Near optimal interval observers bundle for uncertain bioreactors

Marcelo Moisan, Olivier Bernard and Jean-Luc Gouzé

***Abstract*— In this paper we design an interval observer for the estimation of unmeasured variables of uncertain bioreactors. The observer is based on a bounded error observer, as proposed in [1], that considered a loose approximation of the growing rate. We first show how to generate guaranteed upper and lower bounds on the state, provided that a known interval for the initial condition and the uncertainties is available. These so called "framers" depend on a tuning gain. They can be run in parallel and the envelope provides the best estimate. An optimality criterion is introduced leading to the definition of an optimal observer. We show that this criterion provides straighforwardly a gain set containing the best framers. The method is applied to the estimation of the total biomass of an industrial waste water treatment plant, demonstrating its efficiency.**

I. INTRODUCTION

Optimization of biotechnological processes and in particular of wastewater treatment plants is nowadays receiving an increasing attention. It motivated the design of new control strategies [14], [2] that guarantee a better process working and efficiency. However these controllers often require high quality measurements or efficient state estimation procedures. On one hand the difficulty to have online reliable sensors that could provide the key variables is a well known limitation of biotechnological processes. On the other hand, robustness of the controller to state uncertainties becomes a critical requirement in the framework of strong disturbances and high uncertainties that characterize bioprocesses.

Designing a state estimator robust to large process uncertainties is therefore a central issue, and classical methods may be hardly directly applicable to bioprocesses [18]. There was recently many advances on this topic, introducing for example novel uncertainty structures [4], [5] which have helped the development of robust estimation schemes.

Several estimation methods that deal with uncertainty can be found in the current literature. Most of them rely on classical approaches like estimation through ellipsoidal sets [8], robust Kalman filtering [17], $\mathcal{H}_\infty$ or $\mathcal{H}_2$ filtering methods [16].

A popular class of observers for mass balance based models are based on the principle of observers with unknown inputs [15]. The particular structure associated to the mass balance modeling is exploited and the biological kinetics (one of the most critical uncertainties of the system) are considered as an unknown input and eliminated. This leads to the formulation of asymptotic observers [6]. As a consequence, these observers are very robust with respect to the unknown or poorly modeled kinetics. However their convergence rates are fixed and linked to the operating mode of the system. Recently, a hybrid observer combining an asymptotic and a high gain observer has been proposed by [1]. This observer assumes a loose bounding of the bacterial growth rate function and allows to accelerate the convergence rate. A gain θ is tuned at a high positive value when the estimate is far from the real state, and then comes back to the slow but accurate asymptotic observer once the estimate achieved an appropriate region.

An alternative approach to manage uncertainty consists in using interval methods [7]. This approach was mainly developed for discrete time systems and more recently for continuous systems, based on properties of positive differential systems [20], [10], [13]. The idea consists in bounding the state by solutions of dynamical systems that satisfy some cooperativity conditions [11]. If the initial condition is sure to lie in a given interval, then the state is guaranteed to be bounded between an upper and a lower interval.

We propose here an extension of this last approach, in the framework of the nonlinear observers introduced by [1]. We take benefit of the possibility to tune the error dynamics to improve the convergence properties. For this, we run in parallel a broad set of interval observers that all guarantee a bounding of the real state. It is then possible to select the inner envelope of the so called *observer bundle* to obtain a new observer with much better convergence rate and much smaller interval predictions [12]. It is worth noting that among the set of interval observers that are run in parallel, some may be unstable, but providing however a better estimate during first transients associated to the fastest eigenvalues.

Additionally, in this paper we determine a criterion that leads to an optimal gain for providing the best estimates. We show that this unique optimal gain cannot be computed without the state knowledge. However a gain interval which contain the optimal gain can be determined. This allows us to reduce the bundle size and increase its accuracy and convergence rate.

This work is organized as follows. In section II we introduce a bioreactor model and some hypotheses about the system. Section III recalls a bounded error observer and proposes an interval observer highlighting some of its main features. In section IV we explain how to improve the interval estimates by using a bundle of observers and regular reinitialisation. In section V we apply a simple criterion in order to obtain a characterization of the space of gains that performs the best estimates. The application of the method to an industrial wastewater treatment plant is shown in section VII.

Marcelo Moisan, Olivier Bernard and Jean-Luc Gouzé are with the French Institute of Research on Computer Science and Automatic Control, Comore Research Team. 2004 route des Lucioles, Sophia–Antipolis, France. `[mmoisan,obernard,gouze]@sophia.inria.fr`

II. Definitions and Hypotheses

We consider the following widely used model [6], which describes the behavior of the concentrations of a biomass x, a substrate s and a product p in a perfectly mixed bioreactor:

$$\begin{array}{l} \dot{x} = \mu(s)x - ux \\ \dot{s} = u(s_{in} - s) - k\mu(s)x \\ \dot{p} = \lambda\mu(s)x - up \\ y = s \end{array} \quad (1)$$

s_{in} corresponds to the concentration of influent substrate, u is the dilution input and k and λ are conversion yield coefficients. The biological activity of the system are featured by the non negative growth rate function $\mu(s)$ such that $\mu(0) = 0$, known to be highly uncertain. For the sake of simplicity $\mu(s)$ is assumed to be a $\mathcal{C}^1$ function.

Property 1: For any nonnegative initial conditions, trajectories of system (1) remain bounded and positive for any positive time.

Proof:
1) Positivity of the system is trivial if $\mu(0) = 0$ and $\mu(s) \geq 0$, $\forall s \geq 0$.
2) Consider $z_1 = s + kx$ and $z_2 = p + \lambda x$. It is clear that $\lim_{t\mapsto\infty} z_1 = s_{in}$ and $\lim_{t\mapsto\infty} z_2 = 0$, for fixed u and s_{in}, which proves the boundedness of the state vector. ■

In the sequel we assume that the following hypotheses are fulfilled.

Hypothesis 1: The dilution rate u is a persistent input: for $u \geq 0$ there exist α and β positive constants such that:

$$\int_t^{t+\alpha} u(\tau)d\tau \geq \beta > 0 \quad (2)$$

Hypothesis 2: We assume that the growth rate is bounded by two known functions $\underline{\mu}(s)$ and $\overline{\mu}(s)$ and a positive constant a:

1. $0 \leq \underline{\mu}(s) \leq \mu(s) \leq \overline{\mu}(s) \leq a, \quad \forall s \geq 0$
2. $\overline{\mu}(0) = 0$

Now our objective is to develop a robust observer based on interval approach with improved convergence properties in order to estimate the biomass concentration $x(t)$ in a reactor, when monitoring the substrate $s(t)$ (the same analysis holds for the estimation of the variable p, but for the sake of brevity this case is not developed here).

III. Bounded error interval observer

A. Bounded error observers

We consider the class of observers introduced in [1]. These observers do not converge to zero but to a bounded error.

