Reachability in arborescence packings

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18th november, 2020

Joint work with:

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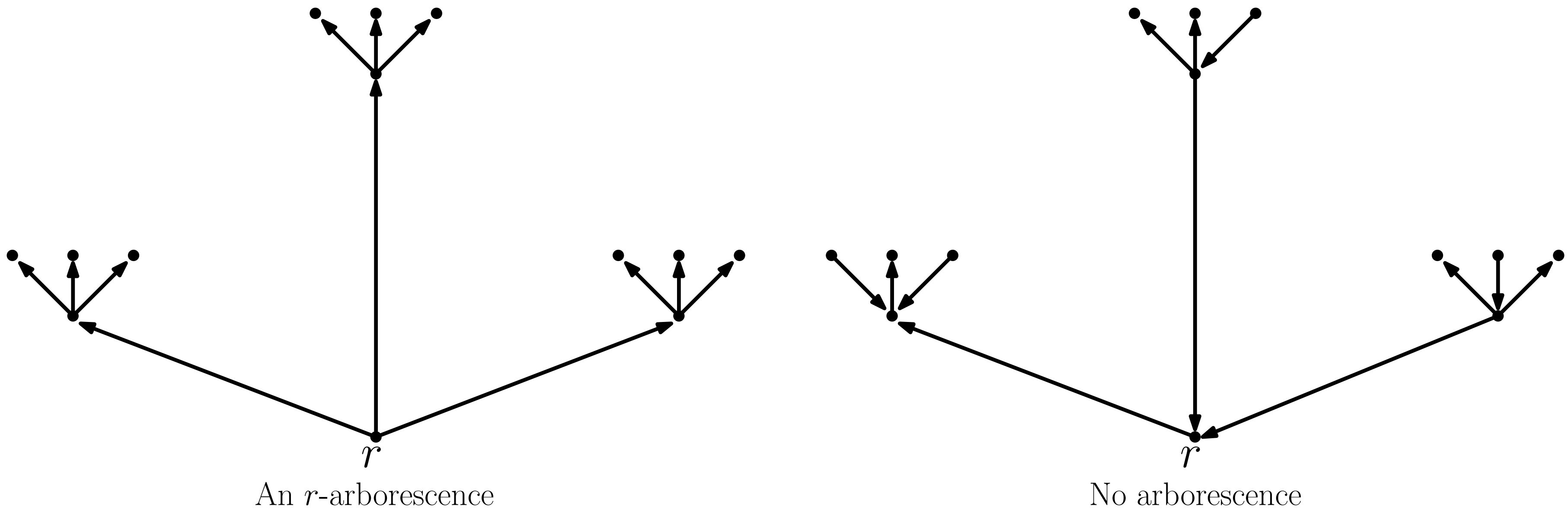
Arborescences

Definition

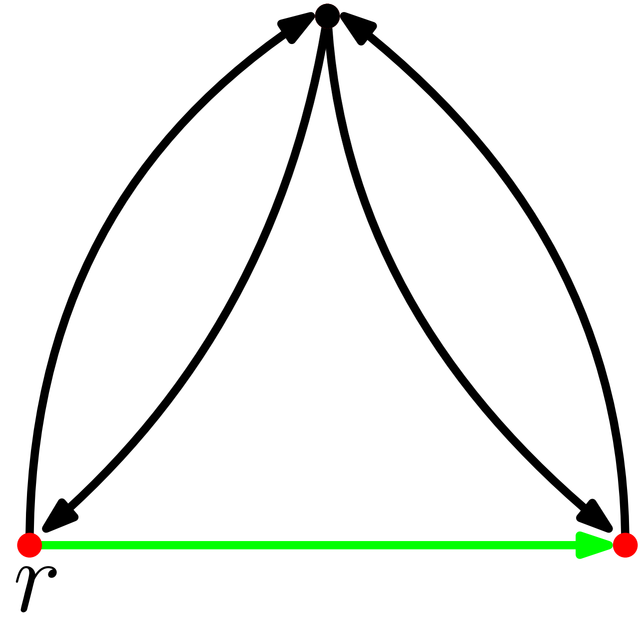
A digraph B is called an *r-arborescence* for some vertex $r \in V(B)$ if the underlying graph of B is a tree and all arcs are directed away from r.



