A computer-assisted discharging procedure: application to 2-distance coloring

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November, 2020

2-distance coloring

A 2-distance k-coloring (Kramer and Kramer, 1969)

A k-coloring such that no pair of vertices at distance at most 2 is monochromatic.

The 2-distance chromatic number

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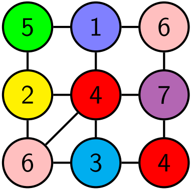

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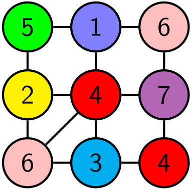

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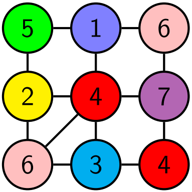

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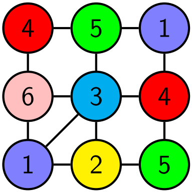

An optimal 2-distance 6-coloring.

$$\chi^2(G) = 6.$$

2-distance coloring

Observation

For any graph G with maximum degree Δ, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$.

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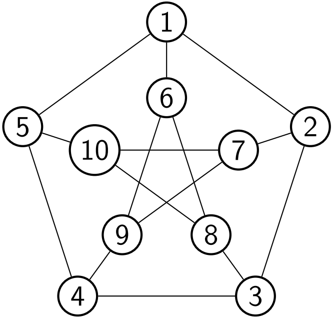
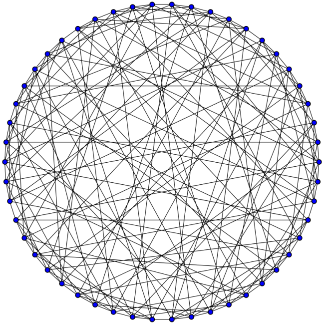


Petersen graph



Hoffman-Singleton graph

$$\chi^2(G) = \Delta^2 + 1.$$

2-distance coloring of planar graphs

Wegner's conjecture, 1977

Let G be a planar graph. Then,

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

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If G is a planar graph with girth $g \geq g_0$ and maximum degree $\Delta \geq \Delta_0$, then $\chi^2(G) \leq \Delta(G) + c_0$.

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If G is a planar graph with $g \geq 9$ and $\Delta = 3$, then $\chi^2(G) \leq \Delta + 3$.

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2-distance coloring of planar graphs

girth \ $\chi^2(G)$	$\Delta+1$	$\Delta+2$	$\Delta+3$	$\Delta+4$	$\Delta+5$	$\Delta+6$	$\Delta+7$	$\Delta+8$
3				$\Delta=3$				
4								
5		$\Delta\geq 10^7$	$\Delta\geq 339$	$\Delta\geq 312$	$\Delta\geq 15$	$\Delta\geq 12$	$\Delta\neq 7,8$	all Δ
6		$\Delta\geq 17$	$\Delta\geq 9$		all Δ			
7	$\Delta\geq 16$			$\Delta=4$				
8	$\Delta\geq 9$		$\Delta=5$ $\Delta\geq 3$					
9	$\Delta\geq 8$	$\Delta=5$	$\Delta=3$					
10	$\Delta\geq 6$							
11		$\Delta=4$						
12	$\Delta=5$	$\Delta=3$						
13								
14	$\Delta\geq 4$							
22	$\Delta=3$							

Results from almost 20 different papers.

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3: Assign charges to vertices and faces so that the sum of all charges is negative thanks to the Euler's formula ($|V| - |E| + |F| = 2$).

4: Redistribute the charges without changing the total sum, and show that we obtain a non-negative final amount, thanks to the structural properties, which is a contradiction.

The discharging method: Step 1

Theorem

If G is a planar graph with girth $g \geq 8$ and maximum degree $\Delta = 3$, then $\chi^2(G) \leq 6$.

Step 1: Take a minimal counter-example G, with $\Delta = 3$, $g \geq 8$, and $\chi^2(G) \geq 7$.

