More on Implicitization Using Moving Surfaces

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Outline

- Implicitization
- Moving planes and moving surfaces
- Ruled surfaces
- Tensor product surfaces
- Surfaces of revolution

Implicitization

Implicitization

- Given a rational surface

$$\mathbf{P(s,t)=(a(s,t),b(s,t),c(s,t),d(s,t))} \quad \mathbf{(1)}$$

- Find an irreducible homogenous polynomial f(x,y,z,w) such that

$$\mathbf{f(a(s,t),b(s,t),c(s,t),d(s,t)) \equiv 0} \quad \mathbf{(2)}$$

Implicitization

- **Traditional methods:**
- Resultants
- Groebner bases
- Wu’s method
- Undetermined coefficients
- ……

Implicitization

- **Problems with the traditional Methods**
  - Groebner bases and Wu's method are computational very expensive.
  - Resultant-based methods fail in the presence of base points.
  - Most methods give a very huge polynomial which is problematic in numerical computation (The implicit equation of a bicubic surface is a polynomial of degree 18 with 1330 terms!)

Implicitization

- **Moving planes and moving surfaces**

[Sederberg and Chen, Implicitization using moving curves and surfaces, SIGGRAPH, 1995]

- **Advantages**
  - Efficient
  - Never fails
  - Simplifies in the presence of base points
  - The implicit equation is a determinant

Implicitization

- **Moving planes and moving surfaces**

[Sederberg and Chen, Implicitization using moving curves and surfaces, SIGGRAPH, 1995]

- **Problems**
  - Lack explicit constructions
  - No rigorous proofs

Implicitization

- **Goal**

  Try to provide a general framework for implicitizing rational surfaces via syzygies.

Moving Planes and Surfaces

Moving planes and surfaces

- A *moving plane* is a family of planes with parameter pairs (s,t):

$$A(s,t)x + B(s,t)y + C(s,t)z + D(s,t) \equiv 0 \quad (3)$$

It is denoted by

$$\mathbf{L}(s,t)=(A(s,t),\ B(s,t),\ C(s,t),\ D(s,t))\in\mathbf{R}[s,t]^4$$

Moving planes and surfaces

- A moving plane is said to *follow* the rational surface **P**(s,t) if

$$Aa + Bb + Cc + Dd \equiv 0 \quad (4)$$

- Geometrically, "follow" means that, for each parameter pair (s,t), the point **P**(s,t) is on the moving plane L(s,t).

Moving planes and surfaces

- Let $\mathbf{L}_{s,t}$ be the set of moving planes which follow the rational surface $\mathbf{P}(s,t)$. Then $\mathbf{L}_{s,t}$ is exactly the syzygy module over $\mathbf{R}$[s,t]

$$Syz(a,b,c,d) :=$$
$$\left\{(A,B,C,D) \in R[s,t]^4 \mid aA + bB + cC + dD \equiv 0\right\} \quad (5)$$

Moving planes and surfaces

- A *moving surface* of degree *l* is a family of algebraic surfaces with parametric pairs (s,t):

$$M(x,y,z,w;s,t) := \sum_{i=1}^{\sigma} f_i(x,y,z,w)\, b_i(s,t) = 0$$

- $f_i(x,y,z,w)$, $i=1,\ldots,\sigma$ are degree *l* homogeneous polynomials, $b_i(s,t)$ are blending functions.

≡

Moving planes and surfaces

- A moving surface is said to *follow* the rational surface P(s,t) if

$$M(a(s,t), b(s,t), c(s,t), d(s,t)) \equiv 0.$$

- The implicit equation f(x,y,z) of the rational surface **P**(s,t) is a moving surface.

Rational Ruled Surfaces

Rational ruled surfaces

- A bi-degree $(n, 1)$ tensor product rational surface

$$\mathbf{P}(s,t) = \mathbf{P}_0(s) + t\mathbf{P}_1(s) \quad (5)$$

where $\mathbf{P}_i(s)=(a_i(s), b_i(s), c_i(s), d_i(s)), i=0, 1.$

- Moving planes involving only the parameter s

$$\mathrm{L}(s) := \{(A,B,C,D)\in R[s]^4 \mid aA + bB + cC + dD \equiv 0\} \quad (6)$$

Rational ruled surfaces

- (6) is equivalent to

$$\begin{pmatrix} a_0(s) & b_0(s) & c_0(s) & d_0(s) \\ a_1(s) & b_1(s) & c_1(s) & d_1(s) \end{pmatrix} \begin{pmatrix} A(s) \\ B(s) \\ C(s) \\ D(s) \end{pmatrix} \equiv 0 \quad (7)$$

i.e.,

$$\mathbf{L}(s) = Syz\begin{pmatrix} a_0(s) & b_0(s) & c_0(s) & d_0(s) \\ a_1(s) & b_1(s) & c_1(s) & d_1(s) \end{pmatrix} \quad (8)$$

Rational ruled surfaces

- **L**(s) is a free module of dimension 2.

- Mu-bais: a basis **p**(s) and **q**(s) of the module **L**(s) with minimum degree.

- Write

$$\mathbf{p}(s) = (p_1(s), p_2(s), p_3(s), p_4(s))$$
$$\mathbf{q}(s) = (q_1(s), q_2(s), q_3(s), q_4(s))$$

Rational ruled surfaces

- $\deg(\boldsymbol{p}(s))+\deg(\boldsymbol{q}(s))=m$, *where m* is the implicit degree of the rational ruled surface. Let $\deg(\boldsymbol{p}(s))=\mu$, then $\deg(\mathbf{q}(\mathrm{s}))=m-\mu$ ($\mu\leq[m/2]$).