Arborescences

Definition

Given a digraph D and a subgraph B of D which is an r-arborescence for some $r \in V(D)$, we say that B spans $V(B)$.



Rooted digraphs

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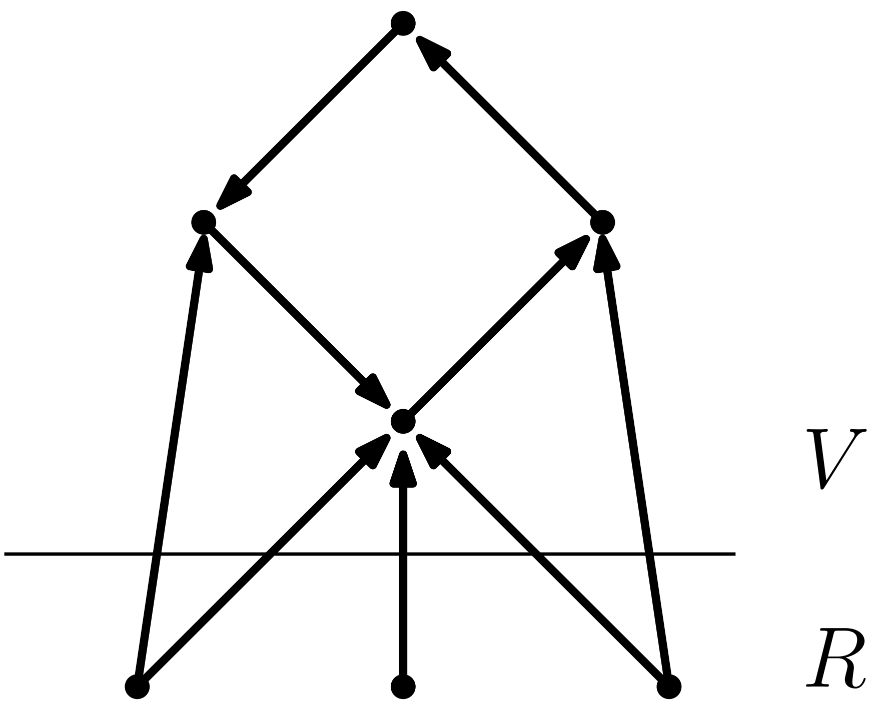
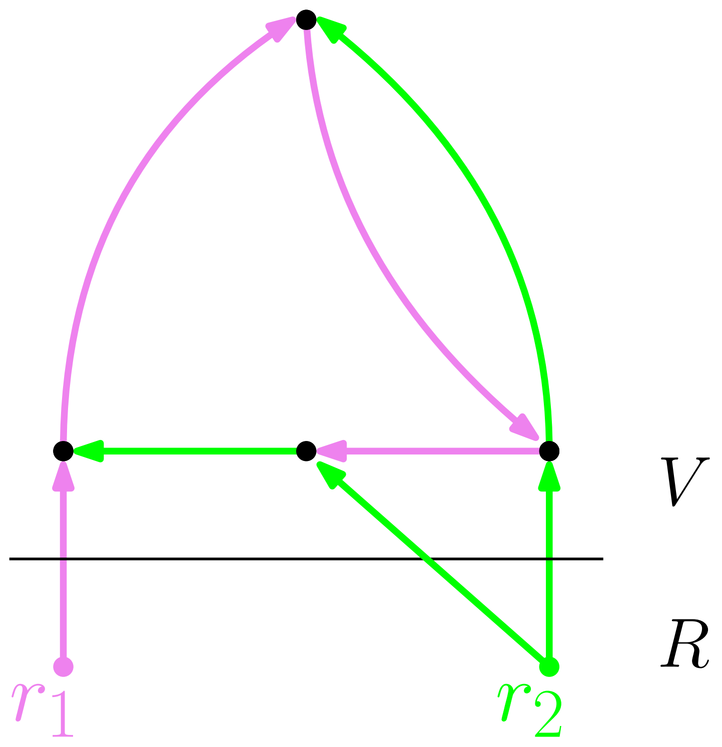
- A rooted digraph is a digraph $(V \cup R, A)$ with R being a set of roots.

