Secant Varieties of Classically Studied Varieties

(joint work with Giorgio Ottaviani and Chris Peterson)

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Tensors

- Let $\mathbb{K}$ be a field such as $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.
- The collected data can be represented as elements which have more than two indices.

  (e.g., facial images= people $\times$ views $\times$ illuminations $\times$ pixcels)
- Such high order equivalents of vectors $v \in \mathbb{K}^n$ and matrices $A \in \mathbb{K}^{m\times n}$ are called **tensors**.

Tensor Rank

- Let V_i be a $(n_i + 1)$-dimensional vector space over $\mathbb{K}$.
- A tensor in $\bigotimes_{i=1}^{k} V_i$ of the form $v_1 \otimes \cdots \otimes v_k$, $v_i \in V_i$, is called a **rank one tensor**.
- Any tensor in $\bigotimes_{i=1}^{k} V_i$ can be expressed as a linear combination of rank one tensors.
- A tensor in $\bigotimes_{i=1}^{k} V_i$ has **rank** r if it can be written as a linear combination of r rank-one tensors (but not fewer).

Segre Varieties

- The set of rank one tensors can be "embedded" into $\bigotimes_{i=1}^k V_i$ as follows:

$$\begin{array}{ccc} \prod_{i=1}^k V_i & \longrightarrow & \bigotimes_{i=1}^k V_i \\ (v_1, \ldots, v_k) & \longmapsto & v_1 \otimes \cdots \otimes v_k \end{array}$$

- Projectivizing, we have a **Segre map**

$$\prod_{i=1}^k \mathbb{P}(V_i) \to \mathbb{P}(\bigotimes_{i=1}^k V_i).$$

- Let $\mathbf{n} = (n_1, \ldots, n_k)$. The image of this map is called a **Segre variety** and denoted by $X_{\mathbf{n}}$.

Secant Varieties

- Let X be a projective variety.
- Let $\sigma_s(X)$ be the Zariski closure of the union of $(s-1)$**-secant planes** $\langle p_1, \ldots, p_s \rangle$, $p_i \in X$.
- $\sigma_s(X)$ is a projective variety called the s^{th} **secant variety** to X.
- If X is a Segre variety, points on a $(s-1)$-secant plane correspond to tensors which can be written as the sum of s rank-one tensors.

Typical Tensor Rank

- $\mathbf{n} = (n_1, \ldots, n_k)$.
- What is the least integer $R(\mathbf{n})$ such that a generic tensor in $\bigotimes_{i=1}^{k} V_i$ has rank $\leq R(\mathbf{n})$?
- $R(\mathbf{n})$ is called the **typical rank** of $\bigotimes_{i=1}^{k} V_i$.
- The problem of finding $R(\mathbf{n})$ is equivalent to the problem of finding a positive integer s such that

$$\sigma_s(X) = \mathbb{P}^{N-1},$$

where $N = \prod_{i=1}^{k}(n_i + 1)$.

Typical Tensor Rank (cont'd)

- A simple dimension count says the expected dimension of $\sigma_s(X_{\mathbf{n}})$ is

$$\min\left\{ s\left(\sum_{i=1}^{k} n_i + 1\right) - 1,\ \prod_{i=1}^{k}(n_i+1) - 1 \right\}.$$

- We have the following inequality:

$$\left\lceil \frac{\prod_{i=1}^{k}(n_i+1)}{1+\sum_{i=1}^{k} n_i} \right\rceil \leq R(\mathbf{n}).$$

Defective Varieties

- Let X be a projective variety.
- We say that $\sigma_s(X)$ is **defective** if $\sigma_s(X)$ does not have the expected dimension.
- X is said to be **defective** if $\sigma_s(X)$ is defective for some s.
- Most of the Segre varieties of the form $\mathbb{P}^m \times \mathbb{P}^n$ are defective.

Symmetric Tensor, Alternating Tensors, etc

- The definitions of tensor rank and typical rank can be extended to various tensors.