- The implicit equation of the ruled surface

$$\mathrm{Res}(\boldsymbol{p}\cdot\boldsymbol{X},\ \boldsymbol{q}\cdot\boldsymbol{X};\ s)=0,$$

where $\boldsymbol{X}=(x,y,z,w)$. The implicit equation is a $(m-\mu)\times(m-\mu)$ determinant.

Rational ruled surfaces

- **Example** A rational ruled surface

$$\mathbf{P}(s,t)=\mathbf{P}_0(s)+t\mathbf{P}_1(s)$$

$$\mathbf{P}_0(s) =(s^3+2s^2-s+3,\ -3s+3,\ -2s^2-2s+3,\ 2s^2+s+2)$$

$$\mathbf{P}_1(s)=(2s^3+2s^2-3s+7,\ 2s^2-5s+5,$$
$$-6s^2- 8s+4, 5s^2+4s+5)$$

The ruled surface has 2 base points.

Rational ruled surfaces

- *A mu*-basis:

  $\mathbf{p}(s)= -5310xs+(-4797s+2947-2434s^2)y+(-2213s^2+7553s-2105)z +(-1263+6778s+442s^2)w$

  $\mathbf{q}(s)=(-842s+2434)x+(4017+741s)y+(-3217+421s^2+2791s)z+ (842s^2+2416s-4851)w$

- The implicit equation Res(p, q; s)=0 is a 2 by 2 determinant.

Tensor Product Surfaces

Tensor product surfaces

- A bidegree (m,n) rational surface:

$$P(s,t) = (a(s,t), b(s,t), c(s,t), d(s,t)) \quad (9)$$

- Moving planes whose degree in parameter t is $n\text{-}1$:

$$\mathbf{L}_{n-1}(s) := \{(A,B,C,D) \in R_{n-1}[s,t]^4 \mid aA + bB + cC + dD \equiv 0\} \quad (10)$$

Here $R_{n\text{-}1}[s,t]$ refers to the set of polynomials whose degrees in t do not exceed $n\text{-}1$.

Tensor product surfaces

- Let

$$\mathsf{P}(s,t)=\mathsf{P}_0(s)+\mathsf{P}_1(s)t+\cdots+\mathsf{P}_n(s)\,t^n$$

$$\mathsf{m}(s,t)=\mathsf{m}_0(s)+\mathsf{m}_1(s)t+\cdots+\mathsf{m}_{n-1}(s)t^{n-1}\in\mathsf{L}_{\text{n-1}}(s)$$

where $\mathsf{P}_i(s),\mathsf{m}_i(s)\in\mathsf{R}[s]^4$

From $\mathsf{P}(s,t)\bullet\mathsf{m}(s,t)\equiv 0$, one has

Tensor product surfaces

$$\begin{pmatrix} P_0 & & & \\ P_1 & P_0 & & \\ \vdots & P_1 & \ddots & \\ P_n & \vdots & & P_0 \\ & P_n & & P_1 \\ & & \ddots & \vdots \\ & & & P_n \end{pmatrix}_{2n\times 4n} \begin{pmatrix} m_0^T \\ m_1^T \\ \vdots \\ m_{n-1}^T \end{pmatrix} = 0 \qquad (11)$$

Denote the coefficient matrix by M. Thus

$\mathbf{L}_{n-1}(s)=Syz(M)$.

Tensor product surfaces

- **Theorem 1** $\mathbf{L}_{n-1}(s)$ is a free module over $\mathbf{R}[s]$ of dimension 2n. (This was also obtained by Sederberg, Cox, et. al.).

- **Definition 1** A basis of $\mathbf{L}_{n-1}(s)$ with minimum degree is called a "mu-basis" of $\mathbf{L}_{n-1}$(s). Denote it by

$$\mathbf{m}_1(s,t), \mathbf{m}_2(s,t), \cdots, \mathbf{m}_{2n}(s,t)$$

Tensor product surfaces

- **Problems**

1. Determine the implicit degree m of the rational surface **P**(s,t).
2. Determine the sum of the degree of the mu-basis,

$$d = \sum_{i=1}^{2n} \deg(\mathbf{m}_i)$$

3. Determine the relationship between m and d.
4. Generate the implicit equation of $\mathbf{P}(s,t)$ from the mu-basis.

Bidegree (n,2) surfaces

- **Mu-basis**

$$\mathbf{m}_1(s,t), \mathbf{m}_2(s,t), \mathbf{m}_3(s,t), \mathbf{m}_4(s,t)$$

$$\mathrm{m}_i(s,t) = \mathrm{m}_{i0}(s) + \mathrm{m}_{i1}(s)t, \quad i = 1,2,3,4$$

The relations between d and the implicit degree m

$$d = \sum_{i=1}^{2n} \deg(\mathbf{m}_i)$$

$$m = \text{implicit degree}$$

The implicit degree

- Number of intersections of a generic line and the surface.