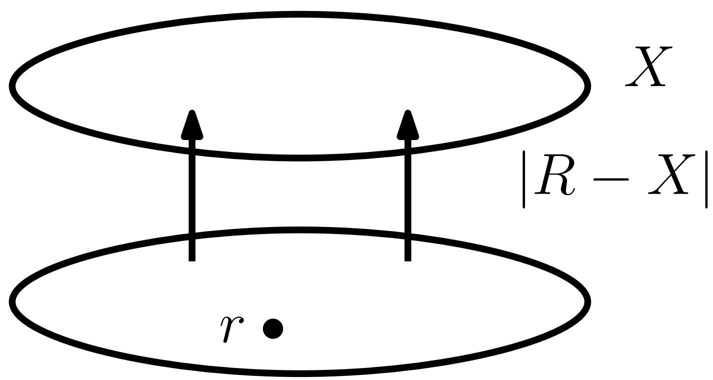


Spanning arborescences

Theorem[Edmonds' strong theorem(1973)]

Let $D = (V \cup R, A)$ be a rooted digraph. Then there is a packing of r-arborescences $\{B_r : r \in R\}$ spanning V if and only if $d_A^-(X) \geq |R - X|$ for all $X \subseteq V \cup R$ with $X - R \neq \emptyset$.

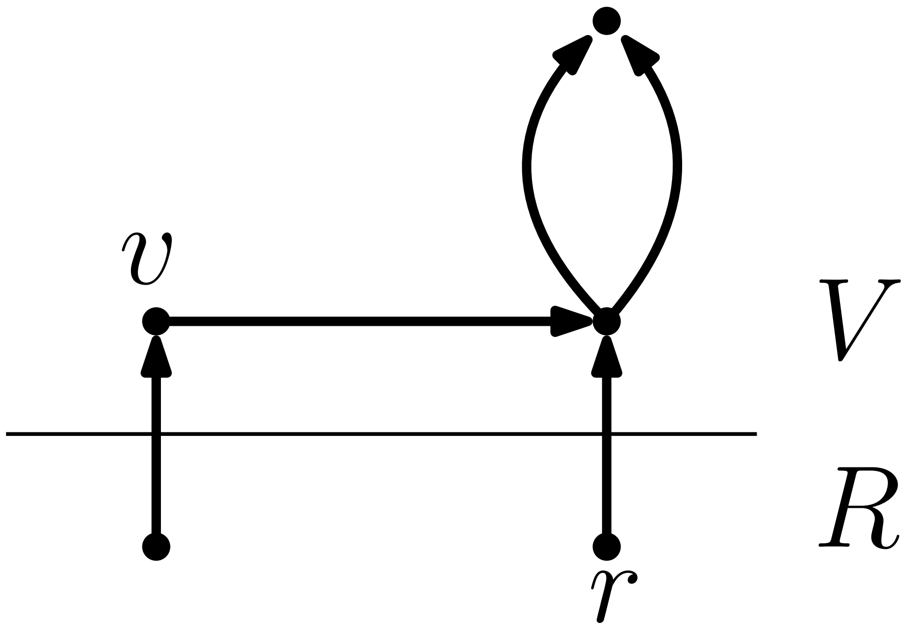




Reachability

Problem

Little information is obtained when some $v \in V$ is not reachable from some $r \in R$.



