# Effective Methods in Algebraic Geometry

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Methods
Methoden
Методы
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in Algebraischer
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Algebrica
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Algebraica

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Geometrie
Геометрии
in Geometria
en Géométrie
en Geometria


## Barcelona, June 15–19 2009

`http://www.imub.ub.es/mega09`

**(CoCoA school: June 9-13)**

# Computing the Newton polygon of offsets to plane algebraic curves


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# Joint work with

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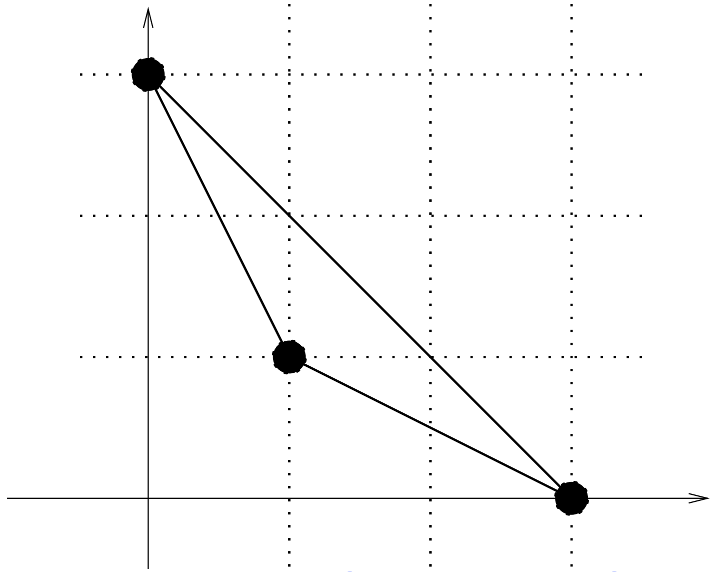
- **Fernando San Segundo** *(Alcalá)*
- **Rafael Sendra** *(Alcalá)*

