Inner Approximation of Distance Constraints with Existential Quantification of Parameters

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Abstract

We present and compare two methods for checking if a *box* is included inside the solution set of an equality constraint with existential quantification of its parameters. We focus on distance constraints, where each existentially quantified parameter has only one ocurrence.
The first method relies on a Specific Quantifier Elimination (SQE) algorithm based on geometric consideration whereas the second method relies on computations with Generalized Intervals Evaluation (GIE).

Introduction

An Euclidean Distance Constraint $c_{a,r}(x)$ between two points $a, x \in \mathbb{R}^n$ can be expressed as following:

$$c_{a,r}(x) : \sum_{k=1}^{n}(x_k - a_k)^2 = r^2 \qquad (1)$$

If $\mathbf{a}$ and $\mathbf{r}$ are existentially quantified parameters, we have the following quantified distance constraint (QDC):

$$c_{\mathbf{a},\mathbf{r}}(x) : (\exists a \in \mathbf{a})(\exists r \in \mathbf{r})(\sum_{k=1}^{n}(x_k - a_k)^2 = r^2) \qquad (2)$$

Some examples of QDC and their solution set are presented in Figure 1. A branch and prune algorithm can be used for computing the solution set of $c_{\mathbf{a},\mathbf{r}}(x)$, but this algorithm will bisect again and again the boxes included inside the solution set, leading to inefficient computation.

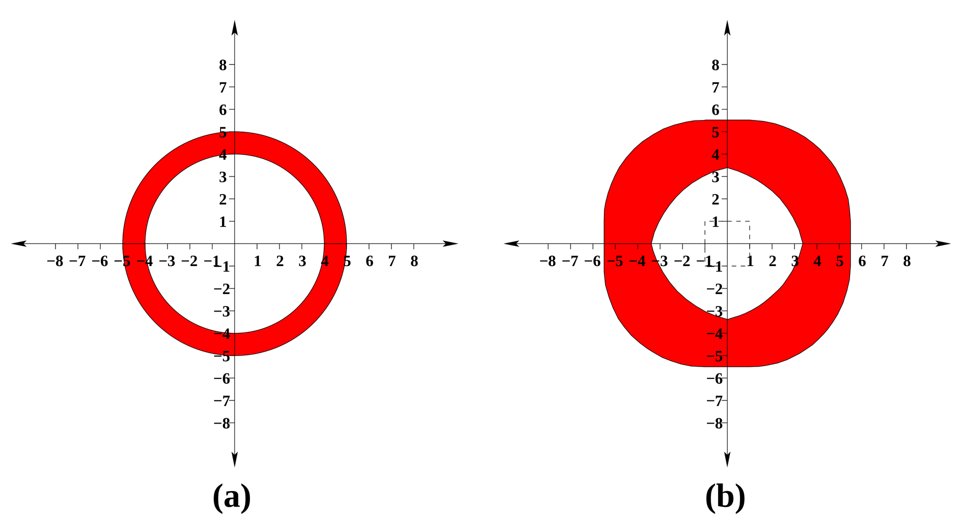

FIGURE 1: *Two quantified distance constraints and their solutions. In (a), the distance* $\mathbf{r} = [4, 5]$*. In (b) the center* $\mathbf{a} = ([-1, 1], [-1, 1])$*.*

Problem Statement

Let $\rho_{c_{\mathbf{a},\mathbf{r}}}$ be the set of x that satisfies (2). Given a box $\mathbf{x} \in \mathbb{IR}^n$, we are interested in a condition for:

$$(\forall x \in \mathbf{x})(x \in \rho_{c_{\mathbf{a},\mathbf{r}}}) \qquad (3)$$

A box that satisfies (3) is called an *inner box* (Figure 2).

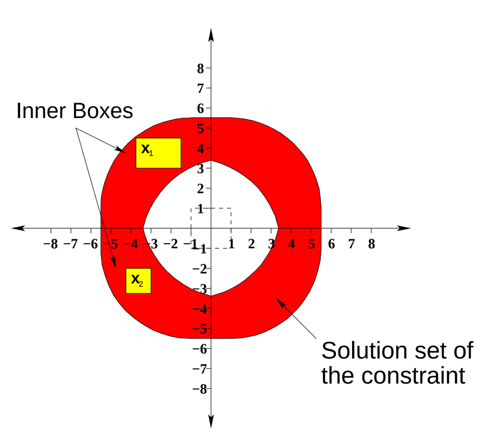

FIGURE 2: *Some examples of inner boxes (boxes included inside the solution set of* $c_{\mathbf{a},\mathbf{r}}(x)$*).*

Specific Quantifier Elimination (SQE)

A quantifier elimination algorithm transforms a quantified formula F into an *equivalent* quantifier-free formula F'. Our SQE algorithm ([2]) decomposes the constraint $c_{\mathbf{a},\mathbf{r}}(x)$ into two auxiliary constraints $c'_{\mathbf{a},\mathbf{r}}(x)$ and $c''_{\mathbf{a},\mathbf{r}}(x)$ (Figure 3).