Reachability arborescences

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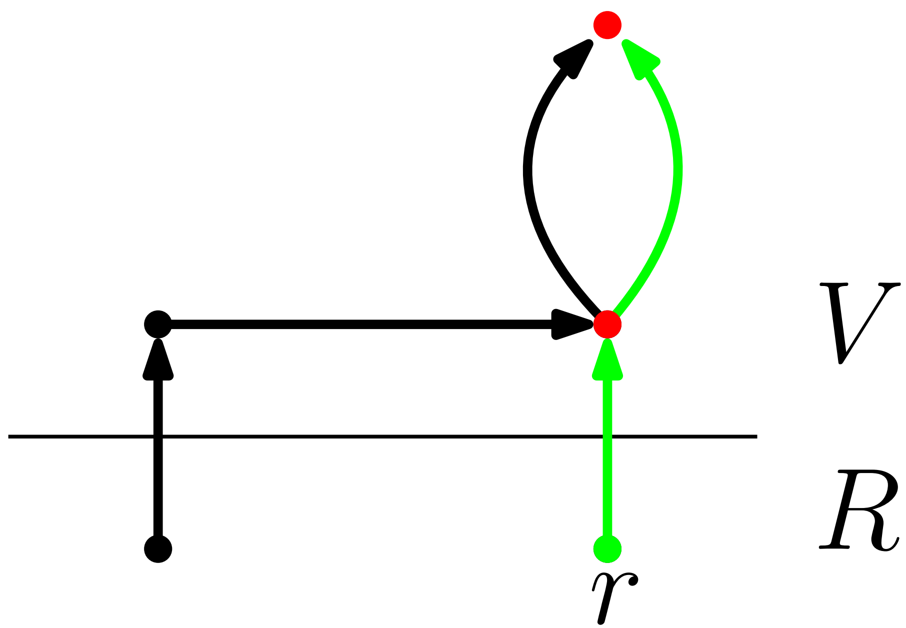
For some rooted digraph $D = (V \cup R, A)$ and $r \in R$, an r-arborescence spanning all vertices reachable from r in D is a reachability r-arborescence.



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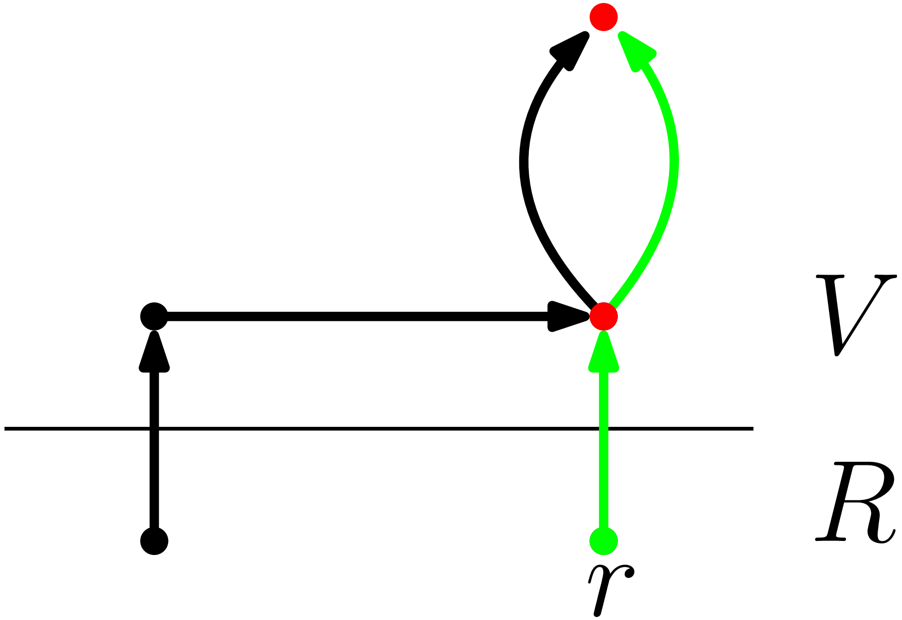