# The Newton Polygon

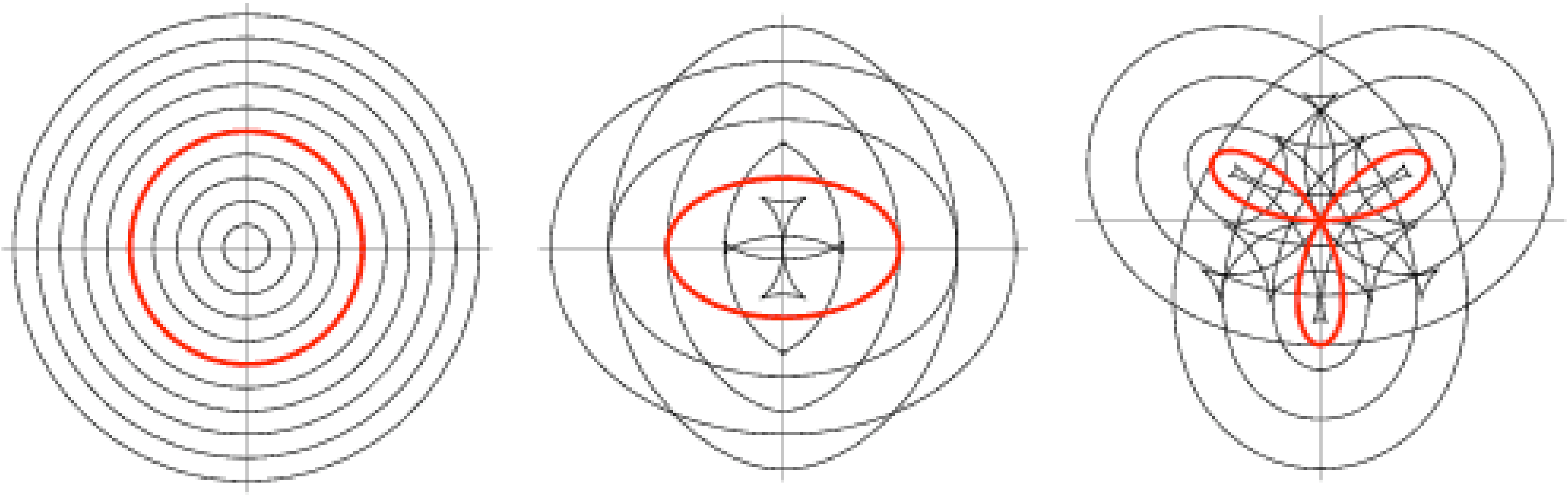
**(of a plane curve)**



$$N(\mathcal{C}) := N(X^3 + Y^3 - 3XY)$$

# Offsets or parallel curves

(to plane curves)



## Parametric equation of the offset

$$O_d(\mathcal{C})(t) = \rho(t) \pm d\frac{N(t)}{\|N(t)\|}$$

- $\rho$ is a parametrization of $\mathcal{C}$
- $d \in \mathbb{R}$ is the distance
- $N(t)$ is a normal field to $\rho(t)$

## Known facts about offsets

- If $\mathcal{C}$ is a plane algebraic curve, then $O_d(\mathcal{C})$ is also an algebraic curve with at most two components

  `(Sendra-Sendra 2000)`

- $\mathcal{C}$ rational does not imply $O_d(\mathcal{C})$ rational

## Parametric equations of the offset

$$\begin{cases} X^{\pm}(t) = \dfrac{A_1(t) \pm \sqrt{h(t)}\,B_1(t)}{D_1(t)} \\[2ex] Y^{\pm}(t) = \dfrac{A_2(t) \pm \sqrt{h(t)}\,B_2(t)}{D_2(t)} \end{cases}$$

# Computational Problem

Given $\mathcal{C}$, compute $O_d(\mathcal{C})$

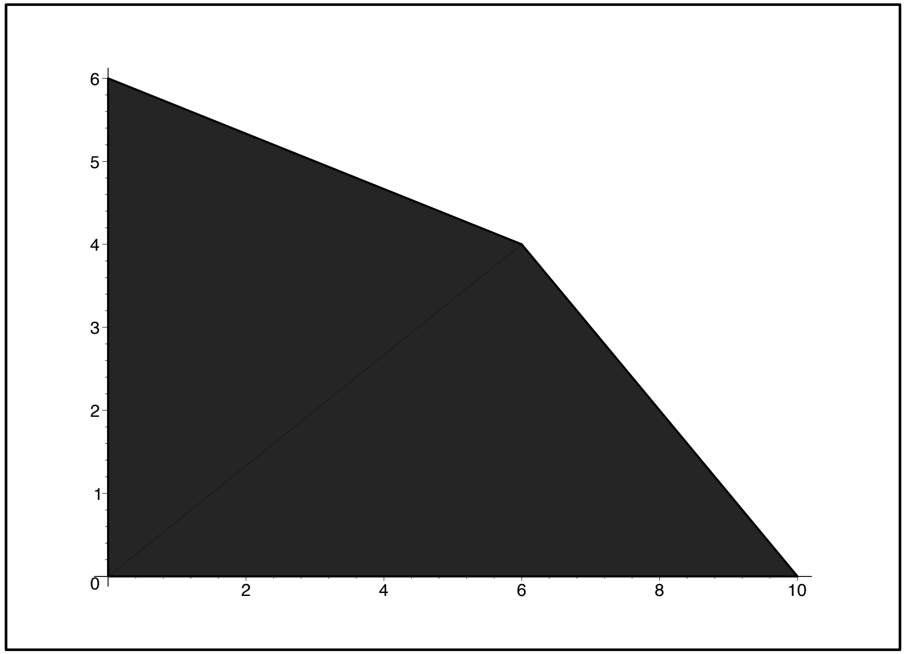
## Solution

Eliminate $y_1, y_2$ from

$$\begin{cases} f(y_1, y_2) & = & 0 \\ (x_1 - y_1)^2 + (x_2 - y_2)^2 - d^2 & = & 0 \\ -\frac{\partial f}{\partial y_2}(x_1 - y_1) + \frac{\partial f}{\partial y_1}(x_2 - y_2) & = & 0 \end{cases}$$