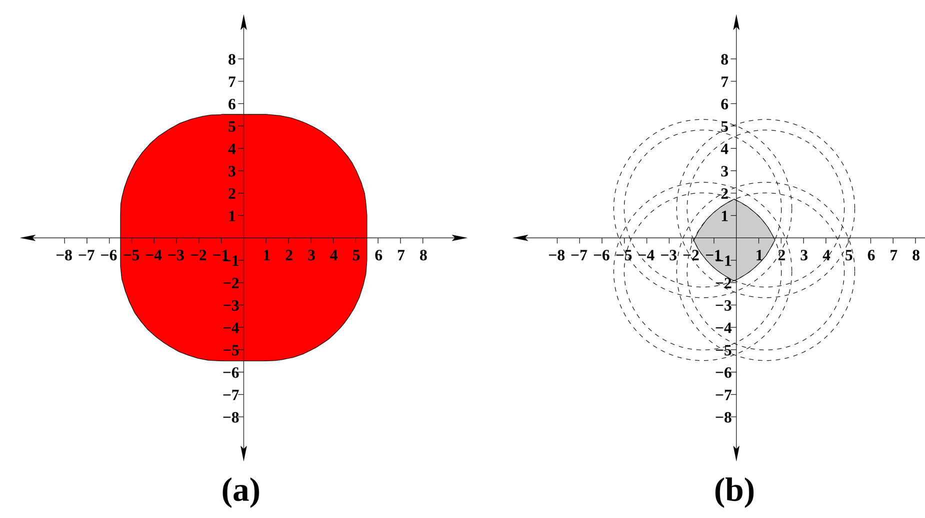

FIGURE 3: *A decomposition of the constraint* $c_{\mathbf{a},\mathbf{r}}(x)$ *into* $c'_{\mathbf{a},\mathbf{r}}(x)$ *(left side) and* $\neg c''_{\mathbf{a},\mathbf{r}}(x)$ *(right side).*

$c'_{\mathbf{a},\mathbf{r}}(x)$ is described as the union of six non-quantified constraints (Figure 4), while $c''_{\mathbf{a},\mathbf{r}}(x)$ is described as the intersection of four constraints. Classic interval arithmetics ([4]) is used for probing that $\mathbf{x} \subseteq \rho_{c'_{\mathbf{a},\mathbf{r}}(x)}$ and $\mathbf{x} \cap \rho_{c''_{\mathbf{a},\mathbf{r}}(x)} = \emptyset$.

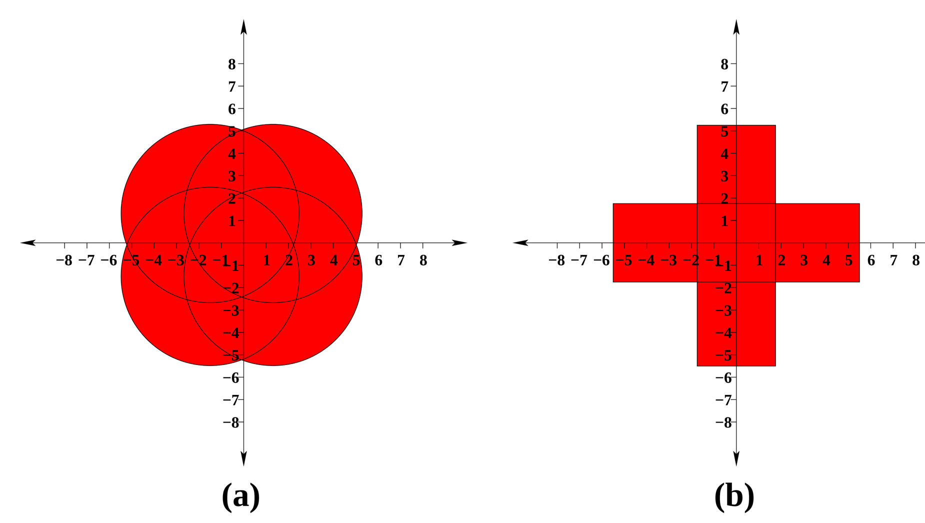

FIGURE 4: *A description of* $c'_{\mathbf{a},\mathbf{r}}(x)$ *with (a) four constraints of the form* $f(x) \leq \overline{r}$ *and (b) two inclusion constraints of the form* $\mathbf{x} \subseteq \mathbf{box}$*.*