The discharging method: Step 2

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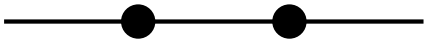
  - G has no 

The discharging method: Step 2

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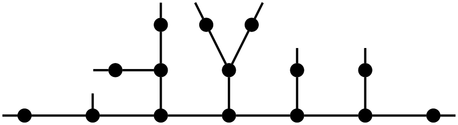
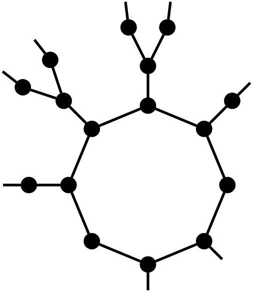
- And almost 50 other complicated reducible configurations...





The discharging method: Step 3

Step 3: Assign charges to vertices.

Charge assignment

$$\mu : v \mapsto 3.5d(v) - 9 \text{ and } \mu : f \mapsto d(f) - 9$$

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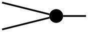
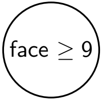
Since $|V| - |E| + |F| = 2$,

$$\sum_{v \in V} (3.5d(v) - 9) + \sum_{f \in F} (d(f) - 9) = -18 < 0$$

Comparison

	"Human" proof $g \geq 9$	"Human" proof ? $g \geq 8$	Our proof $g \geq 8$
Reducible configurations	Few "local" configs		
Charge distribution	 : -2  : 1.5  : ≥ 0 $\mu(v) = 3.5d(v)$ $\mu(f) = d(f) - 9$		

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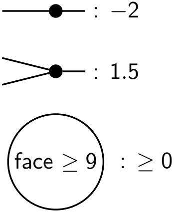
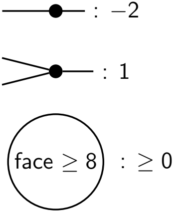
Charge distribution	 $\mu(v) = 3.5d(v)$ $\mu(f) = d(f) - 9$	 $\mu'(v) = 3d(v)$ $\mu'(f) = d(f) - 8$	

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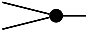
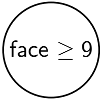
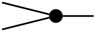
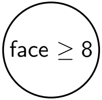
Charge distribution	 : -2  : 1.5  : ≥ 0 $\mu(v) = 3.5d(v)$ $\mu(f) = d(f) - 9$	 : -2  : 1  : ≥ 0 $\mu'(v) = 3d(v)$ $\mu'(f) = d(f) - 8$	

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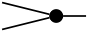
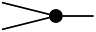
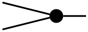
Charge distribution	 : -2  : 1.5 face ≥ 9 : ≥ 0 $\mu(v) = 3.5d(v)$ $\mu(f) = d(f) - 9$	 : -2  : 1 face ≥ 8 : ≥ 0 $\mu'(v) = 3d(v)$ $\mu'(f) = d(f) - 8$	 : -2  : 1.5 face ≥ 9 : ≥ 0

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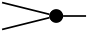
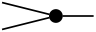
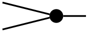
Charge distribution	 : -2  : 1.5 face ≥ 9 : ≥ 0 $\mu(v) = 3.5d(v)$ $\mu(f) = d(f) - 9$	 : -2  : 1 face ≥ 8 : ≥ 0 $\mu'(v) = 3d(v)$ $\mu'(f) = d(f) - 8$	 : -2  : 1.5 face ≥ 9 : ≥ 0 face $= 8$: $= -1$

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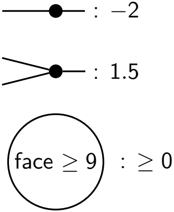
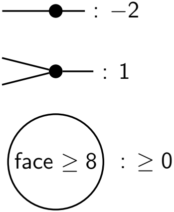
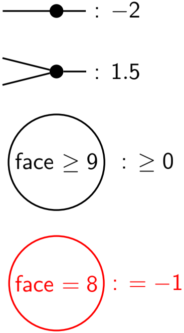
Charge distribution	 $\mu(v) = 3.5d(v)$ $\mu(f) = d(f) - 9$	 $\mu'(v) = 3d(v)$ $\mu'(f) = d(f) - 8$	