## Tropical associated problem

Given $\mathcal{C}$, compute $N\left(O_d(\mathcal{C})\right)$



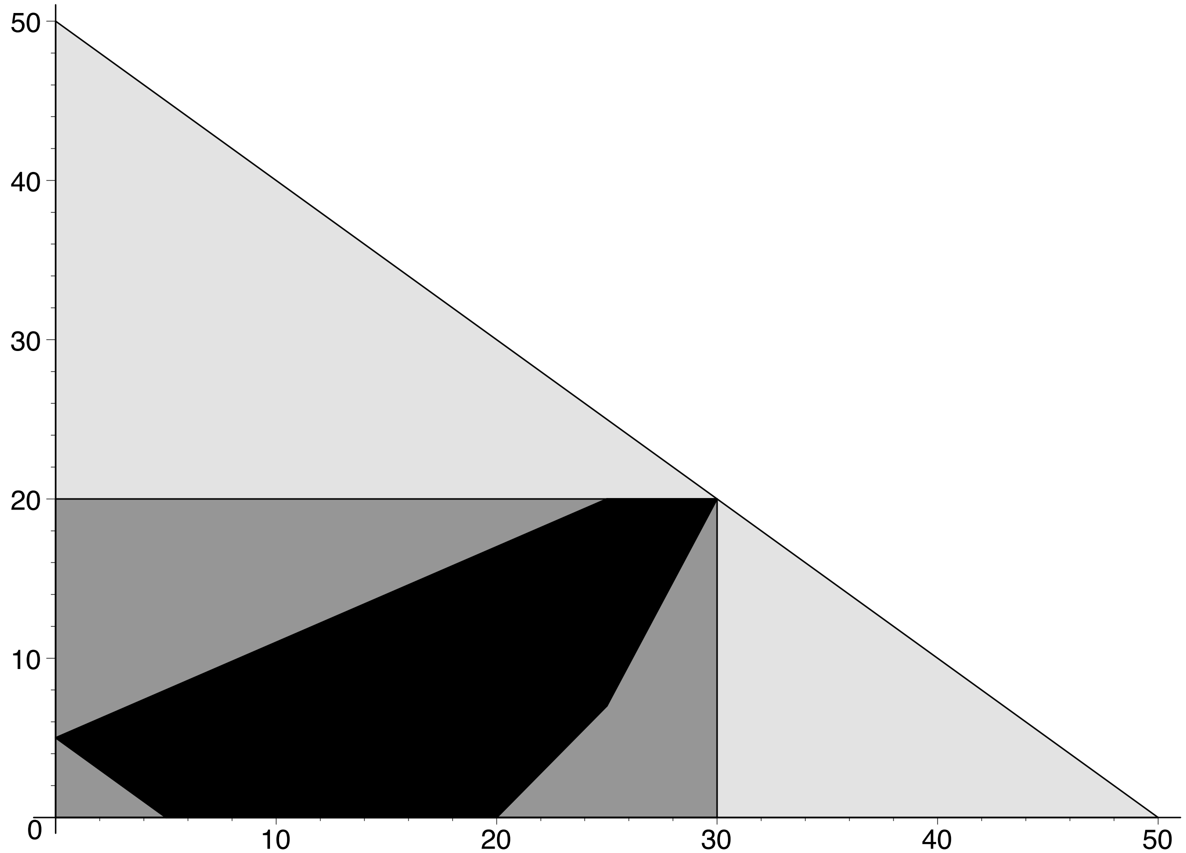
## Known results (offsets)

- The degree of $\mathcal{O}_d(\mathcal{C})$

(San Segundo-Sendra 2004)

- The partial degrees of $\mathcal{O}_d(\mathcal{C})$

(San Segundo-Sendra 2006)