Generalized Interval Evaluation (GIE)

A generalized interval $\mathbf{x} \in \mathbb{KR}$ is an interval whose bounds are not constrained to be ordered. An interval $\mathbf{x} = [a, b]$ is called *proper* if $a \leq b$ and *improper* if $a \geq b$.
Kaucher arithmetics ([3]) and the *(g,x)-interpretability* ([1]) allows one to compute generalized interval evaluation with special interpretation. For example:

$$[2,5] + [-1,1] = [1,6] \;\rightarrow\; \text{(classic interpretation)}$$
$$[2,5] + [2,-2] = [4,3] \;\rightarrow\; \text{(special interpretation)}$$

Here are the interpretations of the above three examples (based on the type of their intervals: *proper/improper*).

$$(\forall x \in [2,5])\ (\forall y \in [-1,1])\ (\exists z \in [1,6]) \quad (x + y = z)$$
$$(\forall x \in [2,5])\ (\forall z \in [3,4])\ \ (\exists y \in [-2,2])\ (x + y = z)$$

Considering each existentially quantified parameter as an improper interval and each variable as a proper one, we have the following inner box text:

$$\sum_{k=1}^{n}(x_k - a_k)^2 - r^2 \;\subseteq\; [0,0] \quad\Longrightarrow\quad \mathbf{x} \subseteq \rho_{c_{\mathbf{a},\mathbf{r}}} \qquad (4)$$

Preliminary Results

We notice that if existentially quantified parameters are not shared between different constraints, we have the following implication:

$$\bigwedge_{k\in[1..m]} \mathbf{x} \subseteq \rho_{c^{(k)}} \quad\Longrightarrow\quad \mathbf{x} \subseteq \bigcap_{k\in[1..m]} \rho_{c^{(k)}} \qquad (5)$$

We can use (5), for solving a system of distance constraints:

$$x^2 + y^2 = [2, 2.25]^2$$
$$(x - [3, 3.5])^2 + y^2 = [2.95, 3.05]^2$$
$$(x - [-2.5, -2.25])^2 + (y - 2)^2 = [3.25, 3.5]^2$$

where intervals denote existentially quantified parameters. Table 1 shows a comparison of a branch and prune algorithm using different inner box tests. The test based on SQE and GIE obtained the same solutions (Figure 5).

	No Test	SQE Test	GIE Test
Boxes	451655	5481	5481
Inner	–	2550	2550
Volume	0.21236	0.21236	0.21236
IVolume	–	0.21103	0.21103
Time (s)	36,08	0.53	0.43

TABLE 1: *A comparison of a branch and prune algorithm using different inner box tests.*

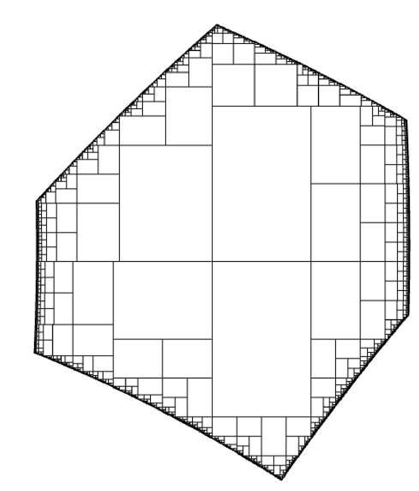

FIGURE 5: *Graphic results of a problem formed by three quantified distance constraints.*

Conclusion

Although both methods are very different, they raise very similar results about both computation times and description of the solution set, but the test based on GIE presents two advantages: it is much simpler to implement, and it can be trivially extended to QDC in arbitrary dimensions without loss of performance.
Both test must be considered as a sufficient condition for (3). As forthcoming work, a new inner test combining the two presented tests will be studied aiming to obtain an optimal test in all situations.

References

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