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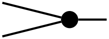
 : -2  : 1.5 face ≥ 9 : ≥ 0 face $= 8$: $= -1$

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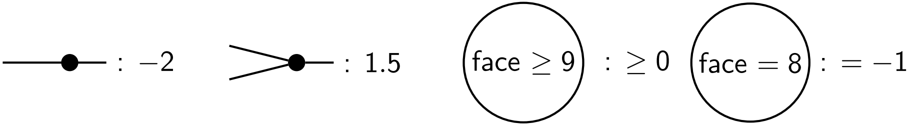

- Choose rules to assure the vertices have non-negative charges.

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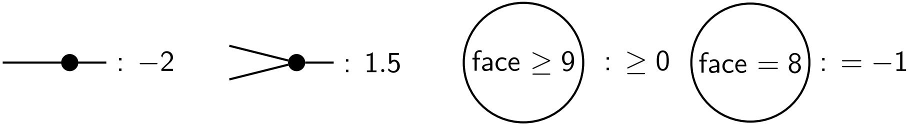

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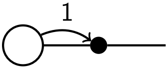
  - For example, 

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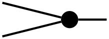
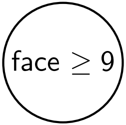
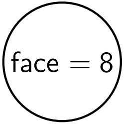
 : -2  : 1.5 

: ≥ 0 

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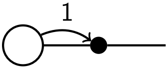
  - For example, 

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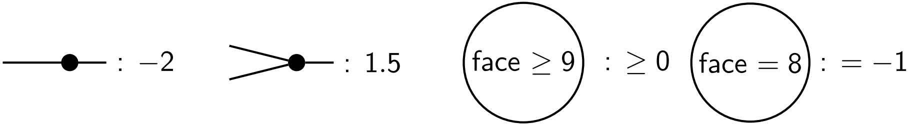

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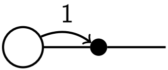
  - For example, 

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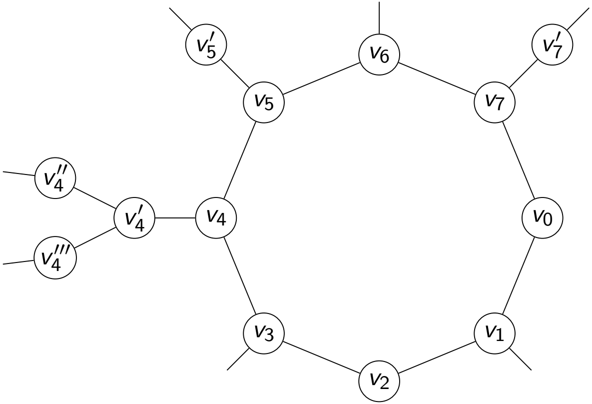
- Choose rules to assure the 8-faces have non-negative charges.

Computer Assistance: Encoding Cycles



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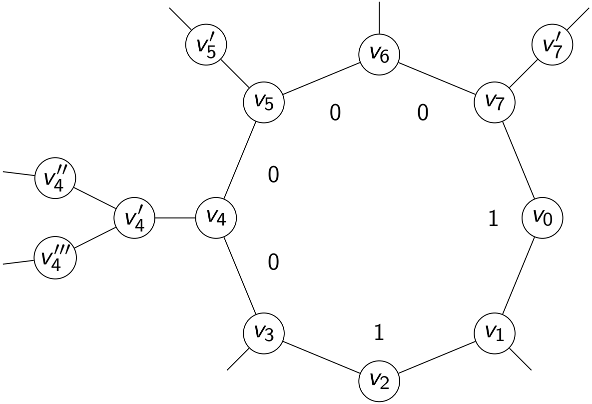

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Count the 2-vertices in between.