50
40
30
20
10
0
10
20
30
40
50

# Known results (tropicalization)

The Newton polygon of a rational plane curve

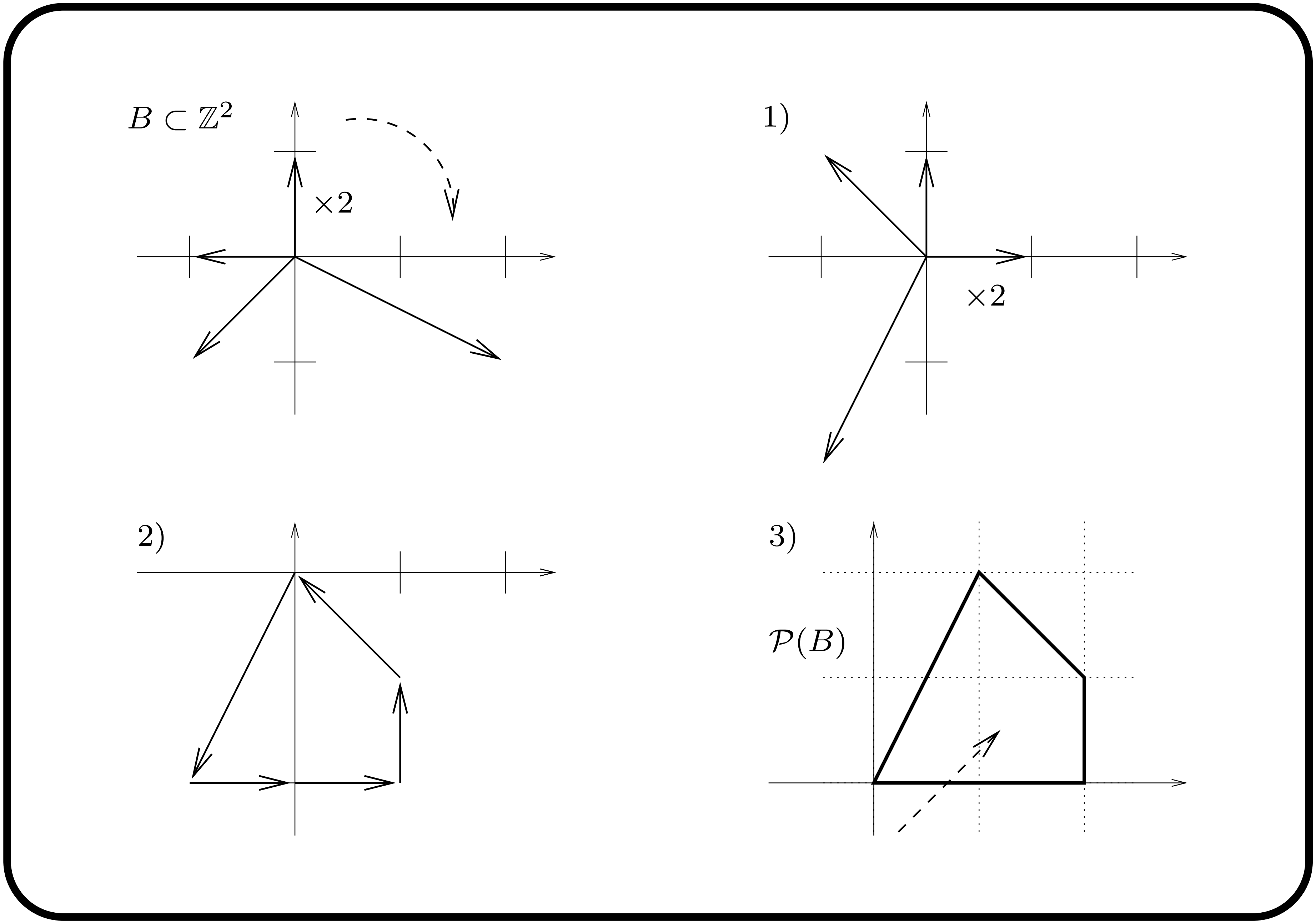
- Dickenstein-Feichtner-Sturmfels 2007
- Sturmfels-Tevelev 2007
- D-Sombra 2007