- Choose a generic line defined by two planes

$$l_0 = A_0 x + B_0 y + C_0 z + D_0 w = 0$$
$$l_1 = A_1 x + B_1 y + C_1 z + D_1 w = 0$$

The implicit degree

- Substitute the parametric equaiton of the surface into the equations of the generic lines, we can get

$$f(s,t) = \mathbf{P}_0 \cdot \mathbf{L}_0 + \mathbf{P}_1 \cdot \mathbf{L}_0\, t + \mathbf{P}_2 \cdot \mathbf{L}_0\, t^2 = 0$$

$$g(s,t) = \mathbf{P}_0 \cdot \mathbf{L}_1 + \mathbf{P}_1 \cdot \mathbf{L}_1\, t + \mathbf{P}_2 \cdot \mathbf{L}_1\, t^2 = 0$$

with

$$\mathbf{L}_0 = (A_0, B_0, C_0, D_0), \quad \mathbf{L}_1 = (A_1, B_1, C_1, D_1)$$

The implicit degree

$$Syl(f,g;t)=\begin{pmatrix} \mathbf{P}_0\cdot\mathbf{L}_0 & \mathbf{P}_1\cdot\mathbf{L}_0 & \mathbf{P}_2\cdot\mathbf{L}_0 & \\ & \mathbf{P}_0\cdot\mathbf{L}_0 & \mathbf{P}_1\cdot\mathbf{L}_0 & \mathbf{P}_2\cdot\mathbf{L}_0 \\ \mathbf{P}_0\cdot\mathbf{L}_1 & \mathbf{P}_1\cdot\mathbf{L}_1 & \mathbf{P}_2\cdot\mathbf{L}_1 & \\ & \mathbf{P}_0\cdot\mathbf{L}_1 & \mathbf{P}_1\cdot\mathbf{L}_1 & \mathbf{P}_2\cdot\mathbf{L}_1 \end{pmatrix}$$

The implicit degree

- **Theorem 2** Consider all the order 4 minors formed by choosing two columns from the first four columns and two columns form the rest four columns of the matrix

$$H := \begin{pmatrix} a_0 & b_0 & c_0 & d_0 & 0 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & d_1 & a_0 & b_0 & c_0 & d_0 \\ a_2 & b_2 & c_2 & d_2 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & 0 & 0 & a_2 & b_2 & c_2 & d_2 \end{pmatrix} \quad (12)$$

The implicit degree

- **Theorem 2 (continued)**

  Let g be the gcd of the minors, then the implicit degree of the rational surface **P**(s,t) is

$$m = 4n - \deg(g) \qquad (13)$$

Degree sum of the mu-basis

- **Theorem 3** Consider all the 4×4 minors of

$$H = \begin{pmatrix} a_0 & b_0 & c_0 & d_0 & 0 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & d_1 & a_0 & b_0 & c_0 & d_0 \\ a_2 & b_2 & c_2 & d_2 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & 0 & 0 & a_2 & b_2 & c_2 & d_2 \end{pmatrix}$$

Let g' be the gcd of all the 4 by 4 minors, then

$$d = 4n - \deg(g') \qquad (14)$$

Degree sum of the mu-basis

- Obviously we have

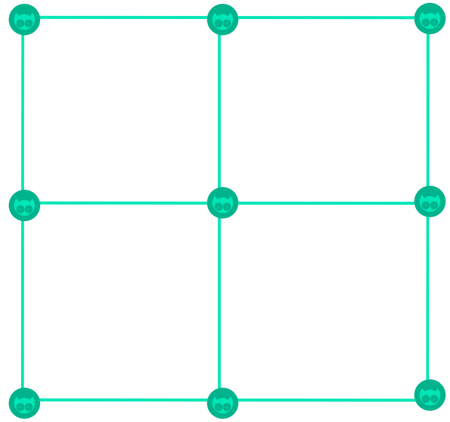
$$d \geq m$$

and the equality holds if and only if g= g' .

Example 1

- A (2,2) tensor product rational surface without base point

$$\mathbf{P}(s,t) = (13 + 6t + 4t^2 + s + 11st + 8st^2 + 10s^2 + 2s^2t + 12s^2t^2,$$
$$(s+t)(10s+t-3), \ (s+t)(1-s+4t),$$
$$13 + 12t + 11t^2 + 10s + 4st + 9st^2 + 13s^2 + 10s^2t + 11s^2t^2)$$



Newton Polygon

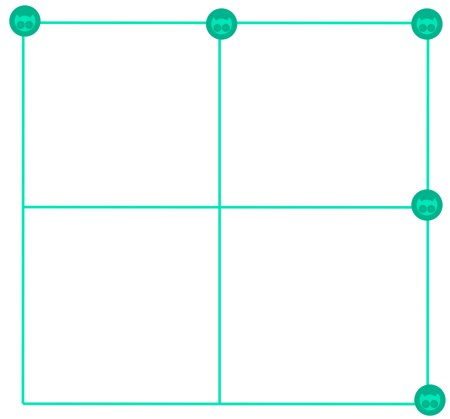
- d =1+2+2+3=8
- m=8

Example 2

- A (2,2) tensor product rational surface with one 2×2 base point at the origin

$$\mathbf{P}(s,t) = (4t^2+8st^2+10s^2+2s^2t+12s^2t^2,\ 6t^2+11st^2+13s^2+11s^2t+9s^2t^2,$$
$$5t^2+3st^2+12s^2+8s^2t+13s^2t^2,\ 11t^2+9st^2+13s^2+10s^2t+11s^2t^2)$$