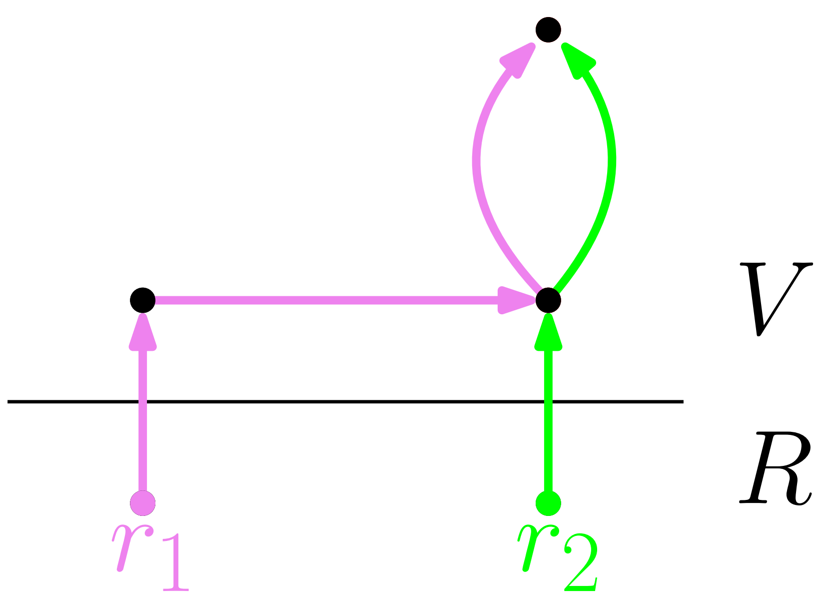
Definition

For $X \subseteq V \cup R$, P_D^X is the set of vertices in R from which X is reachable.

Packing reachability arborescences

Theorem[Kamiyama, Katoh, Takizawa(2009)]

Let $D = (V \cup R, A)$ be a rooted digraph. Then there is a packing of reachability r-arborescences $\{B_r : r \in R\}$ if and only if $d_A^-(X) \geq |P_D^X| - |X \cap R|$ for all $X \subseteq V \cup R$.



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Packing reachability arborescences

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- Certain simplifications were found by Frank.
- We give a simpler inductive proof.
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Proof setup

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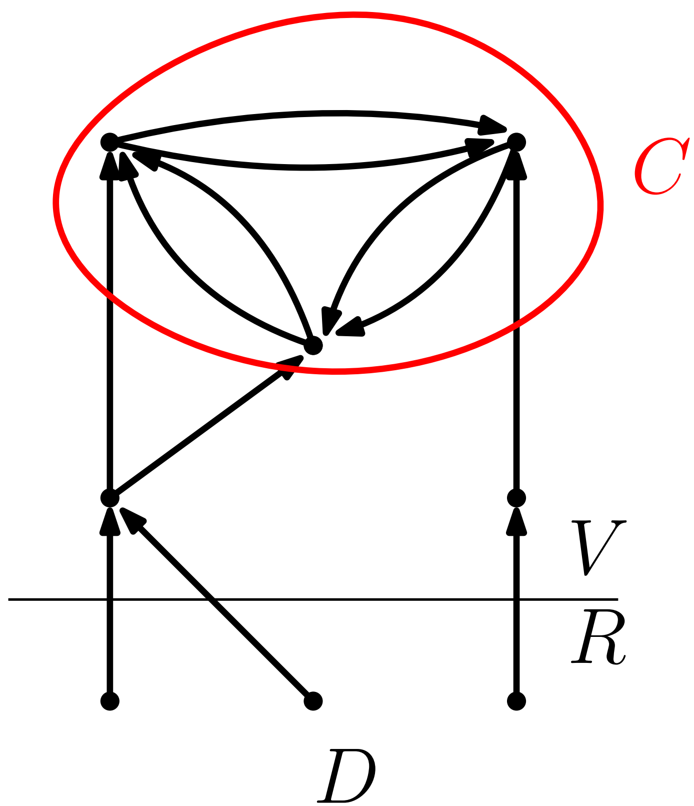
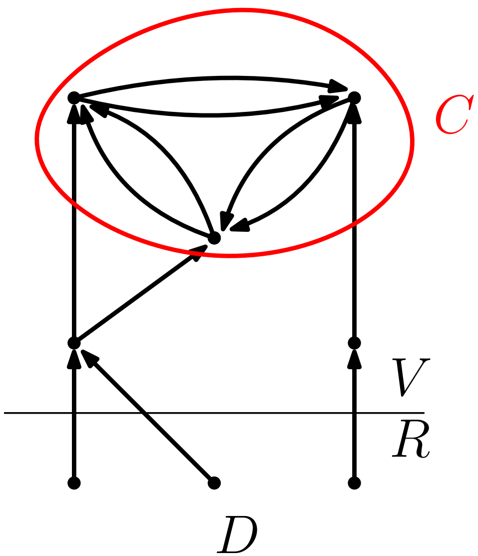
- We choose a a strongly connected component $C \subseteq V$ which has no arc leaving.