Definition 1: A bounded error observer of system (1) is a dynamical system

$$\dot{\hat{x}} = f(\hat{x}, u, y) \quad \text{with} \lim_{t\to\infty} ||\hat{x} - x|| \leq r \quad (3)$$

where r is positive real constant (depending on the uncertainty on μ) such that $k = 0$ if $\mu(s)$ is perfectly known.

A bounded error observer for the variable x of system (1) can be derived, considering that the system dynamics (growth rate) are poorly known. Let us introduce the transformation:

$$z = kx + \theta(t)s \quad (4)$$

where $\theta(t) \in \mathcal{C}^1(\mathbb{R})$ is a gain that will be discussed later on. Considering equation (1), the dynamics of z can be written as follows:

$$\dot{z} = (1-\theta)\mu(s)(z - \theta s) + u(\theta s_{in} - z) + \dot{\theta}s \quad (5)$$

Then, since the influent substrate s_{in} is well known, the following bounded error observer can be derived:

Proposition 1: The following system is a bounded error observer of (1)

$$\begin{array}{l} \dot{\hat{z}} = (1-\theta)\hat{\mu}(s)(\hat{z} - \theta s) + u(\theta s_{in} - \hat{z}) + \dot{\theta}s \\ \hat{x} = (\hat{z} - \theta s)/k \end{array} \quad (6)$$

where the function $\hat{\mu}(s)$ is such that $|\hat{\mu}(s) - \mu(s)| < a$, and a is a positive real.

Proof: See [1]. ■

One can note that function $\hat{\mu}(s)$ can be *e.g.* any of the known functions $\underline{\mu}(s)$ or $\overline{\mu}(s)$ introduced in hypothesis 2, fulfilling the previously mentioned condition.

Remark 1: If $\theta = 1$ then equation (6) becomes the classical asymptotic observer (see [6]).

The idea used in [1] was to implement an *hybrid* observer based on equation (6). The estimations start with a high and positive value of θ which decreases down to one along the time. From a methodological point of view, the hybrid approach combines two type of observers: a bounded error observer with high convergence rate but poor accuracy and an asymptotic observer with fixed convergence rate but high accuracy. Even though this observer showed to improve the properties of asymptotic observers, adjusting the gain θ seems rather complicated. This last issue can be overcome using properties of interval estimates, as explained in the next sections. In what follows we consider the following hypothesis.

Hypothesis 3: The influent substrate s_{in} is an unknown bounded input of system (1). However, we know bounds for this input such that:

$$s_{in}^-(t) \leq s_{in}(t) \leq s_{in}^+(t) \quad (7)$$

We denote $e_{in}^- = s_{in} - s_{in}^-$, $e_{in}^+ = s_{in}^+ - s_{in}$ and $e_{in} = s_{in}^+ - s_{in}^-$.

B. Interval observers

The objective of an interval observer is to generate guaranteed bounds of the unknown state variables. For this we will introduce the concept of *framer*:

Definition 2: A *framer* for system (1) is a pair of coupled dynamical systems

$$\begin{array}{ll} \dot{\overline{x}} = \overline{f}(\overline{x}, \underline{x}, \theta, y), & \overline{x}(0) = \overline{x}_0 \\ \dot{\underline{x}} = \underline{f}(\overline{x}, \underline{x}, \theta, y), & \underline{x}(0) = \underline{x}_0 \end{array} \quad (8)$$

such that, for an initial conditions verifying $\underline{x}_0 \leq x_0 \leq \overline{x}_0$ we have $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$.

Note that this definition is rather general, highlighting the fact that a framer is conceived to give an upper and a lower bound of the unknown state. Stability can be considered as a second feature of a framer, to guarantee a bounded interval and obtain an interval observer. This point is more deeply discussed hereafter.

Let us propose analytical expressions for framers of the variable x of system (1). These expressions are deduced from equation (6).

Proposition 2: Given $\underline{x}_0, \overline{x}_0$ such that $x_0 \in [\underline{x}_0, \overline{x}_0]$, then for a gain $\theta \in \mathcal{C}^1(\mathbb{R})$, the following system defines a framer for system (1). The framer depends on the value of θ as follows:

- for $\theta < 0$

$$\begin{aligned}\dot{\overline{z}}_\theta &= (1-\theta)(\overline{\mu}(s)\overline{z}_\theta - \theta\overline{\mu}(s)s) + u(\theta s_{in}^- - \overline{z}_\theta) + s\dot{\theta}\\ \dot{\underline{z}}_\theta &= (1-\theta)(\underline{\mu}(s)\underline{z}_\theta - \theta\underline{\mu}(s)s) + u(\theta s_{in}^+ - \underline{z}_\theta) + s\dot{\theta}\end{aligned} \tag{9}$$

- for $0 \leq \theta < 1$

$$\begin{aligned}\dot{\overline{z}}_\theta &= (1-\theta)(\overline{\mu}(s)\overline{z}_\theta - \theta\underline{\mu}(s)s) + u(\theta s_{in}^+ - \overline{z}_\theta) + s\dot{\theta}\\ \dot{\underline{z}}_\theta &= (1-\theta)(\underline{\mu}(s)\underline{z}_\theta - \theta\overline{\mu}(s)s) + u(\theta s_{in}^- - \underline{z}_\theta) + s\dot{\theta}\end{aligned} \tag{10}$$

- for $\theta \geq 1$

$$\begin{aligned}\dot{\overline{z}}_\theta &= (1-\theta)(\underline{\mu}(s)\overline{z}_\theta - \theta\overline{\mu}(s)s) + u(\theta s_{in}^+ - \overline{z}_\theta) + s\dot{\theta}\\ \dot{\underline{z}}_\theta &= (1-\theta)(\overline{\mu}(s)\underline{z}_\theta - \theta\underline{\mu}(s)s) + u(\theta s_{in}^- - \underline{z}_\theta) + s\dot{\theta}\end{aligned} \tag{11}$$

with

$$\overline{x}_\theta = (\overline{z} - \theta s)/k \quad \text{and} \quad \underline{x}_\theta = (\underline{z} - \theta s)/k \tag{12}$$

Proof: The proof is carried out for $\theta \geq 1$ (the same arguments hold for the other framers). Let us consider the difference $\overline{e}$ between the upper candidate estimate and the unknown state, this is:

$$\overline{e} = \overline{x}_\theta - x = (\overline{z}_\theta - z)/k \tag{13}$$

Its dynamics is:

$$\begin{aligned}\dot{\overline{e}} \;=\; & (1-\theta)((\underline{\mu}(s)\overline{z}_\theta - \mu(s)z) - \theta(\overline{\mu}(s) - \mu(s))s)\\ & + u(\theta e_{in}^+ - k\overline{e})\end{aligned} \tag{14}$$