- $d = 1+1+1+1 = 4$
- m=d=4

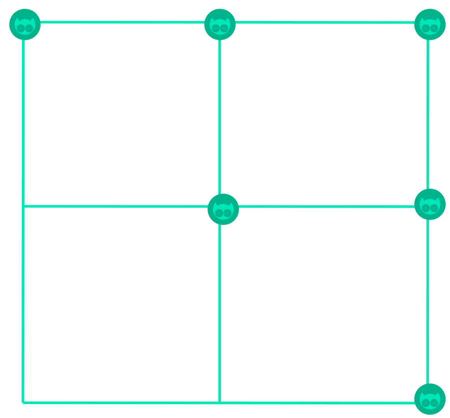


Newton Polygon

Example 3

- A (*2,2*) tensor product rational surface with one 2-ple base point at the origin

$$\mathbf{P}(s,t) = (8t^2+9st+10st^2+7s^2+5s^2t+3s^2t^2,\ 5t^2+10st+9st^2+7s^2+3s^2t+s^2t^2,$$
$$10t^2+2st+st^2+3s^2+8s^2t+s^2t^2,\ 2t^2+8st+6st^2+3s^2+10s^2t+3s^2t^2)$$



Newton Polygon

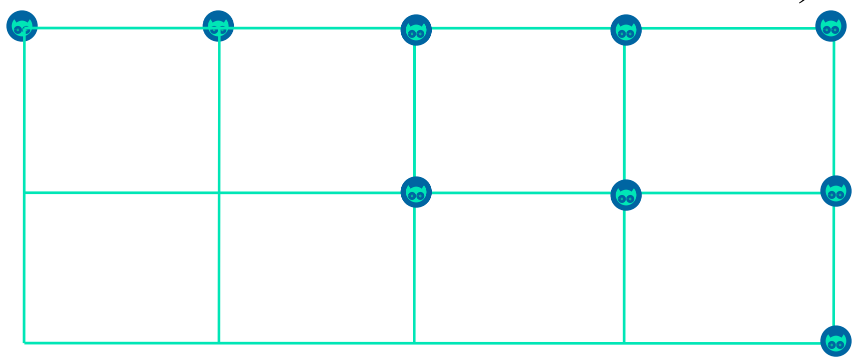
- d =1+1+1+2=5
- m=4

Example 4

- A (4,2) tensor product rational surface with a complicated base point

$$\mathbf{P}(s,t) = (11t^2+19st^2+28s^2t+14s^2t^2+15s^3t+13s^3t^2+2s^4+14s^4t+18s^4t^2,$$
$$3t^2+23st^2+13s^2t+7s^2t^2+6s^3t^2+13s^4+18s^4t+7s^4t^2,$$
$$5t^2+10st^2+17s^2t+9s^2t^2+5s^3t+27s^3t^2+28s^4+20s^4t+25s^4t^2,$$
$$4t^2+18st^2+26s^2t+13s^2t^2+12s^3t+6s^3t^2+14s^4+11s^4t+2s^4t^2)$$

- d =2+2+3+3=10
- m=8

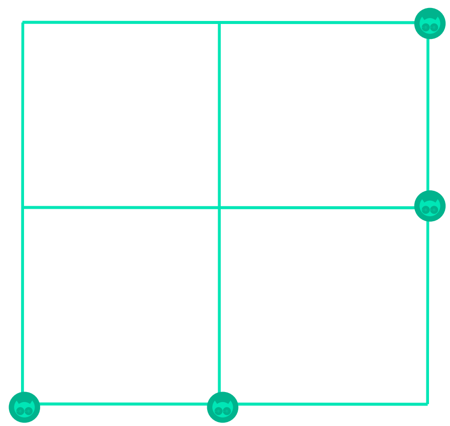


Newton Polygon

Example 5

- A (2,2) tensor product rational surface with two base points

$$\mathbf{P}(s,t) = (28 + 26s + 28s^2 + 14s^2t^2, 22 + 16s + 13s^2t + 7s^2t^2, 14 + 7s + 17s^2t + 9s^2t^2, 25 + 4s + 26s^2t + 13s^2t^2)$$



Newton Polygon

- $d = 0+1+1+1=3$
- m=3

Relationship between *m* and *d*

Base point cases	Relations
no base point	$d=m$
$k\times l$ base point Simple base points	$d=m$ $d=m$
k-ple base point(or more complicated)	$d>m$

Relationship between *m* and *d*

- **Questions:**
  1. What's the relationship between d, m and the Newton polygon of base points?
  2. When does d=m?
  3. What is the relationship between *d* and *m* for complicated base points?

Implicitization via *mu*-basis

Using moving planes

- Let the mu-basis be

$$\mathbf{m}_1(s,t),\ \mathbf{m}_2(s,t),\ \mathbf{m}_3(s,t),\ \mathbf{m}_4(s,t)$$

- Let

$(m_1, m_2, m_3, m_4)=$

$(\mathbf{m}_1(s,t).\mathbf{X},\ \mathbf{m}_2(s,t).\mathbf{X},\ \mathbf{m}_3(s,t).\mathbf{X},\ \mathbf{m}_4(s,t).\mathbf{X})$

with $\mathbf{X}=(x,y,z,1)$.

Using moving planes

Then

$$\begin{pmatrix} m_1 \\ \vdots \\ m_1 s^{l-d_1} \\ \vdots \\ m_4 \\ \vdots \\ m_4 s^{l-d_4} \end{pmatrix} = G \cdot \begin{pmatrix} 1 \\ \vdots \\ s^l \\ t \\ \vdots \\ ts^l \end{pmatrix}$$

Using moving planes

- G has a size $(4l+4-d) \times (2l+2)$. If

$$4l+4-d=2l+2=m$$

i.e., $m=d=2l+2$ is even, then

$$f(x,y,z)=\det(G)$$

would be a good candidate for the implicit equation.

