Digraph Coloring and Distance to Acyclicity

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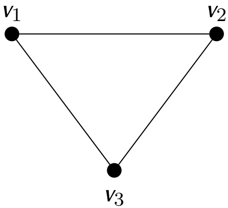
Graph and Digraph Coloring

Graph coloring:



Graph and Digraph Coloring

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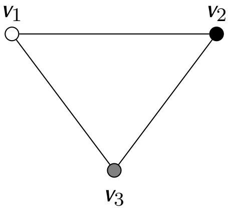

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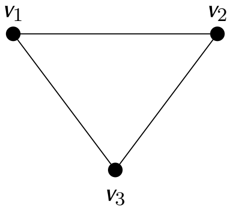

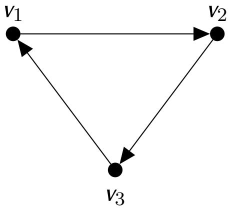

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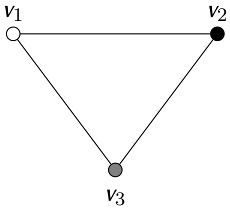

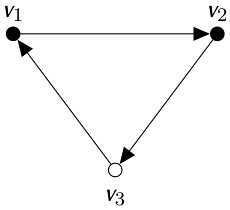

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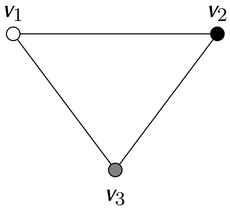

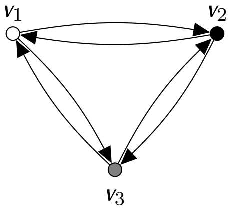

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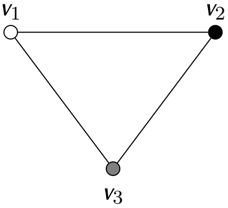
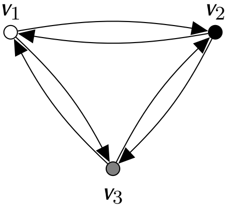




In 1982 V. Neumann-Lara introduced the the notion of dichromatic number of a digraph.

The problem

In this work:

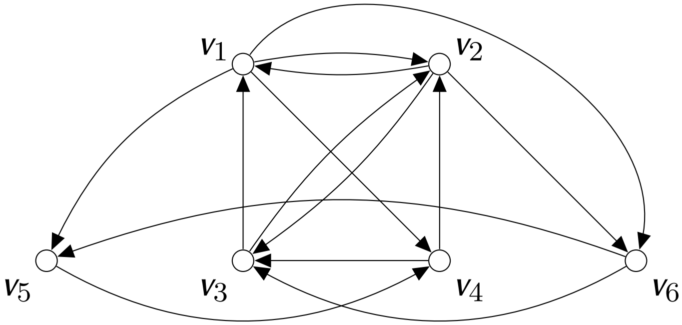
- Instance: Digraph and $k \in \mathbb{N}$.
- Assign colors on the vertices.
- Every color inducers a Directed Acyclic Graph (DAG).

Definition (k-Digraph Coloring)

Given a digraph $D = (V, E)$ and a number $k \in \mathbb{N}$, is there a partition $\{V_1, \ldots, V_k\}$ of V s.t. $\forall i \in [k]$, $D[V_i]$ is a DAG.

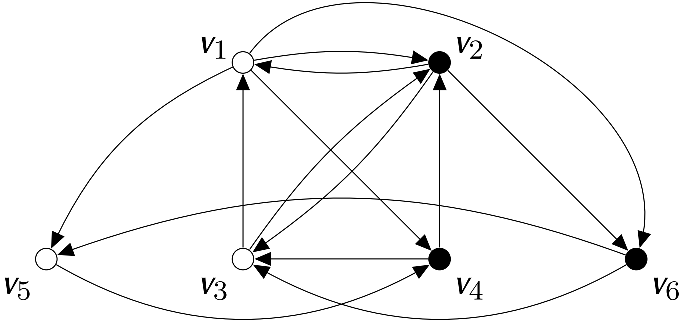
Example

Can we color this digraph with 2 colors and each color induces a DAG?



Example

Can we color this digraph with 2 colors and each color induces a DAG?
(Answer: YES)



Directed Feedback Vertex Set

Observation 1.

- Let D be a DAG then D is 1-colorable.

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Given a digraph $D = (V, E)$ a set $F \subseteq V$ is a DFVS iff $D[V \setminus F]$ is a DAG.

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Observation 1.

- Let D be a DAG then D is 1-colorable.

Definition (Directed Feedback Vertex Set)

Given a digraph $D = (V, E)$ a set $F \subseteq V$ is a DFVS iff $D[V \setminus F]$ is a DAG.