Packing reachability arborescences

Applying induction

- Consider $D - C$.



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Applying induction

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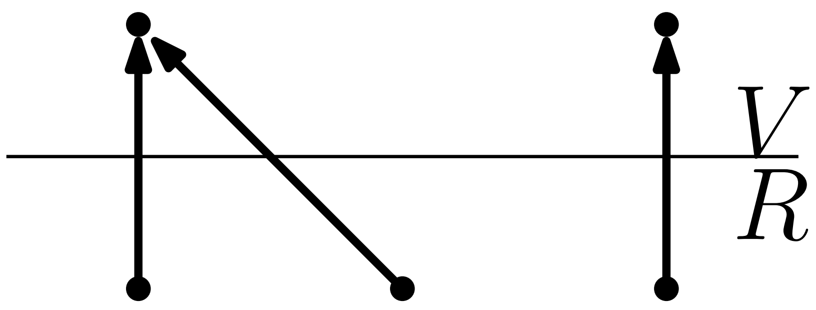

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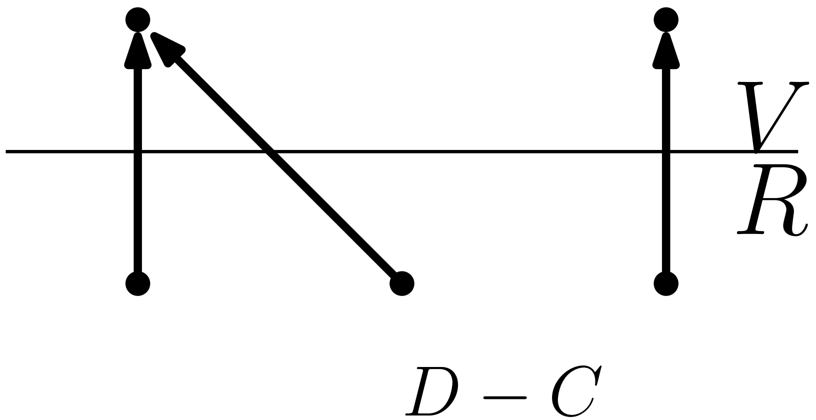