Using moving planes

- In general, we choose l such that

$$4l+4-d \geqq m, \quad 2l+2 \geqq m$$

Let

$$l = \left\lceil \frac{m+d}{4} \right\rceil - 1$$

Then the maximum minor of matrix G would be a candidate for the implicit equation, but it may contain an extraneous factor.

Example 6 (implicitization)

- A (2,2) tensor product rational surface without base point

$$\mathbf{P}(s,t) = (15+12t+17t^2+18st+14st^2+16s^2+14s^2t+14s^2t^2,$$
$$13+4t+8t^2+15st+12st^2+s^2+7s^2t+15s^2t^2,$$
$$6+6t+5t^2+8st+9st^2+6s^2+7s^2t+15s^2t^2,$$
$$2+5t+2t^2+10st+13st^2+13s^2+3s^2t+19s^2t^2)$$

- $d=8$, $m=8$. The basis (m_1,m_2,m_3,m_4) has a monomial support $(1,s,s^2,t,ts,ts^2)$

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ sm_1 \\ sm_2 \\ sm_3 \\ sm_4 \end{pmatrix} = G_{8\times 8} \cdot \begin{pmatrix} 1 \\ s \\ s^2 \\ s^3 \\ t \\ ts \\ ts^2 \\ ts^3 \end{pmatrix}$$

- f(x,y,z)=det(G) is the implicit equation.

Example 7 (implicitization)

- A $(2,2)$ tensor product rational surface with a 2-ple base point

$$
\begin{aligned}
\mathbf{P}(\mathrm{s},\mathrm{t}) = (&8t^2+9st+10st^2+7s^2+5s^2t+3s^2t^2,\\
&5t^2+10st+9st^2+7s^2+3s^2t+s^2t^2,\\
&10t^2+2st+st^2+3s^2+8s^2t+s^2t^2,\\
&2t^2+8st+6st^2+3s^2+10s^2t+3s^2t^2)
\end{aligned}
$$

Example 7 (implicitization)

- d =1+1+1+2=5, m=4, l=2. The basis has a monomial support $(1, s, s^2, t, ts, ts^2)$

- $(m_1, m_2, m_3, m_4, sm_1, sm_2) = (1, s, s^2, t, ts, ts^2) \cdot G_{6\times 6}$

- Each element in the first column is just a constant multiple of each other. Thus

Example 7 (implicitization)

$$G \sim \begin{pmatrix} e & * \\ \mathbf{0} & G_1 \end{pmatrix}$$

- G_1 is a 5 by 5 matrix. $\det(G_1)=h\, f(x,y,z)$, where $f(x,y,z)$ is the implicit equation, h is a linear extraneous factor .

- **Questions**

1. When d=m is even, does det(G) always give the implicit equation (i.e., det(G) doesn't vanish)?
2. When d>m or d=m is not even, under what conditions, the maximum minor of det(G) gives the implicit equation (without extraneous factor)?
3. If the maximum minor contains an extraneous factor, can we know it in advance?

Using moving quadrics

- $(m_1, m_2, m_3, m_4)=$
  $(\mathbf{m}_1(s,t).\mathbf{X}, \mathbf{m}_2(s,t).\mathbf{X}, \mathbf{m}_3(s,t).\mathbf{X}, \mathbf{m}_4(s,t).\mathbf{X})$

- Write

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \sum_{i=0}^{\sigma} (\mathbf{M}_{1,i}(x,y,z)\, t + \mathbf{M}_{0,i}(x,y,z))\, s^i$$

Using moving quadrics

- Find the blending functions

$\mathbf{B}(x,y,z)= (B_0(x,y,z), B_1(x,y,z), B_2(x,y,z), B_3(x,y,z))$

with total degree one in x,y,z, such that

$$\mathbf{B}(x,y,z) \cdot \mathbf{M}_{0,\delta}(x,y,z) \equiv 0$$

$$\mathbf{B}(x,y,z) \cdot \mathbf{M}_{1,\delta}(x,y,z) \equiv 0$$

Using moving quadrics

Then

$$\mathbf{B}(x, y, z) \cdot (m_1, m_2, m_3, m_4)^T = \sum_{i=1}^{\delta-1} Q(x,y,z; t)\ s^i$$

will generate moving quadrics whose degree in s is at most σ-1 and degree one in t.

Using moving quadrics

- In general, we can use blending functions

$$\mathbf{B}(x, y, z, s) = \sum_{j=0}^{k} \mathbf{B}_j(x, y, z)\, s^j$$

to generate more moving quadrics, where $\mathbf{B}_j(x,y,z)$ has degree on in x,y,z:

$$\mathbf{B}(x, y, z, s) \cdot (m_1, m_2, m_3, m_4)^T = \sum_{i=1}^{\delta-1} Q(x,y,z; t)\, s^i$$

Example 8 (implicitization)

- A (2,2) tensor product rational surface without base point

$$\mathbf{P}(s,t) = (15+12t+17t^2+18st+14st^2+16s^2+14s^2t+14s^2t^2,$$
$$13+4t+8t^2+15st+12st^2+s^2+7s^2t+15s^2t^2,$$
$$6+6t+5t^2+8st+9st^2+6s^2+7s^2t+15s^2t^2,$$
$$2+5t+2t^2+10st+13st^2+13s^2+3s^2t+19s^2t^2)$$

- $d=2+2+2+2=8$, $m=8$. The basis has support

$$(1,s,s^2,t,ts,ts^2)$$

Example 8 (continued)

- Using blending function

$$\mathbf{B}(x, y, z; s) = \sum_{j=0}^{1} \mathbf{B}_j(x, y, z)\, s^j$$

We can get four moving quadrics with support monomial $(1, s, t, ts)$, then

$$(mq_1, mq_2, mq_3, mq_4) = (1, s, t, ts) \cdot G_{4\times4}$$

- f(x,y,z)=det(G) gives the implicit equation.

