Computations in tropical algebraic geometry

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Tropical Geometry

Tropical math arises when considering the *tropical semi-ring* $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ where

$\oplus$ is maximum, and

$\odot$ is addition.

Two main directions in tropical geometry:

- Geometric approach
- Algebraic/combinatorial approach

Tropical varieties – Three definitions

The field of Puiseux series $\mathbb{C}\{\{t\}\}$ has a valuation:

$$\mathrm{val} : \mathbb{C}\{\{t\}\}^* \to \mathbb{Q}.$$

The valuation extends to

$$\mathrm{val} : (\mathbb{C}\{\{t\}\}^*)^n \to \mathbb{Q}^n.$$

Definition

If $I \subseteq \mathbb{C}\{\{t\}\}[x_1, \ldots, x_n]$ is an ideal then we define

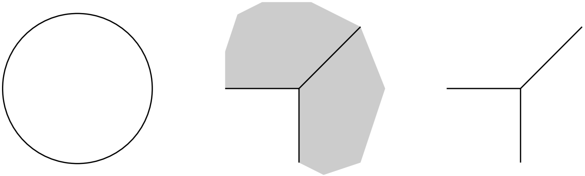
$$T(I) := -\overline{\mathrm{val}(V(I))} \subseteq \mathbb{R}^n$$

The tropical variety $T(I)$ is a “shadow” of the usual variety.

Tropical varieties – Three definitions

Example

$$I = \langle x^2 + y^2 - 1 \rangle$$



Variety Gröbner fan Tropical variety

Tropical varieties – Three definitions

Consider the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. Let $\omega \in \mathbb{R}^n$.

- The *weight* of a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with $a \in \mathbb{N}^n$ is $\langle \omega, a \rangle$.
- The *initial form* $in_\omega(f)$ of a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is the sum of terms with maximal weights.
  Example:

$$in_{(1,2)}(x_1^4 + 2x_2^2 + x_1x_2 + 1) = x_1^4 + 2x_2^2$$

- The *initial ideal* of an ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ is defined as

$$in_\omega(I) = \langle in_\omega(f) \rangle_{f \in I}$$

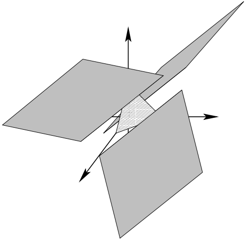
Theorem (Speyer, Sturmfels, 2003)

$$T(I) = \{\omega \in \mathbb{R}^n : \text{in}_\omega(I) \text{ is monomial-free}\}$$

Tropical varieties – Three definitions

Example

The tropical variety of a principal ideal is called a *tropical hypersurface*.
$T(\langle x_1 + x_2 + x_3 \rangle) \subseteq \mathbb{R}^3$ is the union of three 2-dimensional cones:



Lemma

Any tropical variety is an intersection of hypersurfaces:

$$T(I) = \bigcap_{f \in I} T(\langle f \rangle)$$

The tropical variety as a subfan of a Gröbner fan

Let $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a homogeneous prime ideal.

- Bieri, Groves (1984):
  The dimension of $T(I)$ is the dimension of $V(I)$.
- Mora, Robbiano (1988):
  Define an equivalence relation $\sim$ on $\mathbb{R}^n$.

  $$u \sim v \Leftrightarrow \text{in}_u(I) = \text{in}_v(I)$$

  Equivalence classes form the cones in the Gröbner fan GF(I).
- Collart, Kalkbrener, Mall (1993):
  GF(I) may be computed with the Gröbner walk.
- Bogart, J., Speyer, Sturmfels, Thomas (2005):
  The tropical variety as subfan of $GF(I)$ is connected and can be traversed with Gröbner walk like methods

Generalized Newton-Puiseux

The classical Newton-Puiseux algorithm:

Input $f \in \mathbb{C}\{\{t\}\}[x]$

Output All roots of f up to some degree

Maurer's generalization (1980):

Input $f_1, \ldots, f_m \in \mathbb{C}\{\{t\}\}[x_1, \ldots, x_n]$

Output All common roots of $f_1, \ldots, f_m$ up to some degree

- Find a tropism of the system, i.e. a vector in $T(\langle f_1, \ldots, f_m \rangle)$.
- Let this be the first exponent of a series solution. Find the coefficients by solving over $\mathbb{C}$.
- Substitute and find next term recursively.