Now we show that $\overline{e}$ stays positive. We consider the first time instant $t^\star$ where $\overline{e} = 0$ *i.e.* $\overline{x} = x = x^\star$ and $\overline{z}_\theta = z = z^\star$, then:

$$\dot{\overline{e}}(t^\star) = (1-\theta)((\underline{\mu}(s) - \mu(s))z^\star - \theta(\overline{\mu}(s) - \mu(s))s) + u\theta e_{in}^+ \tag{15}$$

Considering that $z^\star$ is positive (because of the positivity of the state and the gain θ) and hypothesis (2), we have:

$$\dot{\overline{e}}(t^\star) \geq 0 \tag{16}$$

which guarantees that the error will stay positive after $t^\star$. Using similar arguments it is possible to show that $\underline{e}(t) \geq 0$ and therefore the interval that contains the state is positive:

$$e = \overline{x}_\theta - \underline{x}_\theta = \overline{e} + \underline{e} \geq 0 \tag{17}$$

These arguments are well known in the field of positive and monotone differential systems [11]. ■

Let us detail two cases of specific interest.

C. The cases $\theta = 0$ and $\theta = 1$

The framer for $\theta = 1$ is the interval version of the classical asymptotic observer [6]: it does not depends on the biological kinetics, however it has a fixed convergence rate given by u, provided that hypothesis (1) is satisfied.

Property 2: The framer $[\underline{x}_{\theta=1}(t), \overline{x}_{\theta=1}(t)]$ provides a bounded interval estimation of variable $x(t)$.

Proof: This property is straightforward and relies on the boundness of s_{in}. ■

Property 3: The framer $\underline{x}_{\theta=0}$ is positive, if $\underline{x}_{\theta=0}(0) \geq 0$.

Proof: The proof is trivial. ■

Note that this framer may be unstable.

As a consequence, the framers obtained for $\theta = 0$ and $\theta = 1$ will provide a guaranteed upper and lower bound for the state.

IV. Observers bundle and reinitialization

One of the key properties related to an interval observers is that we can compare the solutions generated by two or more framers. We run therefore several framers in parallel (for different values of the gain θ), which all provide guaranteed interval estimates for the state.

Definition 3: An observer bundle is a set of interval estimates generated by a finite set of framers, using different values of the gain θ.

$$\begin{aligned}\overline{\mathcal{B}}(t) &= \{\overline{x}_\theta(t) : \theta(t) \in \Theta\}\\ \underline{\mathcal{B}}(t) &= \{\underline{x}_\theta(t) : \theta(t) \in \Theta\}\end{aligned} \tag{18}$$

where Θ is a subset of $\mathcal{C}^1 : \mathbb{R} \mapsto \mathbb{R}$ functions.

$\overline{\mathcal{B}}$ is the upper bundle and $\underline{\mathcal{B}}$ is the lower bundle. Each bundle has an envelope that provides the best bounds:

$$\overline{\mathcal{B}}_{\text{inf}}(t) = \min\{\overline{\mathcal{B}}(t)\} \tag{19}$$

$$\underline{\mathcal{B}}_{\text{sup}}(t) = \max\{\underline{\mathcal{B}}(t)\} \tag{20}$$

That is, we take the inner envelope from all the set of estimates generated by different gain values. It is worth noting that we combine transient behavior of some unstable framers (that may improve transient estimations), with the asymptotic stability of others (that guarantees the boundness of the envelope).

Property 4: The interval $\mathcal{I} = [\underline{\mathcal{B}}_{\text{sup}}, \overline{\mathcal{B}}_{\text{inf}}]$ is bounded if $\theta(t) \equiv 1 \in \Theta$.

Proof: First, it is worth to point out that if one framer is bounded, then $\underline{\mathcal{B}}_{\text{sup}}$ and $\overline{\mathcal{B}}_{\text{inf}}$ are bounded too. This is the main advantage of running several guaranteed estimates in parallel. Boundedness of the interval $\mathcal{I}$ is a direct consequence of Property 2. ■

Property 5: The lower bound best value $\underline{\mathcal{B}}_{\text{sup}}$ is positive if $\theta \equiv 0 \in \Theta$.

Proof: This is a straight consequence of Property 3, which provides a lower positive framer $\forall t$. ■

A. Reinitialisation

A regular reinitialization of the bundle can be performed to restart all the framers with the best available interval predicted by $\underline{\mathcal{B}}_{\text{sup}}$ and $\overline{\mathcal{B}}_{\text{inf}}$. We consider the time interval $[t_k, t_k + \Delta_t]$ (we denote by Δ_t the reinitialisation time interval) where the framers run. Then at the time instant t_k we take the best interval estimates performed by the previous estimation period to reinitialize the whole bundle.

$$[\underline{x}_\theta^0(t_k), \overline{x}_\theta^0(t_k)] = [\underline{\mathcal{B}}_{\text{sup}}(t_k), \overline{\mathcal{B}}_{\text{inf}}(t_k)] \tag{21}$$

The objective behind the regular reinitialisation of the interval estimates is to improve the framer efficiency by feeding it with the best available estimate, and thus take benefit of the transients of some of them.

B. Convergence index

One of the advantages of obtaining interval estimates is that we can assess the observer convergence by comparing upper and lower bound estimates. This leads to a convergence index ϑ, which corresponds to a normalized comparison of the best estimates:

$$\vartheta(t) = \frac{\overline{\mathcal{B}}_{\text{inf}}(t) - \underline{\mathcal{B}}_{\text{sup}}(t)}{\overline{\mathcal{B}}_{\text{inf}}(t) + \underline{\mathcal{B}}_{\text{sup}}(t)} \tag{22}$$

V. Computing the optimal gain

Running in parallel a dense enough bundle (*i.e.* with a large number of values of θ) in order to obtain the best interval $[\underline{\mathcal{B}}_{\text{sup}}, \overline{\mathcal{B}}_{\text{inf}}]$ can be time consuming without guarantee of optimal estimation. Therefore, a characterization of the gain set that can generate the best solutions is proposed.