Example 9 (implicitization)

- A $(2,2)$ tensor product rational surface with a 2-ple base point

$$\mathbf{P}(s,t) = (8t^2+9st+10st^2+7s^2+5s^2t+3s^2t^2,$$
$$5t^2+10st+9st^2+7s^2+3s^2t+s^2t^2,$$
$$10t^2+2st+st^2+3s^2+8s^2t+s^2t^2,$$
$$2t^2+8st+6st^2+3s^2+10s^2t+3s^2t^2)$$

- d =1+1+1+2=5, m=4. The basis has a support

$$(1,s,s^2,t,ts,ts^2)$$

Example 9 (continued)

- Using blending function

$$\mathbf{B}(x,y,z;s)=\sum_{j=0}^{2}\mathbf{B}_j(x,y,z)\,s^j$$

  we can get a moving quadrics with support $(1,s,t,ts)$.
- Choose three moving planes and one moving quadric, we can get

$$(mp_1,mp_2,mp_3,mq_4)=(1,s,t,ts)\cdot \mathrm{G}_{4\times 4}$$

Example 9 (continued)

- The first row of the matrix G has the following property:

$$g_{12} = c_1 \cdot g_{11}$$

$$g_{13} = c_2 \cdot g_{11}$$

$$g_{14} = l(x, y, z) \cdot g_{11}$$

where $G=(g_{ij})$, and l is linear in x,y,z.

Example 9 (continued)

- Thus

$$G \sim \begin{pmatrix} e & \mathbf{0} \\ * & G_1 \end{pmatrix}$$

  G_1 is a 3 by 3 matrix with two linear rows and one quadratic rows.

- Det(G_1) is the correct implicit equation.

Example 10 (implicitization)

- A (2,2) tensor product rational surface without base point

$$\mathbf{P}(\mathrm{s},\mathrm{t}) = (13 + 6t + 4t^2 + s + 11st + 8st^2 + 10s^2 + 2s^2t + 12s^2t^2, (s+t)(10s+t-3), (s+t)(1-s+4t), 13 + 12t + 11t^2 + 10s + 4st + 9st^2 + 13s^2 + 10s^2t + 11s^2t^2)$$

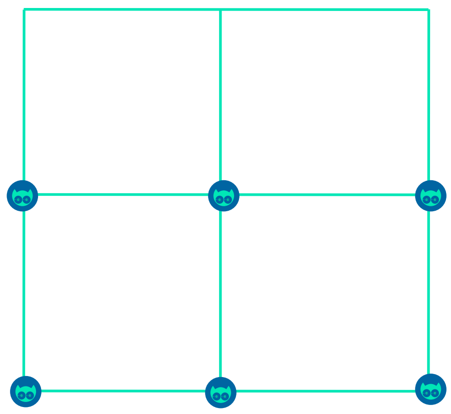
- d =1+2+2+3=8, m=8.

Example 10 (implicitization)

- Then

$$(mp_1, mp_2, mp_3, mp_4, mq_1, mq_2) =$$
$$(1, s, s^2, t, ts, ts^2)G_{6\times 6}$$

- Det(G) is the implict equation.



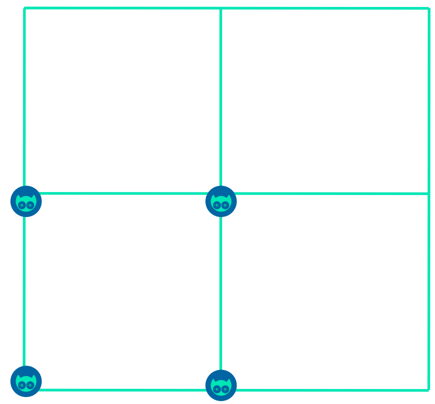
Newton Polygon for moving planes and quadrics

Example 10 (implicitization)

- Another expression

$$(mp_1, mq_1, mq_2, mc_1) = (1, s, t, st)G_{4\times 4}$$

- f(x,y,z)=det(G).



Newton Polygon for moving planes (quadrics, cubic)

- **Questions:**
- Can we always generate right number of moving planes and moving quadrics to form a square matrix with right degree?
- Prove the determinant doesn’t vanish.

Surfaces of Revolution

Surfaces of Revolution

- Let

$$C(s) = \left(\frac{y(s)}{w(s)}, \frac{z(s)}{w(s)} \right)$$

be a parametrization of the curve C in yz-plane. The roation of C around the z-axis results in a surface of revolution with parametrization

$$\mathbf{P}(s,t) = \left(y(s)(1-t^2), y(s)2t, z(s)(1+t^2), w(s)(1+t^2) \right)$$

Surfaces of Revolution

- Surface of revolution is a bidegree (n,2) tensor product surface. Let

$$\mathbf{p}(s) = (p_1(s), p_2(s), p_3(s))$$
$$\mathbf{q}(s) = (q_1(s), q_2(s), q_3(s))$$

be a mu-basis of the planar curve C(s). Assume deg(**p**)=μ, then deg(**q**)=n- μ.