Tensors	**Varieties**
regular	Segre
symmetric	Veronese
alternating	Grassmann
hybrid of regular tensors and symmetric tensors	Segre-Veronese

- Higher rank tensors of each type are constructed as points on proper secants to the corresponding variety.

Classification of Defective Varieties

Theorem (Alexander-Hirschowitz)

Let $\nu_d : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$ be the Veronese map and let $X_{n,d}$ be the Veronese variety $\nu_d(\mathbb{P}^n)$. Then $\sigma_s(X_{n,d})$ has the expected dimension except:

(a) $d = 2$ and $2 \leq s \leq n$;

(b) $d = 3$, $n = 4$ and $s = 7$;

(c) $d = 4$ and $(n, s) = (2, 5)$, $(3, 9)$ and $(4, 14)$.

Conjecture (Ottaviani-Peterson-A, 2006)

Let $X_{\mathbf{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a Segre varietiy. If $k \geq 3$, then $X_{\mathbf{n}}$ is defective if and only if $X_{\mathbf{n}}$ falls into one of the following cases:

(a) $X_{\mathbf{n}}$ is unbalanced, i.e.,
$n_k - 1 \geq \prod_{i1}^{k-1}(n_i + 1) - \sum_{i=1}^{k-1} n_i$;

(b) $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3$;

(c) $\mathbb{P}^2 \times \mathbb{P}^n \times \mathbb{P}^n$ with n even;

(d) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n \times \mathbb{P}^n$.

Conjecture

Let $\mathbb{G}(k,n)$ be the Grassmann variety of k-planes in $\mathbb{P}^n$. Then $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension unless:

(a) $k = 1$, $n \geq 5$ and $1 < s < \left\lfloor \frac{n+1}{2} \right\rfloor$.

(b) $(s,k,n) = (3,2,6)$, $(3,3,7)$, $(4,3,7)$ and $(4,2,8)$.

Remark.

This conjecture was made by Baur-Draisma-de Graaf and Ottaviani independently.

Conjecture (Brambilla-A, 2008)

Let $\mathbf{n} = (m, n) \in \mathbb{N}^2$, let $\mathbf{a} = (a, b) \in \mathbb{N}^2$ and let $X_{\mathbf{n},\mathbf{a}}$ be the Segre-Veronese variety $\mathbb{P}^m \times \mathbb{P}^n$ embedded by $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(\mathbf{a})$. Then $X_{\mathbf{n},\mathbf{a}}$ is defective if and only if $(\mathbf{n}, \mathbf{a})$ falls into one of the following cases:

(a) $b = 1$ and $n \geq \binom{m+a}{a} - m + 1$;

(b) $\mathbf{n} = (1, n)$ and $\mathbf{a} = (2k, 2)$ with $k \geq 1$;

(c) $\mathbf{n} = (4, 3)$, $(2, n)$ with n odd and $\mathbf{a} = (1, 2)$;

(d) $\mathbf{n} = (1, 2)$ and $\mathbf{a} = (1, 3)$;

(e) $\mathbf{n} = (2, 2)$, $(3, 3)$, $(3, 4)$ and $\mathbf{a} = (2, 2)$.

Terracini's Lemma

- Let $X \subset \mathbb{P}^{N-1}$ be a non-singular variety
- Let $\mathbb{T}_p(X)$ be the projective tangent space to X at p.

Theorem (Terracini's lemma)

Let $p_1, \ldots, p_s$ be generic points of X and let q be a generic point of $\langle p_1, \ldots, p_s \rangle$. Then

$$\mathbb{T}_q(\sigma_s(X)) = \langle \mathbb{T}_{p_1}(X), \ldots, \mathbb{T}_{p_s}(X) \rangle.$$

Inductive Approach

- Let $V_1 = V_{1,1} \oplus V_{1,2}$ with

$$\dim V_{1,1} = n_{1,1} + 1 \text{ and } \dim V_{1,2} = n_{1,2} + 1.$$

- Let $\mathbf{n}_i = (n_{1,i}, n_2, \ldots, n_k)$ for $i \in \{1, 2\}$.
- Consider the following linear transformations:

$$\pi_i : V_{1,i} \otimes \bigotimes_{i=2}^{k} V_i \longrightarrow \bigotimes_{i=1}^{k} V_i.$$

