient method yet for large, sparse systems
- Bertini software provides regeneration
  - Adaptive multiprecision is important