Surfaces of Revolution

- Then

$$\mathbf{m}_1(s,t) = (p_1(s), p_1(s)t, p_2(s), p_3(s))$$
$$\mathbf{m}_2(s,t) = (-p_1(s)t, p_1(s), p_2(s)t, p_3(s)t)$$
$$\mathbf{m}_3(s,t) = (q_1(s), q_1(s)t, q_2(s), q_3(s))$$
$$\mathbf{m}_4(s,t) = (-q_1(s)t, q_1(s), q_2(s)t, q_3(s)t)$$

is a mu-basis of the surface of revolution.

- d= μ+ μ+(n- μ)+(n- μ)=2n, m=2n.

Surfaces of Revolution

- Using moving planes, the implicit equation can be written as a 2n by 2n determinant.

$$\begin{pmatrix} m_1 \\ \vdots \\ m_1 s^{n-\mu-1} \\ \vdots \\ m_4 \\ \vdots \\ m_4 s^{\mu-1} \end{pmatrix} = G_{2n\times 2n} \begin{pmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \\ t \\ \vdots \\ ts^{n-1} \end{pmatrix}$$

Surfaces of Revolution

- Using moving planes and moving quadrics

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix} = \begin{pmatrix} yp_1 \\ yq_1 \\ -xp_1 + zp_2 + p_3 \\ -xq_1 + zq_2 + q_3 \end{pmatrix} t + \begin{pmatrix} xp_1 + zp_2 + p_3 \\ xq_1 + zq_2 + q_3 \\ yp_1 \\ yq_1 \end{pmatrix}$$

- Use blending functions to eliminate the highest degree terms in s (degree n-μ).

Surfaces of Revolution

- We use m_2 and m_4 to generate n-μ moving quadrics of degree n-μ-1 in s, and use m_1 and m_3 to generate μ moving quadrics of degree μ-1 in s.

  Finally, we get 2n-2μ moving quadrics.

- We can also generate 2n-4μ moving planes:

$$m_1, m_1 s, \cdots, m_1 s^{n-2\mu-1}$$

$$m_3, m_3 s, \cdots, m_3 s^{n-2\mu-1}$$

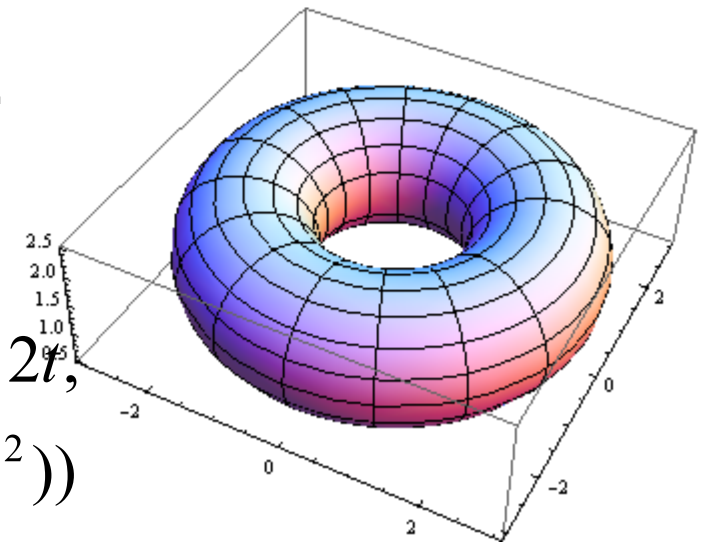
Surfaces of Revolution

- Finally, we have

$$\begin{pmatrix} mp_1 \\ \vdots \\ mp_{2n-4\mu} \\ mq_1 \\ \vdots \\ mq_{2\mu} \end{pmatrix} = G_{(2n-2\mu)\times(2n-2\mu)} \begin{pmatrix} 1 \\ s \\ \vdots \\ s^{n-\mu-1} \\ \vdots \\ ts^{n-\mu-1} \end{pmatrix}$$

Example 12 (Torus)

- The torus



$$\mathbf{P}(s,t) = ((3+s^2)(1-t^2), (3+s^2)\cdot 2t, (3+2s+3s^2)(1+t^2), (1+s^2)(1+t^2))$$

- m=d =1+1+1+1=4. Using moving planes with monomial support (1,s,t,ts), the implicit equation can be written as a determinant of order 4.

Example 12 (Torus)

- We can also implicitize the torus using two moving quadrics with support (*1,s*).

  Therefore the implicit equation can be also written as the determinant of a 2×2 matrix.

Example 13

- The planar curve in *yz*-plane

$$y(t) = \frac{1 - 2s - 2s^2 + 5s^3}{11 + s - s^2 + 3s^3}$$

$$z(s) = \frac{6s^2 + 2s + 1 + 2s^3}{11 + s - s^2 + 3s^3}$$

The corresponding surface of revolution is a $(3,2)$ tensor product rational surface.

Example 13

- m=d=1+1+2+2=6, so the implicit equation can be written as 6 by 6 determinant.
- The implicit equation can also be derived from two moving planes and two moving quadrics with support $(1,s,t,ts)$, which is a 4 by 4 determinant.
- we can also get 3 moving quadrics with support $(1,s, s^2)$, therefore the implicit equation can also be expressed as a 3 by 3 determinant.

- **Questions:**

  Prove that the determinants formed by moving planes and moving quadrics do not vanish.

Thank you for your attention!