A. Definition of an optimality criterion

Let us assume that at time t the best estimate $[\underline{x}_\theta^0(t), \overline{x}_\theta^0(t)]$ is provided. The best upper [resp. lower] framer at t will be the one with the minimal [resp. maximal] slope (see fig. 1). It will thus be given by the value of θ that minimizes [resp. maximizes] $\dot{\overline{x}}$ [resp. $\dot{\underline{x}}$]. The proposed criterion consists then in finding a pair of gains $\overline{\theta}$ and $\underline{\theta}$ which respectively minimizes and maximizes $\dot{\overline{x}}$ and $\dot{\underline{x}}$ at any time instant t.

$$\begin{array}{l} \dot{\overline{x}}_\theta = (\dot{\overline{z}}_\theta - \theta(t)\dot{s} - \dot{\theta}(t)s)/k \\ \dot{\underline{x}}_\theta = (\dot{\underline{z}}_\theta - \theta(t)\dot{s} - \dot{\theta}(t)s)/k \\ \text{such that:} \\ \dot{\overline{x}}_{\overline{\theta}(s,x)} = \min_\theta\{\dot{\overline{x}}_\theta\} \\ \dot{\underline{x}}_{\underline{\theta}(s,x)} = \max_\theta\{\dot{\underline{x}}_\theta\} \end{array} \tag{23}$$

Functions $\dot{\underline{z}}_\theta$ and $\dot{\overline{z}}_\theta$ have a piecewise form given by equations (9), (10) and (11). Using these expressions, we can write $\dot{\overline{x}}_\theta = \overline{f}(x, \overline{x}, s, \theta)$ and $\dot{\underline{x}}_\theta = \underline{f}(x, \underline{x}, s, \theta)$ as polynomials with respect to the gain θ:

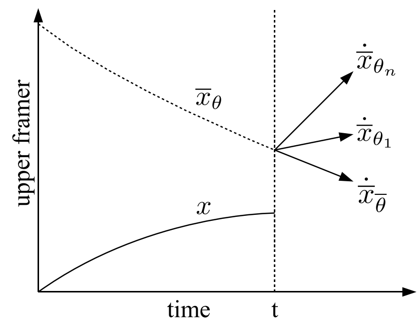

Fig. 1. Concept associated to the optimization criterion (23).

$$\begin{array}{l} \overline{f}_i(x, \overline{x}, s, \theta) = \overline{a}_i\theta^2 + \overline{b}_i\theta + \overline{c}_i \\ \underline{f}_i(x, \underline{x}, s, \theta) = \underline{a}^i\theta^2 + \underline{b}^i\theta + \underline{c}^i \end{array} \tag{24}$$

where $i = 1$ for $\theta < 0$, $i = 2$ for $0 \le \theta < 1$ and $i = 3$ for $\theta \ge 1$. See the appendix for a detailed computation of the coefficients of equation (24). Fig. 2 provides a graph of function $\overline{f}$. It can be easily verified that for $\theta \ge 1$ there exists a global nontrivial solution that minimizes $\dot{\overline{x}}$ [resp. maximizes $\dot{\underline{x}}$] denoted $\underline{\beta}$ [resp. $\overline{\beta}$]. On the other hand, for $\theta < 1$, the optimal gain value is $\theta = 0$. Thus, there are two candidates to solve the optimization problem (23) for $\theta \in \mathbb{R}$, (see fig. 2):

$$\underline{\theta}(t) = \{0, \max\{1, \underline{\beta}\}\} \tag{25}$$

$$\overline{\theta}(t) = \{0, \max\{1, \overline{\beta}\}\} \tag{26}$$

where

$$\begin{array}{l} \underline{\beta}(t) = \dfrac{s\Delta_\mu - ue_{in}^- + k(\mu(s)x - \overline{\mu}(s)\underline{x})}{2s\Delta_\mu} \\ \overline{\beta}(t) = \dfrac{s\Delta_\mu - ue_{in}^+ - k(\mu(s)x - \underline{\mu}(s)\overline{x})}{2s\Delta_\mu} \end{array} \tag{27}$$

and $\Delta_\mu = \overline{\mu}(s) - \underline{\mu}(s)$.

Remark 2: All the framers based on gains which belong to the set $\mathcal{A} = \{\theta \in]-\infty, 1[-\{0\}\}$ will never provide the best estimate. As a consequence we eliminate equation (9) and consider equation (10) only for the case $\theta \equiv 0$ to compute the interval observer.

Now it is very important to note that the optimal solution $\underline{\theta}$ and $\overline{\theta}$ cannot be computed because they depend on the unknown state $x(t)$. We must therefore try to localize the region where the optimal values for θ lie, and run as many framers as possible in this region. Indeed, intervals that contain the nontrivial optimal solution $\underline{\beta}$ and $\overline{\beta}$ can be determined on the basis of the present state estimate. Let us focus on the set $\theta \ge 1$ (where $\underline{\beta}$ and $\overline{\beta}$ live) to define the bounds of this interval.

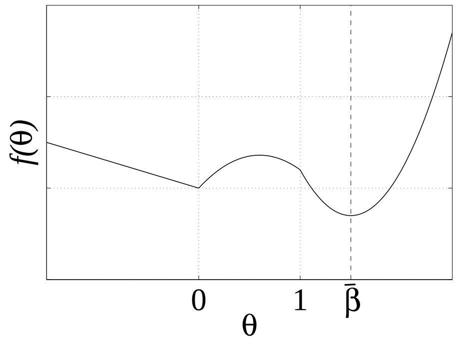

Fig. 2. Piecewise function $\overline{f}(\theta)$.

B. Bounding of the optimal gain at time t

The objective is to compute the sets that contains the unknown optimal gain values, and run an observer bundle considering these values, to get as close as possible to the optimal framer.

The following hypothesis is not mandatory, but it strongly simplifies the computation of the optimal interval:

Hypothesis 4: The unknown function $\mu(s)$ can be written as $\mu(s) = \gamma\rho(s)$, where $\rho(s)$ is a known $\mathcal{C}^\infty$ function and $\gamma = [\underline{\gamma}, \overline{\gamma}]$.

Property 6: The sets $\underline{\Omega}$ and $\overline{\Omega} \subset \mathbb{R}$ defined by:

$$\underline{\Omega}(t) = \left[1, \underline{\varphi}(t)\right] \quad \text{and} \quad \overline{\Omega}(t) = [1, \overline{\varphi}(t)] \tag{28}$$

with

$$\underline{\varphi}(t) = \frac{1}{2} + \frac{k\overline{\gamma}\Delta_{\mathcal{B}}}{2(\overline{\gamma}-\underline{\gamma})} \quad \text{and} \quad \overline{\varphi}(t) = \frac{1}{2} + \frac{k\underline{\gamma}\Delta_{\mathcal{B}}}{2(\overline{\gamma}-\underline{\gamma})} \tag{29}$$

contain $\underline{\beta}$ and $\overline{\beta}$ for any time t, where $\Delta_{\mathcal{B}} = \overline{\mathcal{B}}_{\text{inf}} - \underline{\mathcal{B}}_{\text{sup}}$.

