C*-actions and Kinematics

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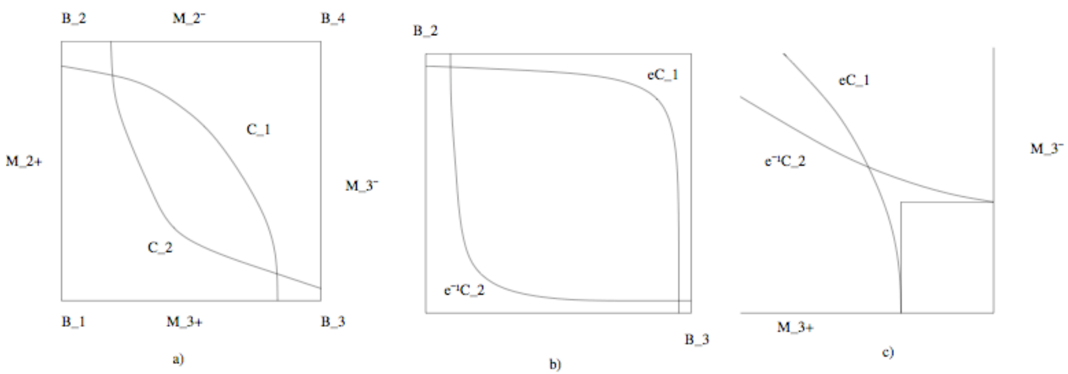

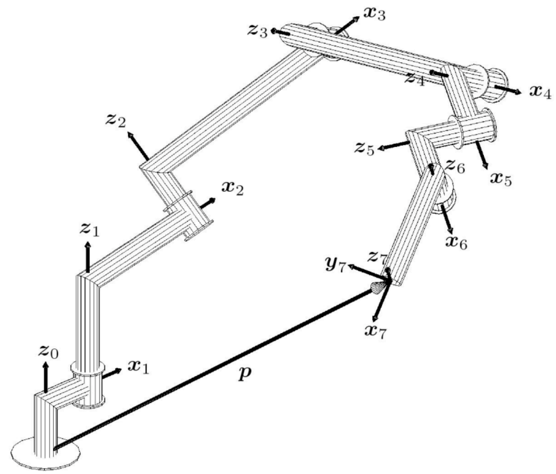

Plan:

- Some facts on complex manifolds with a C^*-action.
- Intersecting two subvarieties of complementary dimension.
- A numerical approximation, the Intersection Algorithm.
- Solving the inverse kinematics problem for a general Six-Revolute Serial-Link Manipulator.

Complex manifolds with a C*-action

Consider a non singular complex projective variety of dimension n.

Suppose that it is equipped with a C*-action having a finite fixed point set.

Data:

$X \subset \mathbb{P}^N$ $\qquad$ $\mathbb{C}^* \times X \to X, (t,x) = tx$

fixed points: $B_1, ..., B_r$

Examples:

$$\mathbb{P}^N$$

$$G(N, r)$$

$$Q \subset \mathbb{P}^7, x_0x_4 + x_1x_5 + x_2x_7 + x_3x_8 = 0$$

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Complex manifolds with a C*-action

- The Bialynicki-Birula decomposition (1973):

The space X can be decomposed in locally closed invariant subsets, in two ways: the “plus” and “minus” decomposition.

There are two distinguished blocks, called the *source* and the *sink*

$$M_i^+ = \{x \in X \mid \lim_{t \to 0} tx = B_i\} \qquad \overline{M_1^+} = X,\ B_1 \text{ source },\ M_k^+ = B_k,\ B_k \text{ sink}$$

$$M_i^- = \{x \in X \mid \lim_{t \to \infty} tx = B_i\} \qquad \overline{M_k^-} = X,\ M_1^- = B_1$$

$$X = \cup_{i=1}^r M_i^+ = \cup_{i=1}^r M_i^-$$

Complex manifolds with a C*-action

- Example: The smooth quadric hypersurface in 3-space, with an action having 4 fixed points