Example 10 (implicitization)

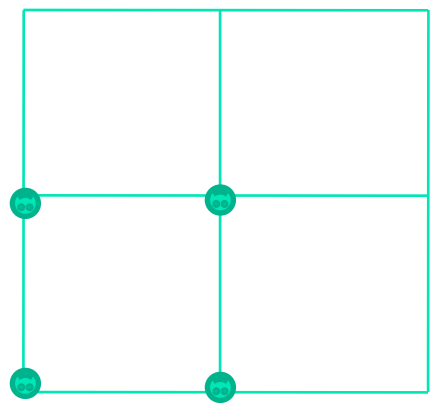
- A (*2,2*) tensor product rational surface without base point

$$\mathbf{P}(s,t) = ((s+t+11)(1-s+4t),\ (s+t)(10s+t-3),\ (s+t)(1-s+4t),\ 13+12t+11t^2+10s+4st +9st^2+13s^2+10s^2t+11s^2t^2)$$

- d =1+1+3+3=8, m=8.

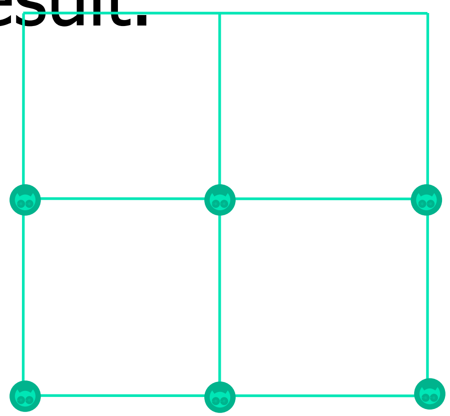
- We cannot get enough moving planes and moving quadrics (even cubic) for implicitization with support:

$(1, s, t, ts)$



Newton Polygon for moving planes (quadrics, cubic)

- However, under the support ($1, s, s^2, t, ts, ts^2$), we can get four moving planes and two moving quadrics, from which we can get the implicit result.



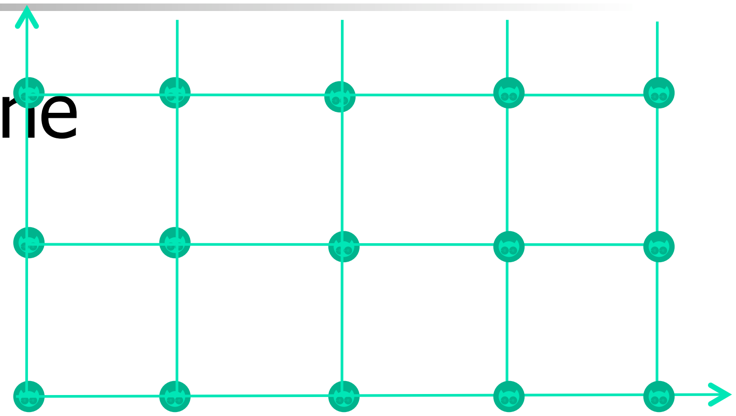
Newton Polygon for moving planes and quadrics

Example 14

- The planar circle in *yz*-plane

$$y(t) = \frac{(1+s)(1-2s-2s^2+5s^3)}{11+s-s^2+3s^3-s^4}$$

$$z(s) = \frac{(1+s)(6s^2+2s+1)}{11+s-s^2+3s^3-s^4}$$

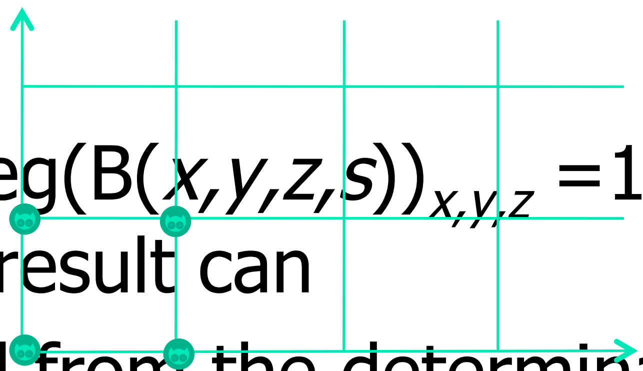


This is a (4,2) tensor product rational surface. We can get

d =2+2+2+2=8=implicit degree.

- From the basis with minimal degree summation, we can also get 4 moving quadrics with support *(1,s,t,ts)* using blending functions

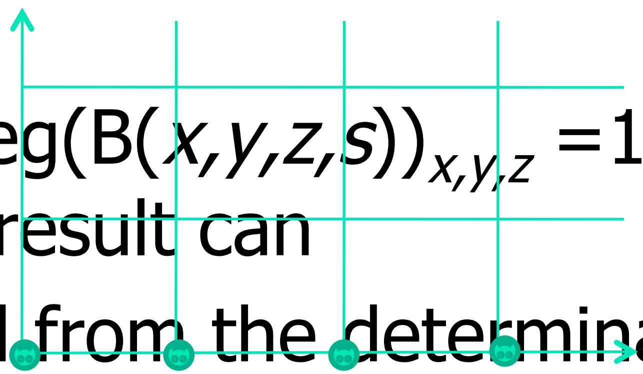
$$\mathbf{B}(x,y,z;s)=\sum_{j=0}^{2}\mathbf{B}_j(x,y,z)s^j$$



where tdeg(B(*x,y,z,s*))$_{x,y,z}$ =1. Therefore, the implicit result can be derived from the determinant of a 4×4 matrix.

- From the basis with minimal degree summation, we can also get 3 moving quadrics with support (*1,s, s*^{2}) using blending functions

$$\mathbf{B}(x,y,z;s)=\sum_{j=0}^{1}\mathbf{B}_j(x,y,z)\,s^j$$



where tdeg(B(*x,y,z,s*))$_{x,y,z}$ =1. Therefore, the implicit result can be derived from the determinant of a 4×4 matrix.