Constructing Numerical Campedelli Surfaces

Wolfram Decker, Michael Kunte, Frank-Olaf Schreyer

Hongkong, June 2008

"... given a system of equations, it can be extremely difficult to analyze its qualitative properties, such as the geometry of the corresponding variety. The theory of syzygies offers a microscope for looking at systems of equations, and helps to make their subtle properties visible."

– David Eisenbud, The Geometry of Syzygies

Birational Invariants of Surfaces

X = smooth connected projective variety of dimension 2 over $\mathbb{C}$
$\omega_X = \bigwedge^2 T_X^*$ its canonical bundle

Canonical ring:

$$R(X) = \mathbb{C} \oplus \bigoplus_{m \geq 1} H^0(X, \omega_X^{\otimes m}) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

Plurigenera: $P_m = \dim_{\mathbb{C}} H^0(X, \omega_X^{\otimes m}), \quad m \geq 1$

Geometric and **arithmetic genus**: $p_g = P_1 \geq p_a = \chi(\mathcal{O}_X) - 1$

Irregularity: $q = p_g - p_a \geq 0$

Kodaira dimension:

$$\text{kod}\,(X) = \begin{cases} -\infty & R(X) \simeq \mathbb{C} \\ \text{trdeg}_{\mathbb{C}}\, R(X) - 1 & \text{otherwise} \end{cases} \in \{-\infty, 0, 1, 2\}$$

characterizes behavior of plurigenera for $m \longrightarrow \infty$

Enriques-Kodaira Classification

Minimal model:

$$\begin{array}{cl} X & \\ \downarrow & \quad \text{contract } (-1)-\text{curves} \\ X_{\min} & \end{array}$$

Theorem. Every surface has a minimal model in precisely one of the classes (1)–(8) below. This is uniquely determined (up to isomorphisms) except for the surfaces with minimal models in the classes (1) and (2).

	class of X	$\text{kod}\,(X)$
(1)	minimal rational surfaces	$-\infty$
(2)	ruled surfaces with $q \geq 1$	$-\infty$
(3)	Enriques surfaces	0
(4)	bielliptic surfaces	0
(5)	K3 surfaces	0
(6)	Tori	0
(7)	minimal elliptic surfaces with $\text{kod}\,(X) = 1$	1
(8)	minimal surfaces of general type	2

□

Castelnuovo criterion: X rational $\iff P_2 = q = 0\,.$ $\square$

Are there nonrational surfaces with $p_g = q = 0$?

Example: Enriques.

Enriques surfaces satisfy $K_X^2 = 0$.

Examples with $p_g = q = 0$ and $K_X^2 \geq 1$ must be of general type.

Surfaces of General Type

X = minimal surface of general type. Then:

- There is a quasiprojective coarse moduli scheme for X's with fixed Chern numbers c_1^2 and c_2.
- $K_X^2 = c_1^2 > 0$ and $c_2 > 0$.
- Miyaoka-Yau inequality: $c_1^2 \leq 3c_2$

Noether's formula: $1 - q + p_g = \frac{1}{12}(c_1^2 + c_2)$
X = minimal surface of general type with $p_g = q = 0$. Then:

- $1 \leq c_1^2 \leq 9$ and $c_2 = 12 - c_1^2$.

Godeaux Construction

$\mathbb{Z}_5$ acts freely on the Fermat quintic

$$F \;=\; \left\{ x_0^5 + x_1^5 + x_2^5 + x_3^5 \;=\; 0 \right\} \;\subset \mathbb{P}^3$$

via

$$\left(x_1, x_2, x_3, x_4 \right) \;\longrightarrow\; \left(x_1, \epsilon x_2,\, \epsilon^2 x_3,\, \epsilon^3 x_n \right),$$

where

$$\epsilon \;=\; e^{2\pi\, i/5}.$$

$X = F/\mathbb{Z}_5$ is a minimal surface of general type with $p_g = q = 0$ and $K_X^2 = 1$.

Definition. Any minimal surface of general type with invariants as above is called a **Numerical Godeaux surface**. □

Fact (Miyaoka, Reid). $H_1(X, \mathbb{Z})$ is either 0, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_5$.

Similarly as above, $\mathbb{Z}_8$ acts freely on the complete intersection T of 4 quadris in $\mathbb{P}^6$ chosen as follows:

- None of the quadrics passes through the coordinate vertices.
- The ideal generated by the 4 quadrics is $\mathbb{Z}_8$-invariant.

$X = T/\mathbb{Z}_8$ is a minimal surface of general type with $p_g = q = 0$ and $K_X^2 = 2$.

Definition. Any minimal surface of general type with invariants as above is called a **Numerical Campedelli surface**. $\square$

Fact (Beauville, Reid). $|H_1(X, \mathbb{Z})| \leq 9$.

Syzygy Construction

To begin, we discuss three examples of Betti diagrams:

Example 1.

```
                0       1       2      3
-----------------------------------------
     0:         1       -       -      -
     1:         -       3       2      -
     2:         -       3       6      3
-----------------------------------------
total:          1       6       8      3   .
```

Example 2.

	0	1	2	3	4	5
0	1	–	–	–	–	–
1	–	–	–	–	–	–
2	–	–	–	–	–	–
3	–	–	–	–	–	–
4	–	10	13	4	–	
5	–	1	3	3	1	

Example 3.

	0	1	2	3	4	5
0	1	–	–	–	–	–
1	–	10	15	$*$	–	–
2	–	–	$5 + *$	26	20	5

Fact. X is a minimal surface of general type with $p_g = q = 0$, then:

- $R(X)$ is a finitely generated Noetherian ring which is Gorenstein if $q = 0$.
- The Hilbert function of $R(X)$ is

$$m \to \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X)) = \frac{m(m-1)}{2} K_X^2 + 1.$$

Idea. Let X be a numerical Campedelli surface. Choose a 3-dimensional subspace $V \subset H^0(X, \mathcal{O}_X(3K_X))$ and bases x_0, x_1, x_2 of $H^0(X, \mathcal{O}_X(2K_X))$ and y_0, y_1, y_2 of V. Consider $R(X)$ as a module over $S = \mathbb{C}[x_0, x_1, x_2, y_0, y_1, y_2]$, where the x_i and y_j have degrees 2 and 3 respectively.

Fact. Assume that $H^0(X, \mathcal{O}_X(2K_X))$ is base-point free Then the graded Betti diagram of $R(X)$ is of type:

	0	1	2	3
0	1	–	–	–
1	–	–	–	–
2	–	–	–	–
3	4	–	–	–
4	7	–	–	–
5	–	18	–	–
6		20	–	–
7			20	–
8			18	–
9			–	7
10			–	4
11			–	–
12			–	–
13			–	1

Fact. Assume moreover that V is base-point free. Let L be the linear system formed by the x_i and y_j. Then L embeds the canonical model of X into the weighted projective space $\mathbb{P}(2^3, 3^3)$ iff the module presented by the morphism

$$S(-3)^4 \oplus S(-4)^7 \xleftarrow{\phi} S(-6)^{18} \oplus S(-7)^{20}$$

is of finite length. Writing $F = \ker \phi$. we then have a Buchsbaum-Eisenbud resolution for the homogeneous coordinate ring of the embedded canonical model of X of type

$$0 \leftarrow S/I(X) \leftarrow S \leftarrow F \leftarrow F^{\vee}(-14) \leftarrow S(-14) \leftarrow 0.$$