Proof: At time t, we have that $\underline{\mathcal{B}}_{\text{sup}}(t) \leq x(t) \leq \overline{\mathcal{B}}_{\text{inf}}(t)$. After some algebraic arrangements on equation (27) the sets $\underline{\Omega}$ and $\overline{\Omega}$ can be written as:

$$\underline{\Omega} \subset \left[\frac{-ue_{in}}{2s\Delta_\mu} + \frac{1}{2}\left(1 - k\frac{\underline{\mathcal{B}}_{\text{sup}}}{s}\right), \frac{1}{2} + \frac{k\overline{\mu}(s)\Delta_{\mathcal{B}}}{2\Delta_\mu}\right] \tag{30}$$

$$\overline{\Omega} \subset \left[\frac{-ue_{in}}{2s\Delta_\mu} + \frac{1}{2}\left(1 - k\frac{\overline{\mathcal{B}}_{\text{inf}}}{s}\right), \frac{1}{2} + \frac{k\underline{\mu}(s)\Delta_{\mathcal{B}}}{2\Delta_\mu}\right] \tag{31}$$

Let us focus on the lower bound of these intervals. Given that $\underline{\mathcal{B}}_{\text{sup}}(t)$, $\overline{\mathcal{B}}_{\text{inf}}(t)$ and $s(t)$ are non negative quantities, then:

$$\frac{1}{2}\left(1 - k\frac{\underline{\mathcal{B}}_{\text{sup}}}{s}\right) \leq \frac{1}{2} \quad \text{and} \quad \frac{1}{2}\left(1 - k\frac{\overline{\mathcal{B}}_{\text{inf}}}{s}\right) \leq \frac{1}{2} \tag{32}$$

As $\frac{-ue_{in}}{2s\Delta_\mu} < 0$, it follows that the computed lower bounds can not be greater than the trivial value $\theta = 1$. Now, considering the upper bound of intervals (30), (31) and hypothesis (4), we have:

$$\frac{\underline{\mu}(s)}{2\Delta_\mu} = \frac{\underline{\gamma}}{2(\overline{\gamma}-\underline{\gamma})} \quad \text{and} \quad \frac{\overline{\mu}(s)}{2\Delta_\mu} = \frac{\overline{\gamma}}{2(\overline{\gamma}-\underline{\gamma})} \tag{33}$$

This leads to the intervals given by equation (29). ■

C. Bounding of the optimal gain on the interval $[t_k,\ t_{k+1}]$

We have provided an interval where the optimal gain is sure to lie at time t, now we must be sure that this interval will still contain the optimal gain for any time in the interval $[t_k,\ t_{k+1}]$.

We will thus recompute the optimal gain interval on $[t_k,\ t_{k+1}]$, on the basis of the state estimation provided by the framers.

Our strategy consists in splitting the interval (28) into N parts in order to get $N+1$ gain values θ_i.

$$\begin{aligned} \underline{\theta}_i &= \frac{i + \underline{\varphi}(t)(N-i)}{N} \quad \text{and} \\ \overline{\theta}_i &= \frac{i + \overline{\varphi}(t)(N-i)}{N} \quad \text{for } i \in \{0,..,N\} \end{aligned} \tag{34}$$

As a consequence, the values of θ_i will not be constants and we need to estimate $\dot{\theta}_i$. The quantity $\dot{\theta}_i$ involves the computation of $\dot{\underline{\varphi}}(t)$ and $\dot{\overline{\varphi}}(t)$. The term $\dot{\Delta}_{\mathcal{B}}$ depends on the output derivative $\dot{s}(t)$ which cannot be properly online computed. To solve this problem, we propose a larger bound for the gain interval whose derivative is independant on $\dot{s}(t)$. This is simply achieved considering the state estimate provided by $\theta = 0$.

Considering the bounds already computed for the non trivial optimal gain of equations (28) and (29), let us define the following sets:

$$\underline{\Omega}_{\theta=0}(t) = \left[1, \underline{\underline{\varphi}}(t)\right] \quad \text{and} \quad \overline{\Omega}_{\theta=0}(t) = \left[1, \overline{\overline{\varphi}}(t)\right] \tag{35}$$

where

$$\begin{aligned} \underline{\underline{\varphi}}(t) &= \frac{1}{2} + \frac{k\overline{\gamma}(\overline{x}_{\theta=0}(t) - \underline{x}_{\theta=0}(t))}{2(\overline{\gamma}-\underline{\gamma})} \\ \overline{\overline{\varphi}}(t) &= \frac{1}{2} + \frac{k\underline{\gamma}(\underline{x}_{\theta=0}(t) - \underline{x}_{\theta=0}(t))}{2(\overline{\gamma}-\underline{\gamma})} \end{aligned} \tag{36}$$

Property 7: $\underline{\Omega}(t) \subset \underline{\Omega}(t)_{\theta=0}$ and $\overline{\Omega}(t) \subset \overline{\Omega}(t)_{\theta=0}$.

Proof: It is clear from the properties of an observer bundle that $\underline{\mathcal{B}}_{\text{sup}}(t) \geq \underline{x}_\theta(t)$ and $\overline{\mathcal{B}}_{\text{inf}}(t) \leq \overline{x}_\theta(t)$, for any θ. In particular for $\theta = 0$ one has straightforwardly that $\Delta_{\mathcal{B}}(t) \leq \overline{x}_{\theta=0}(t) - \underline{x}_{\theta=0}(t)$ and then property 7 holds. ■

Now we can continuously compute $\underline{\underline{\varphi}}(t)$ and $\overline{\overline{\varphi}}(t)$ for all $t \in [t_k, t_{k+1}]$, considering that $\dot{\overline{x}}_{\theta=0}(t) - \dot{\underline{x}}_{\theta=0}(t) = g(\underline{x}_{\theta=0}, \overline{x}_{\theta=0}, s) = (\underline{\mu}(s) - u)\Delta_x + \Delta_\mu \overline{x}$. The dynamics of $\dot{\underline{\underline{\varphi}}}$ and $\dot{\overline{\overline{\varphi}}}$ can be expressed by the system:

$$\begin{aligned} \begin{bmatrix} \dot{\underline{\underline{\varphi}}} \\ \dot{\overline{\overline{\varphi}}} \end{bmatrix} &= \begin{bmatrix} \dfrac{k\overline{\gamma}}{2(\overline{\gamma}-\underline{\gamma})} \\ \dfrac{k\underline{\gamma}}{2(\overline{\gamma}-\underline{\gamma})} \end{bmatrix} g(\underline{x}, \overline{x}, s), \\ \begin{bmatrix} \underline{\underline{\varphi}}(t_k) \\ \overline{\overline{\varphi}}(t_k) \end{bmatrix} &= \begin{bmatrix} \max\{1, \underline{\varphi}(t_k)\} \\ \max\{1, \overline{\varphi}(t_k)\} \end{bmatrix} \end{aligned} \tag{37}$$

It is worth to remark that the system (37) is initialized at every reinitialization time instant t_k using the values computed by the instantaneous gain bounds of equation (28). This becomes critical in order to keep a good gain bounding and therefore obtain a bundle with framers close to the optimal unknown gain.

VI. Biased output

Now we consider a bounded noise affecting the system output.

Hypothesis 5: Online measurement $s(t)$ is perturbed by a noise $\delta(t)$. We assume that this perturbation is of multiplicative nature: $y(t) = s(t)(1 + \delta(t))$.