Inductive Approach (cont'd)

- Let $p \in X_{\mathbf{n}_{1,i}} \subset X_{\mathbf{n}}$. The above-memtioned linear transformation gives rise to

$$0 \to \hat{\mathbb{T}}_p(X_{\mathbf{n}_{1,i}}) \to \hat{\mathbb{T}}_p(X_{\mathbf{n}}) \to g_i\left[\hat{\mathbb{T}}_p(X_{\mathbf{n}})\right] \to 0,$$

  where $\hat{\mathbb{T}}_p(X_{\mathbf{n}})$ is the affine cone over $\mathbb{T}_p(X_{\mathbf{n}})$ and $\dim g_i\left[\hat{\mathbb{T}}_p(X_{\mathbf{n}})\right] = n_1 - n_{1,i}$.

- Let $p_1 \in X_{\mathbf{n}_{1,1}}$ and let $p_2 \in X_{\mathbf{n}_{1,2}}$. Then we have

$$\left\langle \hat{\mathbb{T}}_{p_1}(X_{\mathbf{n}}), \hat{\mathbb{T}}_{p_2}(X_{\mathbf{n}}) \right\rangle =$$
$$\left\langle \hat{\mathbb{T}}_{p_1}(X_{\mathbf{n}_{1,1}}), g_2\left[\hat{\mathbb{T}}_{p_1}(X_{\mathbf{n}})\right] \right\rangle \oplus \left\langle \hat{\mathbb{T}}_{p_2}(X_{\mathbf{n}_{1,2}}), g_1\left[\hat{\mathbb{T}}_{p_2}(X_{\mathbf{n}})\right] \right\rangle.$$

Inductive Approach (cont'd)

- Thus $\sigma_2(X_{\mathbf{n}})$ has the expected dimension, i.e.,

$$\dim\left\langle \hat{\mathbb{T}}_{p_1}(X_{\mathbf{n}}), \hat{\mathbb{T}}_{p_2}(X_{\mathbf{n}})\right\rangle = 2\left(\sum_{i=1}^{k} n_i + 1\right),$$

  if, for each $i \in \{1,2\}$,

$$\dim\left\langle \hat{\mathbb{T}}_{p_i}(X_{\mathbf{n}_{1,i}}), g_i\left[\hat{\mathbb{T}}_{p_i}(X_{\mathbf{n}})\right]\right\rangle = (n_{1,i} + \sum_{j=2}^{k} n_j + 1) + (n_1 - n_{1,i}).$$

- $\mathbb{P}\left\langle \hat{\mathbb{T}}_{p_i}(X_{\mathbf{n}_{1,1}}), g_i\left[\hat{\mathbb{T}}_{p_i}(X_{\mathbf{n}})\right]\right\rangle$ can be viewed as the tangent space of the join of $X_{n_{1,i}}$ and a subvariety of $X_{n_{1,i}}$ of the form $\mathbb{P}(V_{1,i}) \times \{\mathrm{pt}\} \times \cdots \{\mathrm{pt}\}$.

Results

- The classification of defective s^{th} secant varieties of Segre varieties and Grassmann varieties has been completed for small s.

- Bryan Wilson implemented the inductive approach into a computer algebra system and checked that there are no defective secant Segre varieties of Segree varieties $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$ $(n_1 \leq n_2 \leq n_3 \leq 20)$ except for the known defective cases.

Results (cont'd)

- In 1985, Lickteig proved that $R(n,n,n) = \left\lceil \frac{(n+1)^3}{3n+1} \right\rceil$ for $n \neq 2$.

- $\mathbf{n} = \overbrace{n, \ldots, n}^{k}$.

- We proved that $R(\mathbf{n})$ is asymptotically equivalent to $\frac{(n+1)^k}{kn+1}$ as $n \to \infty$ and $k \to \infty$.

Thank you very much for your attention!