Let $X = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, $(x_0, x_1, y_0, y_1) \mapsto (x_0y_0, x_0y_1, x_1y_0, x_1y_1)$

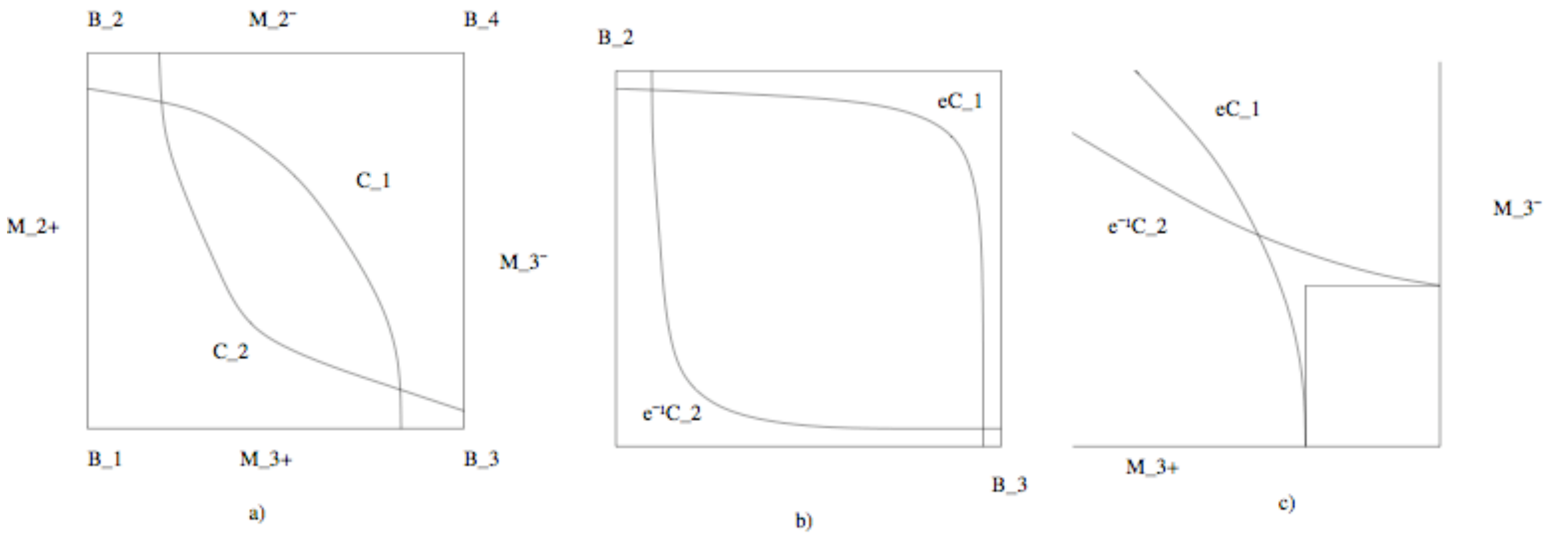
$$(t, (x_0, x_1, y_0, y_1)) \to (x_0, tx_1, y_0, ty_1)$$

$$B_1 = (0,1,0,1), B_2 = (0,1,1,0), B_3 = (1,0,0,1), B_4 = (1,0,1,0)$$

$$\begin{aligned}
M_1^+ = B_1, M_1^- &= \{(x_0, x_1, y_0, y_1) : x_1 \neq 0, y_1 \neq 0\} \\
M_2^+ &= \{(x_0, x_1, y_0, y_1) : x_0 = 0, y_0 \neq 0\} \\
M_3^+ &= \{(x_0, x_1, y_0, y_1) : x_0 \neq 0, y_0 = 0\} \\
M_4^+ = \{(x_0, x_1, y_0, y_1) : x_0 \neq 0, y_0 \neq 0\}, M_4^- &= B_4 \\
M_2^- &= \{(x_0, x_1, y_0, y_1) : x_1 \neq 0, y_1 = 0\} \\
M_3^- &= \{(x_0, x_1, y_0, y_1) : x_1 = 0, y_1 \neq 0\}
\end{aligned}$$

Complex manifolds with a C*-action

- Main idea: use the action to find numerically the intersection of two curves.
- How: pushing one towards the sink and the other towards the source. This will provide starting points and a homotopy to track the points back.