Moreover, the noise is bounded by $\Delta \in \mathbb{R}^+$ such that $|\delta| \leq \Delta < 1$.

Considering that $s(t)$ is a positive variable, the last hypothesis implies that we know two bounds for this quantity:

$$\frac{y(t)}{(1+\Delta)} \leq s(t) \leq \frac{y(t)}{(1-\Delta)} \tag{38}$$

We rewrite the framer equations (11) and (10) (see remark 2) taking into account the output uncertainty:

Proposition 3: Given $\underline{x}_0, \overline{x}_0$ such that $x_0 \in [\underline{x}_0, \overline{x}_0]$, then the following system is a framer of system (1):

- for $\theta \geq 1$

$$\begin{aligned}
\dot{\overline{z}}_\theta &= (1-\theta)(\underline{\nu}(\underline{y},\overline{y})\overline{z}_\theta - \theta\overline{\nu}(\underline{y},\overline{y})\overline{y}) \\
&\quad + u(\theta s_{in}^+ - \overline{z}_\theta) + \dot{\theta}(\varepsilon\overline{y} + (1-\varepsilon)\underline{y}) \\
\dot{\underline{z}}_\theta &= (1-\theta)(\overline{\nu}(\underline{y},\overline{y})\underline{z}_\theta - \theta\underline{\nu}(\underline{y},\overline{y})\underline{y}) \\
&\quad + u(\theta s_{in}^- - \underline{z}_\theta) + \dot{\theta}(\varepsilon\underline{y} + (1-\varepsilon)\overline{y}) \\
\overline{x}_\theta &= (\overline{z}_\theta - \theta\underline{y})/k \quad \text{and} \quad \underline{x}_\theta = (\underline{z}_\theta - \theta\overline{y})/k
\end{aligned} \tag{39}$$

- for $\theta = 0$:

$$\begin{aligned}
\dot{\overline{x}}_{\theta=0} &= (\overline{\nu}(\underline{y},\overline{y}) - u)\overline{x}_{\theta=0} \\
\dot{\underline{x}}_{\theta=0} &= (\underline{\nu}(\underline{y},\overline{y}) - u)\underline{x}_{\theta=0}
\end{aligned} \tag{40}$$

This means that for $\underline{x}_0 \leq x_0 \leq \overline{x}_0 \Rightarrow \underline{x}(t) \leq x(t) \leq \overline{x}(t)$. The functions $\underline{\nu}(.)$ and $\overline{\nu}(.)$ are defined such that:

$$\underline{\nu}(\underline{y},\overline{y}) = \min_{q\in[\underline{y},\overline{y}]}\{\underline{\mu}(q)\} \quad \text{and} \quad \overline{\nu}(\underline{y},\overline{y}) = \max_{q\in[\underline{y},\overline{y}]}\{\overline{\mu}(q)\} \tag{41}$$

and $\varepsilon = \begin{cases} 1 & \text{if } \dot{\theta} \geq 0 \\ 0 & \text{otherwise} \end{cases}$, uses the sign of $\dot{\theta}$ in order to provide the correct bounding.

Proof: The same arguments as proof of proposition 2 can be applied. Considering the upper bound error equation for $\theta \geq 1$ at the time instant $t^\star$ where $\overline{e} = 0$ we have:

$$\begin{aligned}
\dot{\overline{e}}(t^\star) &= (1-\theta)((\underline{\nu}(\underline{y},\overline{y}) - \mu(s))z^\star + \theta(\overline{\nu}(\underline{y},\overline{y})\overline{y} \\
&\quad -\mu(s)s)) + u\theta e_{in}^+ + \dot{\theta}(\varepsilon\overline{y} + (1-\varepsilon)\underline{y} - s)
\end{aligned} \tag{42}$$

which under the stated assumptions verifies $\dot{\overline{e}}(t^\star) \geq 0$ and therefore $\overline{e}(t)$ stays positive after $t^\star$ for $\theta \geq 1$ (the proof for $\theta = 0$ is trivial). ■

The same principle to design an observer bundle can now be applied. The already computed bounds for the optimal gain (equations (35) and (36)) keep the same expressions, as far as they do not depend explicitly on the output $s(t)$.

VII. Application and results

A. System setup

For the application of the method, we have considered a real industrial anaerobic digestion wastewater treatment plant (ADWTP). It consist in a highly efficient process that remove a concentrated polluting organic substrate and can recover energy via methane production. This is performed through several consecutive biological degradation steps involving various groups of anaerobic bacteria. Organic compounds are bioconverted in one hand into biogas (*i.e.* a mixture of CO_2 and CH_4) and, on the other hand, into microbial biomass and residual organic matter. ADWTP has a lot of advantages in comparison with traditional methods: it does not generate too much sludge and produces energy. However, it is known to be highly unstable: accumulation of intermediate acid compounds can lead to the process acidification with complete loss of biomass activity. Therefore, accurate monitoring of the system variables is required in order to ensure the appropriate working of the process.

Bacterial growing rate in AD is often described by a non monotone function of Haldane type:

$$\mu(s) = \frac{\mu_h s}{s + k_s + s^2/k_i} \tag{43}$$

Then we have considered an industrial ADWTP processing raw industrial vinasses of 20000m^3. This plant is owned by the AGRALCO company located in Stella, Spain [3].

In particular for this application, the parameter μ_h is badly known, with an uncertainty of $\pm 15\%$ with respect to a nominal (real) value. Parameters meaning and values are summarized in table I.

TABLE I

System parameters.

parameter	meaning	value
u_h	maximal growing rate	0.9
k_s	saturation constant	95
k_i	inhibition constant	243
k	yield conversion	19.5
$[u_h^-, u_h^+]$	bounds for u_h	[0.72,1.08]

The dilution input u[1/h] as well as the available online measurements of the chemical oxygen demand (which represents the substrate s[g/l] in the reactor) are shown in fig. 3 and 4 respectively.

Bounds for the unknown influent substrate s_{in} are shown in fig. 5, known to fluctuate around $\pm 30\%$ of the real value. Noise on the measurements is featured by $\Delta = 0.03$ (see equation (38)).

B. Observers bundle

The observer was tested considering a reinitialization time $\Delta_t = 1$ [days]. The observer initial condition $\underline{x}_0 = 0$,

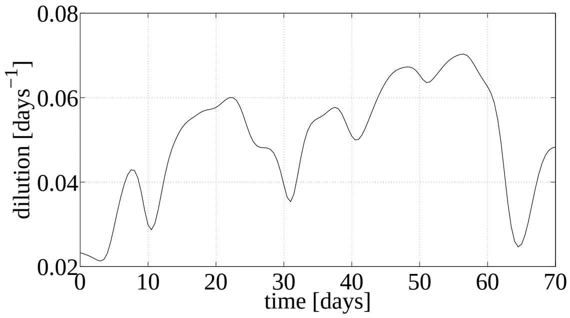

Fig. 3. Dilution input u and measured chemical oxygen demand.