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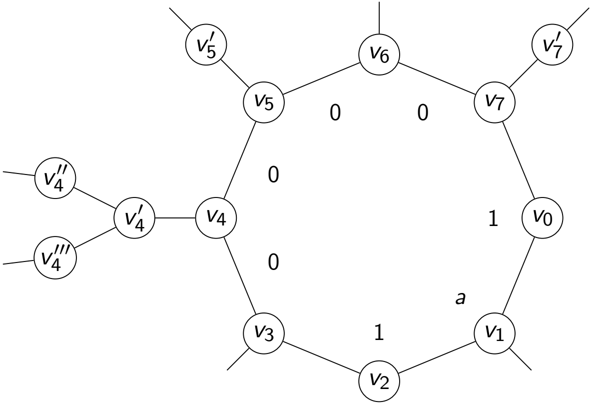

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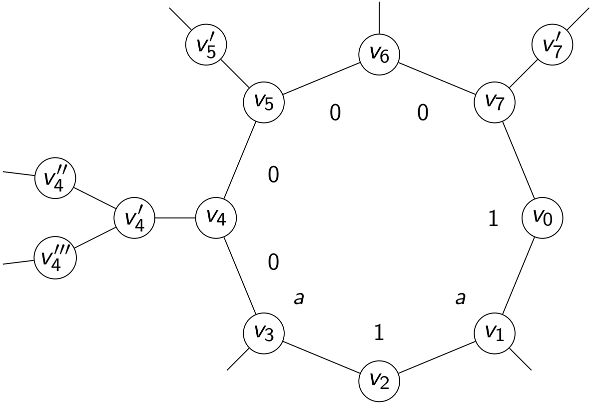

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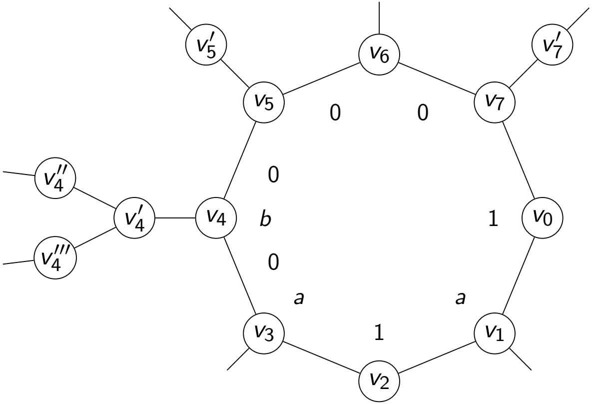

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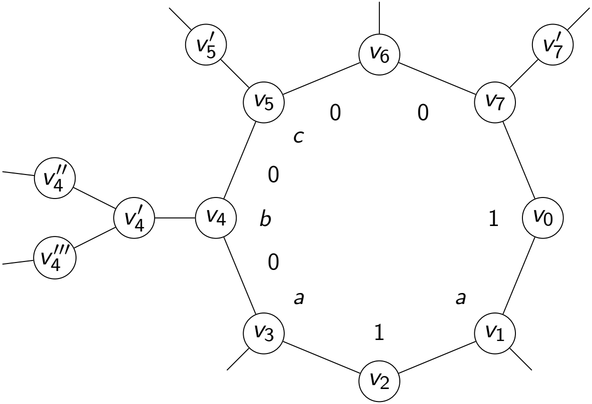

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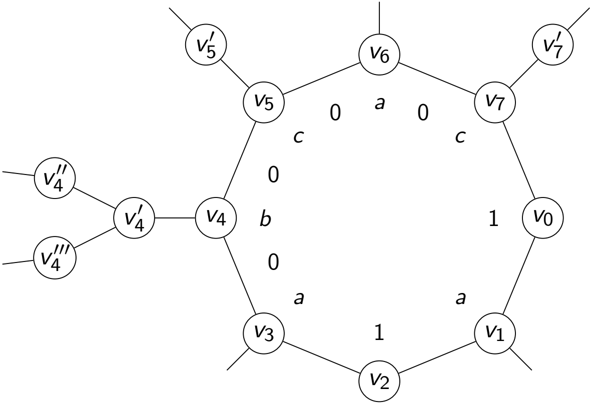
Computer Assistance: Encoding Cycles