Algorithm in a toy-example

- Example with two curves, Y,Z in the quadric.

$$Y = V(f), Z = V(g)$$

$$f := 3x_0y_0 + x_0y_1 + x_1y_0 + x_1y_1$$

$$g := y_0^2x_0 + y_1^2x_0 + y_0^2x_1 + 3y_1^2x_1.$$

Consider:

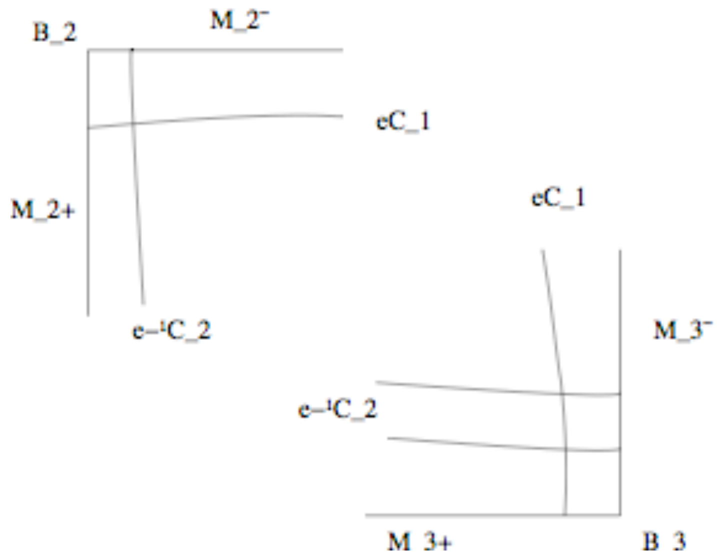
$tY, t^{-1}Z$ as $t \to 0$

the Homotopy is $H : \mathbb{C}^* \times X \to \mathbb{C}^2$

$$H(t, x_0, y_0, y_0, y_1) = \begin{pmatrix} 3t^2x_0y_0 + tx_0y_1 + tx_1y_0 + x_1y_1 \\ y_0^2x_0 + t^2y_1^2x_0 + ty_0^2x_1 + 3t^3y_1^2x_1 \end{pmatrix}$$

Example

- Locally near the other two fixed points:



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Example

- In an (analytic) neighborhood of the other two points we can linearize the action.
- Locally the cells are translates of the coordinate axis. Intersection with the cells give start points.

Near B_3	Near B_2
$t(z,w) = (tz, t^{-1}w)$	$(-\epsilon, -\epsilon)$
$\epsilon Y \cap M_3^+ = (-\epsilon, 0)$	↓
$\epsilon^{-1} Z \cap M_3^- = (0, -\epsilon\sqrt{-1})(0, \epsilon\sqrt{-1})$	RUN BERTINI
start points:	
$(0, -\epsilon\sqrt{-1}), (0, \epsilon\sqrt{-1})$ →	

The problem

- Let X be a non singular complex projective variety, with a C*-action whose fixed-set is finite.
- Let Y,Z be pure-dimensional subvarieties of complementary dimension.
- Assume (for simplicity) that they are in general position with respect to the action and they intersect transversally.
- Give an algorithm to approximate numerically the points of intersection.