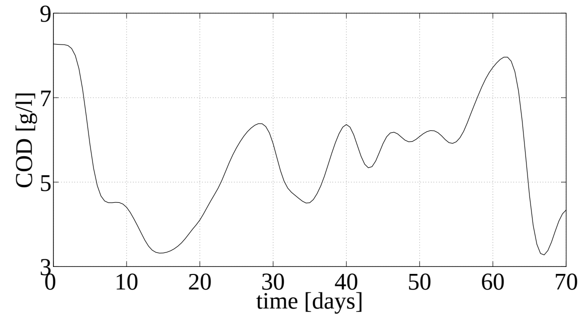

Fig. 4. Measured chemical oxygen demand.

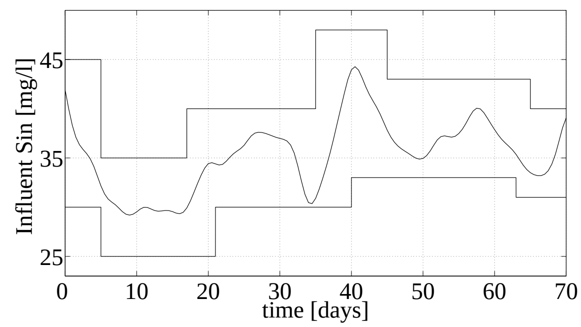

Fig. 5. Influent substrate and its bounds.

$\overline{x}_0 = 100$ has been chosen very large in order to assess the convergence properties of the method. For that, the gains were selected considering a homogeneous partition of 8 framers on the gain interval (in open loop) of equations (35) and (36), including the gains $\theta = 0$ and $\theta = 1$ to guarantee stability and positivity of the predicted intervals. Fig. 7 shows the observer performance. Estimation results have been compared with available samples of a biomass proxy (*i.e.* correlated measurements) given by the so called total suspended solids in the reactor.

The estimation performed for the bundle of observers are compared with the classical asymptotic interval observer ($\theta = 1$). It is possible to assess the good convergence properties of the bundle of observers, whose best estimates become usable much more rapidly than the asymptotic observer. The lower and upper gain sets are shown in fig. 8. The computed gain intervals are quite reduced if we consider that any gain $\theta \in \mathbb{R}$ can provide a valid guaranteed interval of the biomass.

VIII. Conclusion

In this paper we have developed and applied a new interval observer, extending results obtained in [1]. We introduced an optimality criterion that exploits the guaranteed interval nature of the estimates. We showed that the optimal gain value that generates the best interval estimate depends on the unknown state and therefore cannot be directly computed. We proposed an identification of the region where this optimal gain belongs. An observer bundle was considered, with gain values uniformly distributed on the identified region in order to be sure to have a framer close enough the unknown optimal.

The proposed observer considerably increases the properties of the asymptotic observers. Another advantage of the interval based approach is that we can assess the convergence of the observer (through the size of the interval) and thus assess its efficiency. Such an observer can thus be advantageously used in a robust control framework, where state uncertainties are taken into account in the control strategy.

IX. Acknowledgements

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Appendix

The coefficients of each polynomial in equation (24) can be explicitly determined. Denoting $\Delta_\mu = \overline{\mu}(s) - \underline{\mu}(s)$ we have:

- for $\theta < 0$

$$\begin{array}{l} \overline{a}_1 = 0, \quad \underline{a}^1 = 0 \\ \overline{b}_1 = \mu(s)x - \overline{\mu}(s)\overline{x} - ue_{in}^-/k \\ \underline{b}^1 = \mu(s)x - \underline{\mu}(s)\underline{x} + ue_{in}^+/k \\ \overline{c}_1 = (\overline{\mu}(s) - u)\overline{x} \\ \underline{c}^1 = (\underline{\mu}(s) - u)\underline{x} \end{array} \tag{44}$$

- for $\theta \in [0, 1[$

$$\begin{array}{l} \overline{a}_2 = -s\Delta_\mu/k, \quad \underline{a}^2 = s\Delta_\mu/k \\ \overline{b}_2 = \mu(s)x - \overline{\mu}(s)\overline{x} + (s\Delta_\mu + ue_{in}^+)/k \\ \underline{b}^2 = \mu(s)x - \underline{\mu}(s)\underline{x} - (s\Delta_\mu + ue_{in}^-)/k \\ \overline{c}_2 = (\overline{\mu}(s) - u)\overline{x} \\ \underline{c}^2 = (\underline{\mu}(s) - u)\underline{x} \end{array} \tag{45}$$

- for $\theta \geq 1$

$$\begin{array}{l} \overline{a}_3 = s\Delta_\mu/k, \quad \underline{a}^3 = -s\Delta_\mu/k \\ \overline{b}^3 = \mu(s)x - \underline{\mu}(s)\overline{x} - (s\Delta_\mu - ue_{in}^+)/k \\ \underline{b}_3 = \mu(s)x - \overline{\mu}(s)\underline{x} + (s\Delta_\mu - ue_{in}^-)/k \\ \overline{c}_3 = (\underline{\mu}(s) - u)\overline{x} \\ \underline{c}^3 = (\overline{\mu}(s) - u)\underline{x} \end{array} \tag{46}$$

It is possible to verify that:

1. $\overline{f}_2(1) = \overline{f}_3(1)$ and $\underline{f}^1(0) = \underline{f}^2(0)$.
   Then, the piecewise function $\overline{f}(\theta)$ is a continuous function for all θ. The same statement holds for the

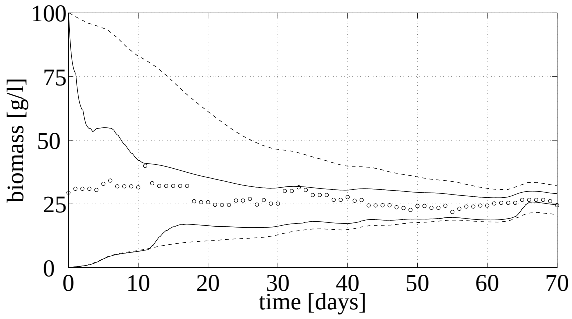

Fig. 6. Biomass interval estimates. Observer bundle (solid lines) is compared with an asymptotic observer (dashed lines). $\circ$ represents real samples of total suspended solids.

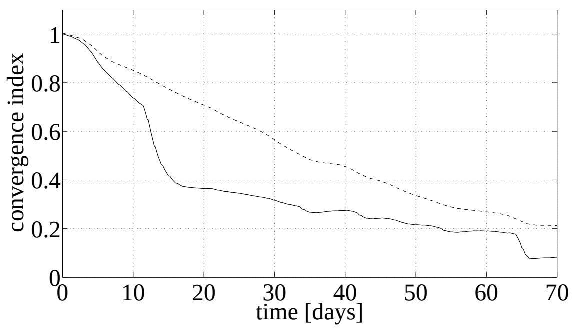

Fig. 7. Convergence index ϑ.

function $\underline{f}(\theta)$.