Problem: How do we find the tropisms?
Solution: Tropical geometry.
Details were worked out in J., Markwig, Markwig (2007).

Tropical bases

Definition

A finite generating set F of I is a *tropical basis* if

$$T(I) = \bigcap_{f \in F} T(\langle f \rangle).$$

Cone description VS tropical basis.

Theorem (Bogart, J., Speyer, Sturmfels, Thomas, 2005)

Every ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ has a tropical basis.

An algorithm was presented that based on $\bigcap_{f \in F} T(\langle f \rangle)$ would add in elements successively to form a basis.

Essentially as hard as computing the Gröbner fan.

Tropical bases by generic projections

- Elimination is projection.
- A $\mathrm{GL}_n(\mathbb{Z})$ multiplicative change of coordinates in $(\mathbb{C}\{\{t\}\}^*)^n$ results in a linear transformation of $T(I)$.
- This allows projection in any direction.
- For any rational subspace $U \subseteq \mathbb{R}^n$, $T(I) + U$ can be computed algebraically.

Theorem (Hept and Theobald, 2007)

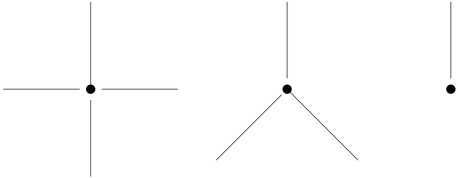
Any finite generating set for a prime ideal can be extended to a tropical basis by adding codimension+1 elements.

Connection to numerical homotopy

The important polyhedral computation

$$\bigcap_i T(\langle f_i \rangle)$$

is closely related to the *mixed volume* computation.
Difficult combinatorial task based on LP-solving.



The Huber-Sturmfels (1995) *polyhedral* homotopy method has this computation as a first step. Carefully studied by T. Y. Li.

Can numerical methods benefit from computational tropical geometry? Or the other way around?

Application in celestial mechanics

Given 4 bodies in space satisfying Newton laws, does there exist an infinite number of relative equilibria?

Hampton and Moeckel (2006): “No”

The set of relative equilibria is given by equations

$$f_1, \ldots, f_9 \in \mathbb{R}[r_{12}, \ldots, r_{34}]$$

in the 6 pairwise distances.
If there was an infinite number of solutions, there exist a curve of solutions parametrizable by a Puiseux series.

Carefully checking the mixed faces of the Minkowski sum of the Newton polytopes and corresponding initial ideals, it is concluded that no such parametrization exists.

Application for implicitization

Let $g_1, \ldots, g_n \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$ and consider

$$g : (\mathbb{C}^*)^{n-1} \to (\mathbb{C})^n$$

$$x \mapsto (g_1(x), \ldots, g_n(x))$$

Fix the Newton polytopes of the g_i's and suppose coefficients are generic.

Sturmfels, Tevelev and Yu (2007) gave a tropical polyhedral method for computing the Newton polytope for a defining equation of $\overline{g((\mathbb{C}^*)^{n-1})}$.

Providing examples in tropical geometry

In ongoing work with Herrmann, Joswig and Sturmfels the tropical Grassmannian $G_{3,7}$ has been computed.

Hereby we classify all 94 combinatorial types of generic tropical 2-planes in tropical projective space TP^6.

The tropical Grassmannian has 252000 maximal cone and lives in $\mathbb{R}^{35}$.

Complexity

Several complexity results were proved by Theobald (2006).