Observation 2.

- Let D be a digraph with DFVS F of size k then the D is $k + 1$ colorable.

Directed Feedback Vertex Set

Based on these observations we have the following question:

- Let D be a digraph of order n with DFVS F of size k. Is it hard to decide if D is k-colorable?

Directed Feedback Vertex Set

Based on these observations we have the following question:

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Theorem

For all $k \geq 2$, it is NP-hard to decide if a digraph $D = (V, E)$ is k-colorable even when the size of its feedback vertex set is k.

Completeness Proof (case $k = 2$)

We will present a reduction form a restricted version of 3-SAT problem.

Definition (RESTRICTED-3-SAT)

We consider the instances of 3-SAT that have the following properties:

1. any clause contains either only positive literals or only negative literals,
2. all variables appear at most 2 times positive and 1 time negative.

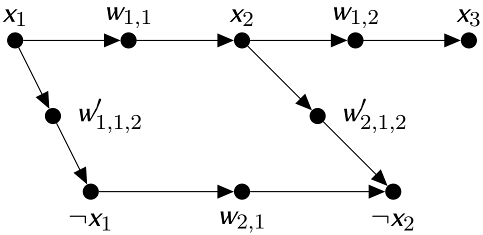
Completeness Proof (case $k = 2$)

Given an instance of RESTRICTED-3-SAT we construct a digraph D by following the next steps:

1. for each clause c_i we construct a path of five vertices $l_{i,1}, w_{i,1}, l_{i,2}, w_{i,2}, l_{i,3}$ (or three vertices if the clause contain two literals);
2. for each variable x_j, if x appears positive in c_{i_1} and negative in c_{i_2} we construct a vertex w'_{j,i_1,i_2} and add an incoming arc from the vertex that represents the positive literal and an outgoing arc to the vertex that represents the negative literal.

Completeness Proof (case $k = 2$, example graph)

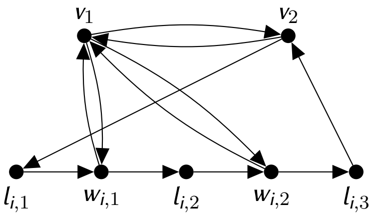
For the formula $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2)$ we have the following digraph:



Completeness Proof (case $k = 2$)

Now we will add a "palette" and we will connect it with the digraph:

- we add two vertices v_1, v_2 connected by a digon;
- we connect v_1 to all existing $w_{i,j}$ with a digons;
- we connect v_2 has an outgoing arc to the first vertex of the path of each clause and incoming arc from the last vertex of the path of each clause.

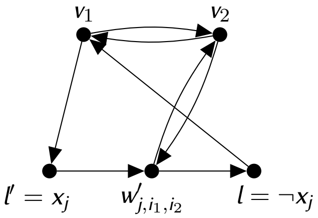


The cycles created by $\{v_1, v_2\}$ and clauses with three literals.

Completeness Proof (case $k = 2$)

Finally,

- we connect v_2 to all existing w'_{j,i_1,i_1} with digons;
- last, v_1 has an outgoing arc to all the vertices that represent positive literals and incoming arcs from all the vertices that represent negative literals.

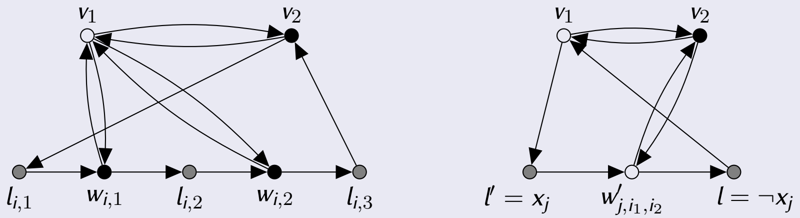


The cycles created by $\{v_1, v_2\}$ and each pair $\{x, \neg x\}$.

Completeness Proof (case $k = 2$)

In order to finish the proof we need to show:

ϕ is satisfiable if and only if the digraph is 2-colorable



Results on Δ and FAS

Bounded DFVS and Δ

- It is *NP*-hard to decide if a digraph is k-colorable, even if it has a DFVS of size k and $\Delta = 4k - 1$ (for all $k \geq 2$).
- Let D be a digraph with DFVS F of size k and $\Delta \leq 4k - 3$. Then, D is k-colorable if and only if $D[N[F]]$ is k-colorable.

Idea of the proof

- First: We change the "palette" in the previous proof.
- Second: We assume a k-coloring of $D[N[F]]$. We can modified it and we show that this new coloring is extendable.