2. For any $\theta < 0$ we have that $\underline{f}^1(\theta)$ and $\overline{f}_1(\theta)$ are straight lines with positive and negative slopes respectively. Indeed:

$$\begin{array}{l} \overline{b}_1 = k(\mu(s)x - \overline{\mu}(s)\overline{x}) - ue_{in}^- \leq 0 \\ \underline{b}^1 = k(\mu(s)x - \overline{\mu}(s)\overline{x}) + ue_{in}^+ \geq 0 \end{array} \tag{47}$$

3. For any $\theta \in [0, 1[$, $\underline{f}^2(\theta)$ and $\overline{f}_2(\theta)$ are convex and concave parabolas respectively.

4. For any $\theta \geq 1$, $\underline{f}(\theta)$ and $\overline{f}(\theta)$ are concave and convex parabolas respectively.

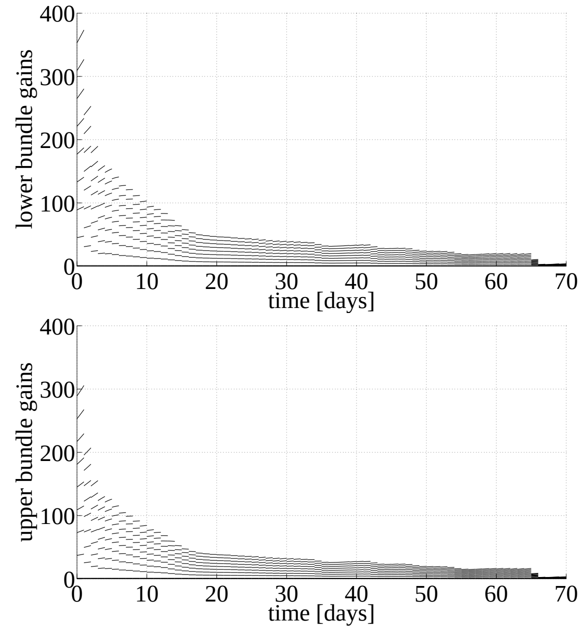

Fig. 8. Gain sets for the lower and upper framers.

REFERENCES

[1] V. Lemesle and J.-L. Gouzé, An hybrid bounded error observer for uncertain bioreactor models, *Bioprocess and Biosystems Engineering*, vol. 27, 2005, pp 311–318.

[2] L. Mailleret, O. Bernard and J.-P. Steyer, Robust Nonlinear Adaptive Control for Bioreactors with Unknown Kinetics, *Automatica*, vol. 40:8, 2004, pp 365–383.

[3] O.Bernard, B. Le Dantec, B.Chachuat, J.-P. Steyer, L. Lardon, S. Lambert, P. Ratini, J. Lema, G. Ruiz, J.Rodriguez, P. Vanrolleghem, U.Zaher, D. De Pauw, K. De Neve, K. Lievens, D. Dochain, O. Schoefs, R. Farina, V. Alcaraz-Gonzalez, V. Gonzlez-Alvarez, P. Lemaire, J.A. Martinez, O.Duclaud and J.F. Lavigne, An integrated system to remote monitor and control anaerobic wastewater treatment plants through the Internet, *Wat. Sci. Tech.* , vol. 52, 2005, pp 457-464.

[4] A. Alessandri, M. Baglietto and G. Battistelli, Design of state estimators for uncertain linear systems using quadratic boundedness, *Automatica*, vol. 42, 2005, pp 497-502.

[5] S.G. Douma and P.M.J. Van den Hof, Relations between uncertainty structures in identification for robust control, *Automatica*, vol. 41, 2005, pp 439–457.

[6] G. Bastin and D. Dochain *On-line estimation and adaptive control of bioreactors*, Elsevier, New York, 1990.

[7] L. Jaulin, M. Kieffer, O. Didrit and E. Walter, *Applied Interval Analysis with Examples in Parameter and State Estimation, Robust Control and Robotics*, Springer, London, 2001.

[8] A. Kurzhanski and I. Valyi, *Ellipsoidal Calculus for Estimation and Control*, Birkhauser, Boston, 1997.

[9] J. P. Gauthier, H. Hammouri and S. Othman, A simple observer for nonlinear systems applications to bioreactors, *IEEE Trans. Autom. Contr.*, vol. 37, 1992, pp. 875–880.

[10] A. Rapaport and J.-L. Gouzé, Parallelotopic and practical observers for nonlinear uncertain systems, *Int. Journal. Control*, vol. 76:3 , 2003, pp 237–251.

[11] H.L. Smith, Monotone Dynamical Systems: An introduction to the theory of competitive and cooperative systems, *American Mathematical Society*, vol. 75, 1995.

[12] O. Bernard and J.-L. Gouzé, Closed loop observers bundle for uncertain biotechnological models, *Journal of Process Control*, vol. 14, 2004, pp. 765–774.

[13] D. Dochain and A. Rapaport, Interval observer for biochemical processes with uncertain kinetics and inputs, *Mathematical Biosciences*, vol. 76:3, 2004, pp. 237–251.

[14] J. Harmand, A. Rapaport and F. Mazenc, Output tracking of continuous bioreactors through recirculation and by-pass, *Automatica*, vol. 76:3, 2004, pp. 237–251.

[15] M. Darouach, M. Zasadzinski and S.J. Xu, Full-order observers for linear systems with unknown inputs, *IEEE Transactions on Automatic Control*, vol. 39:3, 1994, pp. 606–609.

[16] J.C.Geromel, J. Bernussou, G. Garcia and M.C. De Oliveira, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ robust filtering for discrete-time linear systems, *SIAM J. Contr. Optim.*, vol. 38:5, 2000, pp. 1353–1368.

[17] L. Xie and Y. C. Soh and C. E. de Souza, Robust kalman filtering for uncertain discrete-time systems, *IEEE Transactions on Automatic Control*, vol. 39:6, 1994, pp. 1310–1314.

[18] O. Bernard and G. Sallet and A. Sciandra, Nonlinear observers for a class of biological systems. Application to validation of a phytoplanktonic growth model, *IEEE Transactions on Automatic Control*, vol. 43, 1998, pp. 1056-1065.

[19] B. Chachuat and O. Bernard, Probabilistic observers for mass-balance based bioprocess models, *International Journal of Robust and Nonlinear Control*, vol. 16, 2005, pp. 157-171.

[20] J.-L. Gouzé, A. Rapaport and Z. Hadj-Sadok, Interval observers for uncertain biological systems, *Ecological Modelling*, vol. 133, 2000, pp. 45-56.