## Example

$$\rho(t) = \left(\frac{1}{t(t-1)}, \frac{t^2 - 5t + 2}{t}\right)$$

$$1 - 16X - 4X^2 - 9XY - 2X^2Y - XY^2$$

- $ord_0(\rho) = (-1, -1)$
- $ord_1(\rho) = (-1, 0)$
- $ord_\infty(\rho) = (2, -1)$
- for $v^2 - 5v + 2 = 0$ $ord_v(\rho) = (0, 1)$



$B \subset \mathbb{Z}^2$
$\times 2$
1)
$\times 2$
2)
3)
$\mathcal{P}(B)$

## <u>Main result</u>

(D-San Segundo-Sendra-Sombra)

If $\mathcal{C}$ is given parametrically, then the same "recipe" works

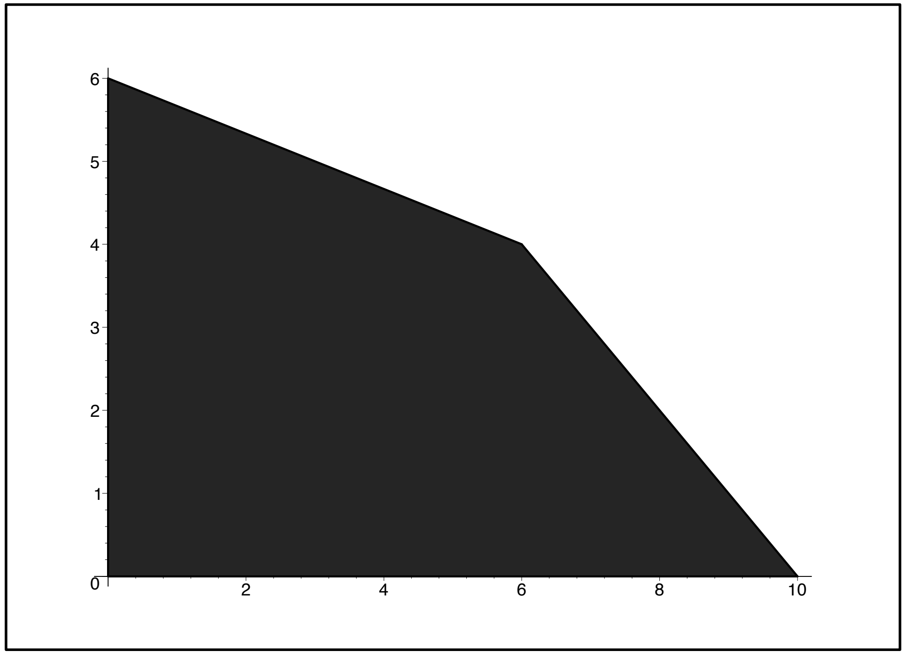
## Example

$$\rho(t) = (t, t^3) \;\; d = 1$$

$$X^{\pm}(t) = t \mp \frac{3t^2}{\sqrt{9t^4+1}}, \quad Y^{\pm}(t) = t^3 \mp \frac{1}{\sqrt{9t^4+1}}$$

$$X^{\pm}(t) = \frac{t\,(9\,t^4+1) \mp 3\,t^2\sqrt{9\,t^4+1}}{9\,t^4+1}$$

$$Y^{\pm}(t) = \frac{t^3\,(9\,t^4+1) \pm \sqrt{9\,t^4+1}}{9\,t^4+1}$$



# Sketch of a proof

**(tropical flavor)**

* Lift the curve to $\mathbb{K}^3$ and consider $\begin{cases} P(t,X) & = & 0 \\ Q(t,Y) & = & 0 \end{cases}$

* Tropicalize the spatial curve

* Compute its multiplicities

* Project

- Sturmfels-Tevelev 2007

- “Puiseux Expansion for Space Curves”, Joseph Maurer (1980)

