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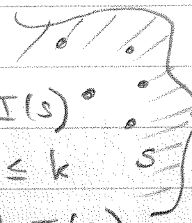
Perfect Point Configurations w/ J. Gouveia, P Parrilo

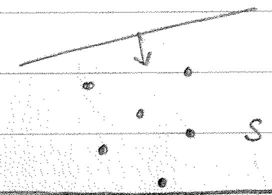
① Defns: $S \subset \mathbb{R}^n$ $\quad I(S) \subset \mathbb{R}[\underline{x}]$ vanishing ideal

$f \in \mathbb{R}[\underline{x}]$
- $f \geq 0$ mod $I(S)$ if $f(s) \geq 0 \ \forall s \in S$
- f k-sos mod $I(S)$ if $f \equiv \sum h_j^2$ mod $I(S)$, $h_j \in \mathbb{R}[\underline{x}]$, deg $h_j \leq k$



- $S/I(S)$ k-sos if $\forall f \geq 0$ mod $I(S)$, f k-sos mod $I(S)$
- $S/I(S)$ k-perfect if $\forall$ LINEAR — " — " —

Main Goal Characterize 1-perfect $S \subseteq \mathbb{R}^n$ (Lovasz)
(k-perfect)



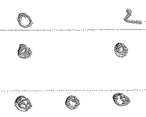
② S finite / $I(S)$ 0-dim:

(i) $f \geq 0$ mod $I(S)$ $\Longleftrightarrow$ f sos mod $I(S)$ (Parrilo)

(ii) perfectness-type of $I(S)$ $\leq$ sos-type of $I(S)$ $\leq$ regularity $I(S)$ $\leq$ degree of interpolators of S $\leq |S|-1$

eg $G = 2k+1$ cycle $\quad S = \{$inc vectors of stable sets in $G\}$
2-perfect

(iii) $f \geq 0$ mod $I(S)$ depends only on conv(S)
but perfectness-type depends on S eg



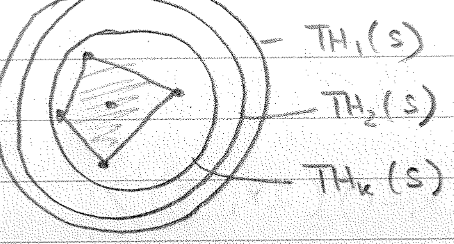
perfect
2-perfect

(iv) perfectness type can be arb. large even when $n=1$
eg $S_k = \{0, 1, 2, \ldots, k\}$ $\quad \frac{k}{2} \leq$ perfectness type $\leq k$

TO .... FROM .... DATE .... NO ....

③ <u>Theta bodies</u> $S \subseteq \mathbb{R}^n$

$TH_k(S) := \{ x \in \mathbb{R}^n : g(x) \geq 0 \ \forall$ linear g k-sos mod $I(S) \}$



<u>Thm</u> $TH_k(S)$ is the (projⁿ of) the feasible region of a semidefinite program.

$\Rightarrow$ can optimize over them in poly time

Concrete repⁿs in terms of moment matrices (Lasserre, Laurent etc)

<u>Lovasz</u>: Characterize $S \subset \mathbb{R}^n$ st $\operatorname{conv}(S) = TH_1(S)$

<u>Motivating Ex</u>: $G = (V, E)$ $\quad S_G = \{$ inc. vectors of stable sets in $G \}$

<u>Thm</u> G perfect $\Leftrightarrow \operatorname{conv}(S_G) = TH_1(S_G) \Leftrightarrow S_G$ perfect

$\rightsquigarrow$ poly time alg to compute largest stable set in a perfect graph

④ <u>Structure Theorems</u>:

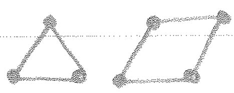
<u>Thm</u> Ⓐ For $S \subseteq \mathbb{R}^n$, TFAE

① S k-perfect
② $\operatorname{conv}(S) = TH_k(S)$
③ $\exists$ linear ineq description of $\operatorname{conv}(S)$ in which every $g(x) \geq 0$ is k-sos mod $I(S)$

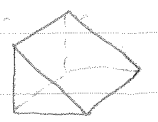
S finite TFAE

① S 1-perfect
② $(TH_1(S))$
③ 1-sos
④ —"— $\quad g(x) \equiv \alpha\, g(x)^2$ mod $I(S)$
⑤ $\forall$ facet ineq $g(x) \geq 0$ of $\operatorname{conv}(S)$, S lies on $g(x) = 0$ and $g(x) = \alpha$ for some α.

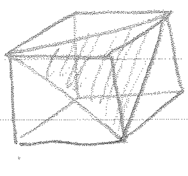
eg $\mathbb{R}^2$



$\mathbb{R}^3$



perfect



not perfect

eg families in $\mathbb{R}^n$: simplices, hypercubes, hypersimplices, cross-polytopes, slices, pyramids/prisms over perfect confs

<u>Thm</u> $S \subseteq \mathbb{R}^n$ $\quad TH_1(S) = \bigcap_{\substack{F \text{ convex quadric} \\ F \in I(S)}} \operatorname{conv}(\mathcal{V}(F))$

Corollaries S perfect $\Rightarrow$

1. ~~S perfect ⇒~~ S affinely eq to a subset of $\{0,1\}^n$
2. # vertices, # facets of $\mathrm{conv}(S) \leq 2^n$ (both sharp)
3. $I(S) = \langle (c_i^T x - \alpha_i)(c_i^T x - \beta_i) \quad i = 1, \ldots, r \rangle$

$$\alpha_i \neq \beta_i \qquad c_i \in \mathbb{R}^n$$

Converse is false

eg $\cancel{I(S)}$ $I = \langle x_i^2 - x_i \quad i = 1, \ldots, 5, \ (\Sigma x_i - 1)(\Sigma x_i - 3) \rangle$

$\mathcal{V}(I)$ not perfect.