The algorithm

Set up:

$$\text{Y defined by } f = f_1, \ldots, f_{N-s}, \dim(Y) = s$$
$$\text{Z defined by } g = g_1, \ldots, g_{N-r}, \dim(Z) = r$$
$$\dim(X) = r + s$$

$$\text{Let } B_1, \ldots, B_k \text{ s.t.}$$
$$\dim(M_i^+) = \dim(Z), \dim(M_i^-) = \dim(Y), i = 1, \ldots, k$$

The algorithm

1) Set the homotopy:

$$H(t,x) = (f(t^{-1}x), g(tx))$$

from ϵ to 1

2)

around $B_i, i = 1 \dots k$

$$t(z_1, \dots, z_s, w_1, \dots, w_r) = (t^{\alpha_1} z_1, \dots, t^{\alpha_s} z_s, t^{\beta_1} w_1, \dots, t^{\beta_r} w_r)$$

$\alpha_i, \beta_i \in \mathbb{Q}_+$, where

$$U \cap M_i^+ = \{w_j = 0, j = 1 \dots r\}$$

$$U \cap M_i^- = \{z_j = 0, j = 1 \dots s\}$$

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The algorithm

3) Via Puiseux expansions and Newton method to increase precision compute

$$(z_1, \ldots, z_s, w_1, \ldots, w_r) \text{ s.t.}$$

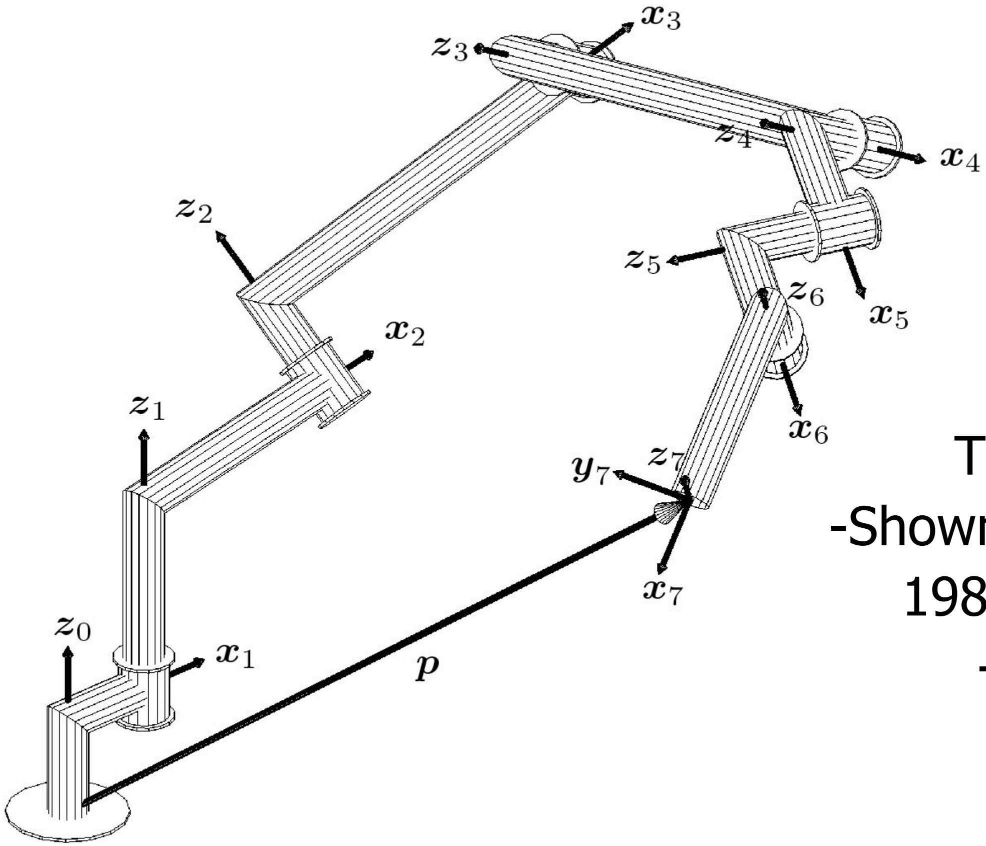
$$(z_1, \ldots, z_s) \in M_i^+ \cap \epsilon Y, (w_1, \ldots, w_r) \in M_i^- \cap \epsilon^{-1} Z$$

4) Use the result as start points run BERTINI

The Six-Revolute Serial-Link Manipulator

- The most common Robot-arm

The Six-Revolute Serial-Link Manipulator



Given p find the positions
and rotation of each joint making
The arm arrive at p.