## Sketch of another proof

**(mediterranean flavor)**

* Stay in $\mathbb{K}^2$

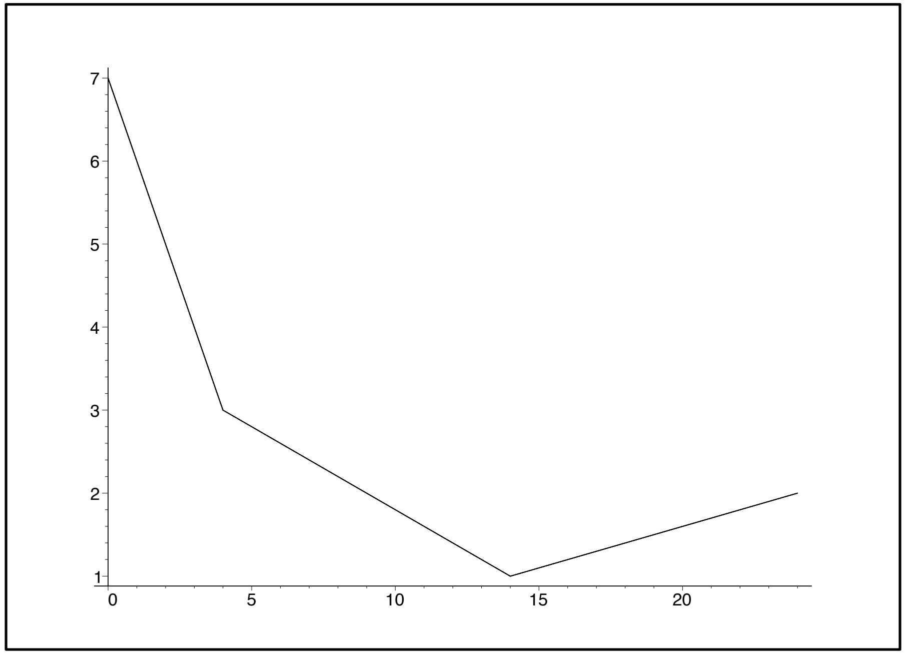
* Use Theorem 4.1 from the book of Walker, combined with the (inverse) Puiseux diagram construction

# Theorem 4.1

**(Algebraic Curves by Robert J. Walker)**

If $f(x,y) \in \mathbb{K}[x,y]$, to each root $\overline{y} \in \mathbb{K}((x))$ of $f(x,y) = 0$ for which $\mathcal{O}(\overline{y}) > 0$ there corresponds a unique place of the curve $f(x,y) = 0$ with center at the origin. Conversely, to each place $(\overline{x}, \overline{y})$ of $f$ with center at the origin there correspond $\mathcal{O}(\overline{x})$ roots of $f(x,y) = 0$, each of order greater than zero.

# (inverse) Puiseux diagram construction




The family $\{(\mathcal{O}(\overline{x}), \mathcal{O}(\overline{y}))\}_{(\overline{x},\overline{y}) \in P(\mathcal{C})}$ with $\mathcal{O}(\overline{x}) \neq 0$ or $\mathcal{O}(\overline{y}) \neq 0$ determines $N(f(x,y))$

## In general

Maurer's results can be applied to projections of curves of the form

$$\begin{cases} P(t,X,Y) = 0 \\ Q(t,X,Y) = 0 \end{cases}$$

And the tropicalization theorem holds also in this case

# Moreover

From ANY formula (algebraic or not) of the form

$$\begin{cases} X &= \Psi_1(t) \\ Y &= \Psi_2(t), \end{cases}$$

if you can extract the data $\left\{\left(\mathcal{O}(\overline{x}), \mathcal{O}(\overline{y})\right)\right\}_{(\overline{x},\overline{y})\in P(\mathcal{C})}$ with $\mathcal{O}(\overline{x}) \neq 0$ or $\mathcal{O}(\overline{y}) \neq 0$, then you can get $N(\mathcal{C})$

# THANKS...

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