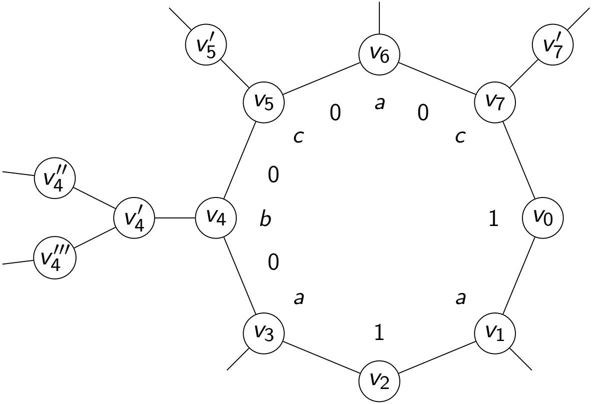


Identify the 3-vertices.

Count the 2-vertices in between.

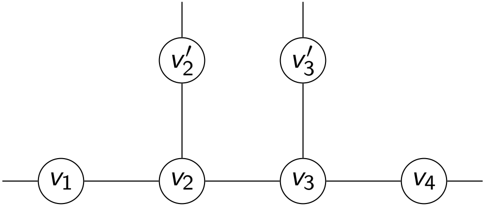
Encode the neighborhood of each 3-vertex.

Computer Assistance: Encoding Cycles



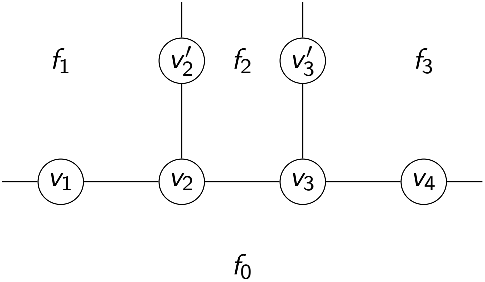
Identify the 3-vertices.
Count the 2-vertices in between.
Encode the neighborhood of each 3-vertex.
We obtain: **1a1a0b0c0a0c**.

Computer Assistance: Encoding Trees



Encode a tree from all incident faces

Computer Assistance: Encoding Trees



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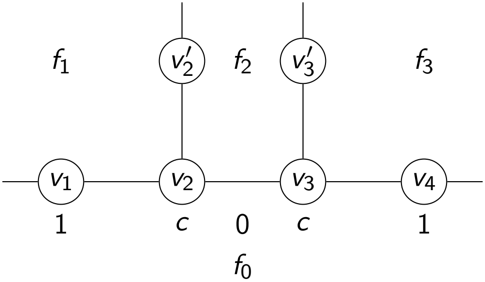
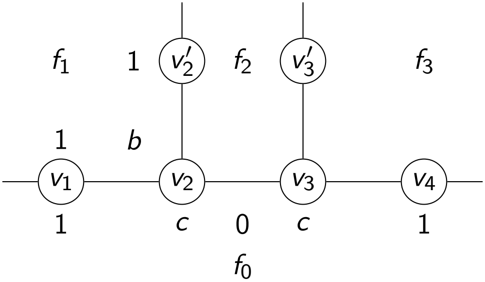


Encode a tree from all incident faces

f_0, f_2 : **1c0c1**

Computer Assistance: Encoding Trees



Encode a tree from all incident faces

f_0, f_2 : **1c0c1**

f_1, f_3 : **1b1**

Computer Assistance

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  - Calculate the charge of each word by its subwords.

Conclusion

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Conclusion

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- Solve the problem of the verifying a huge amount of configurations.
- Find problematic configurations (non reducible and non dischargeable) immediately.
- Verify a proof quickly.