This problem has 16 solutions
-Shown by continuation by Tsai&Morgan
1985, total degree homotopy: 256
- 1988, Li&Liang, degree 16

Geometric setting

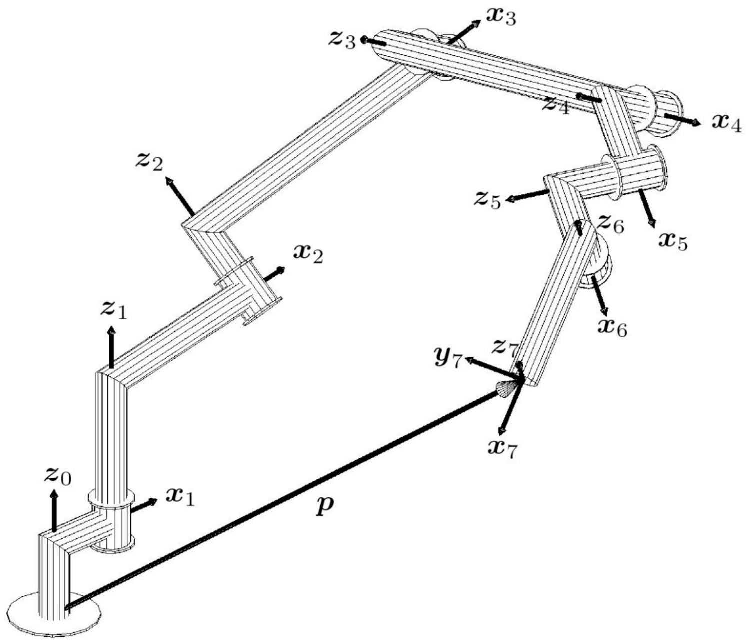
- The solution space:

The space of "special Euclidean transforms in 3-space" is identified with a non singular quadric in 7-projective space, called the Study Quadric.

$$SO(3) \times \mathbb{R}^3 \leftrightarrow Q \subset \mathbb{P}^7$$

$$(R, t) \leftrightarrow (p_0, \ldots, p_4, q_0, \ldots, q_4), p_0q_0 + \ldots + p_4q_4 = 0$$

The 6R IKP



- Split the problem in two 3R IKP
- Each 3R IKP has a 3-dimensional subspace of solutions, X,Y.
- The final solutions are given by intersecting X and Y.

The (general) 6R IKP

- Reduce the problem to two general 3R IKP.

$$f := M_0 \circ R_z(\theta_1) \circ M_1 \circ R_z(\theta_2) \circ M_2 \circ R_z(\theta_3) \circ M_3$$

$$g := M_6 \circ R_z(\theta_6) \circ M_5 \circ R_z(\theta_5) \circ M_4 \circ R_z(\theta_4) \circ M_3$$

M_i : known descriptions of the links

θ_i : joint variables

The intersection algorithm, in MatLab+ Bertini, takes 44 sec. to find the 16 solutions, tracking exactly 16 paths.

Summary

- A C*-action on a non singular algebraic variety, having a finite fixed-set, gives two decompositions.
- The decompositions have two distinguished cells: The source and the sink.
- Given two subvarieties of complementary dimension, by pushing one towards the source and the other towards the sink we force the intersection points to move towards certain fixed points.
- By homotopy continuation we can trace the intersection back and solve the intersection problem.
- This algorithm has natural applications in kinematics, for example it gives a new algorithm to solve a general 6R IKP.

THANKS!