Bounded Feedback Arc Set

Bounded FAS

- For all $k \geq 2$, it is *NP*-hard to decide if a digraph D is k-colorable, even if D has a FAS of size k^2 (and $\Delta = 4k - 1$).
- Let D be a digraph with a FAS F of size at most $k^2 - 1$. Then D is k-colorable if and only if $D[V(F)]$ is k-colorable.

Idea of the proof

- Second: Induction to the number of colors.

Bounded Treewidth

Most of the previous results are negative so we consider an other parameter; the treewidth of the (underlined) graph.

Theorem

There is an algorithm which, given a digraph D on n vertices and a tree decomposition of its underlying graph of width tw *decides if D is k-colorable in time* $k^{\mathrm{tw}}(\mathrm{tw}!)n^{\mathcal{O}(1)}$

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Can we reduce the running time?

Lower Bound

Theorem

Let D be a digraph on n vertices and treedepth td*. If there exist an algorithm decides if D is* 2*-colorable in time* $\mathrm{td}^{o(\mathrm{td})} n^{O(1)}$*, then the ETH is false.*

Lower Bound

Theorem

Let D be a digraph on n vertices and treedepth td*. If there exist an algorithm decides if D is 2-colorable in time* $\mathrm{td}^{o(\mathrm{td})} n^{O(1)}$*, then the ETH is false.*

- Reduction based on 3-SAT.
- We construct a digraph D with $|V(D)| = 2^{\mathcal{O}(n/\log n)} m$ vertices and treedepth $\mathrm{td} = \mathcal{O}(n/\log n)$.
- We show that D is 2-colorable if and only if ϕ is satisfiable.

Lower Bound

Idea of the construction

- We partition the variables into $\log n$ sets $X_1, \ldots, X_{\log n}$.
- For each X_i we construct a set of vertices V_i and we connect all of them with a universal vertex u (same for all X_i).

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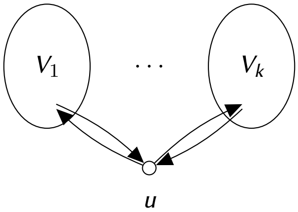
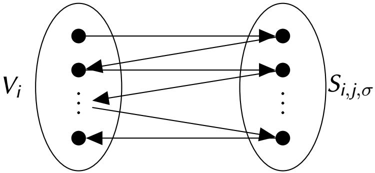
$$X = \{X_1, \dots, X_k\}$$



Lower Bound

Idea of the construction

- For each X_i and truth assignment σ we define (arbitrarily) an ordering $\rho(\sigma)$.
- For each X_i, each clause c_j and truth assignment σ that satisfies c_j we construct a set of vertices $S_{i,j,\sigma}$.
- We construct a path with the vertices $S_{i,j,\sigma} \cup V_i$ (using alternatively vertices of these sets) that respects the ordering $\rho(\sigma)$.



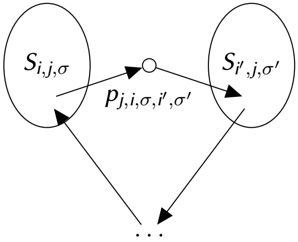
the set $S_{i,j,\sigma}$ related to:
clause c_j, variables of X_i
and the truth assignment σ

the path is based
on the ordering $\rho(\sigma)$

Lower Bound

Idea of the construction

- For each clause c_j we construct a "cycle" with the sets $S_{i,j,\sigma}$ and use some new "connecting" vertices $p_{j,i,\sigma,i',\sigma'}$ between them.
- We connect all the "connecting" vertices $p_{j,i,\sigma,i',\sigma'}$ with arbitrary vertices of V_1 (both ways).



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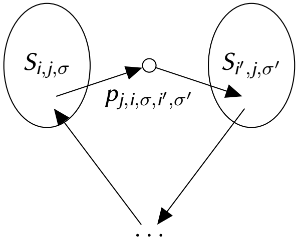

Why ϕ is satisfiabe if and only if D is 2-colorable?

Conclusion

NP-harfnes:

- k-coloring in digraphs with DFVS of size k and $\Delta = 4k - 1$.
- k-coloring in digraphs with FAS of size k^2 and $\Delta = 4k - 1$.

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FPT

- k-coloring for digraphs with DFVS of size k and $\Delta \leq 4k - 3$.
- k-coloring for digraphs with FAS of size $k^2 - 1$.
- k-coloring for digraphs with bounded (undirected) treewidth.

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- k-coloring for digraphs with bounded (undirected) treewidth.

Lower bounds for 2-coloring

- worst than $\mathrm{td}^{o(\mathrm{td})} n^{O(1)}$, unless ETH is false.

Conclusion

Open questions:

- Consider other directed and undirected structural parameters.
- For larger values of the DFVS, how the tractability threshold for the degree evolves from $4k - \Theta(1)$ to $2k + \Theta(1)$.

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Thank You!