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Applying induction

- Consider $D - C$.
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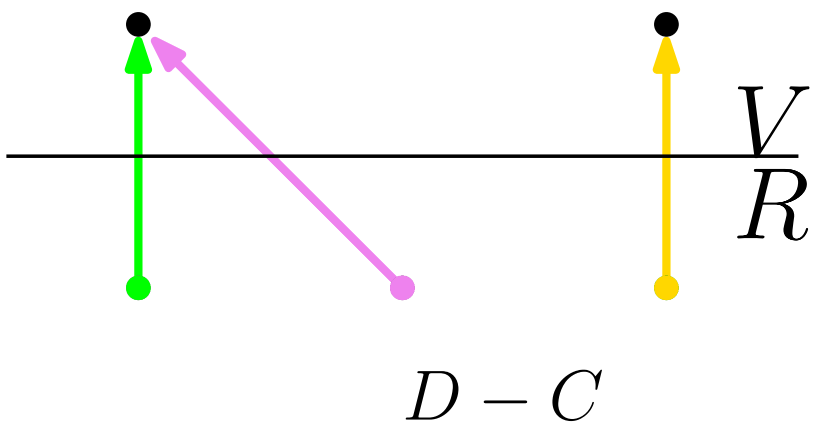
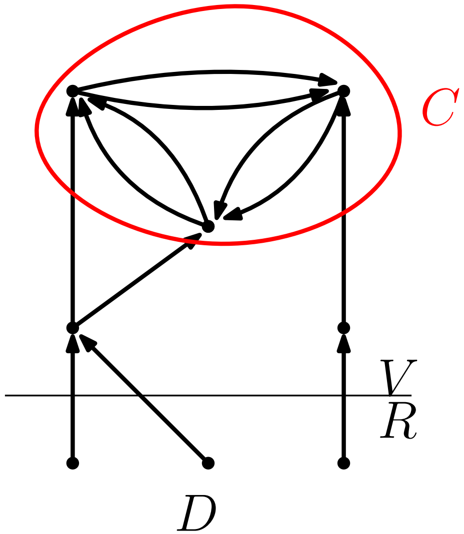
- This yields a packing of reachability r-arborescences $\{B_r^1 : r \in R\}$ in $D - C$.



Packing reachability arborescences

Construction of auxiliary digraph D'



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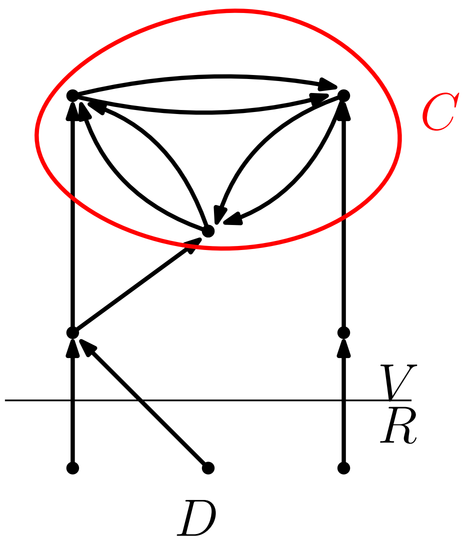
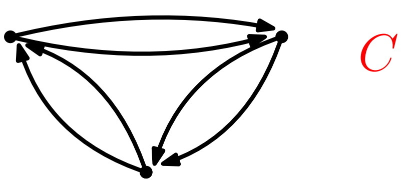
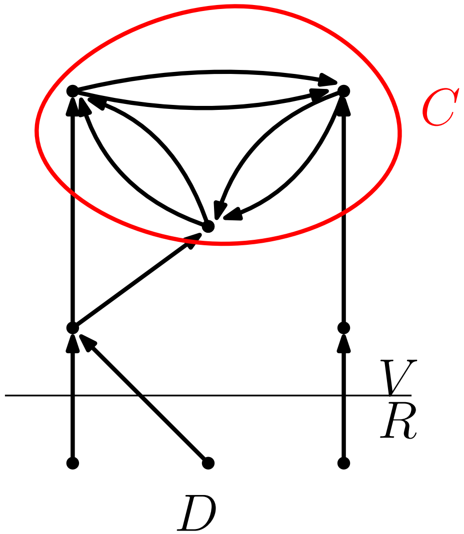


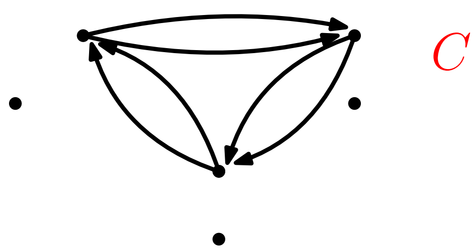


Packing reachability arborescences

Construction of auxiliary digraph D'

- We start from $D[C]$.
- We add one vertex t_a for every arc a entering C.





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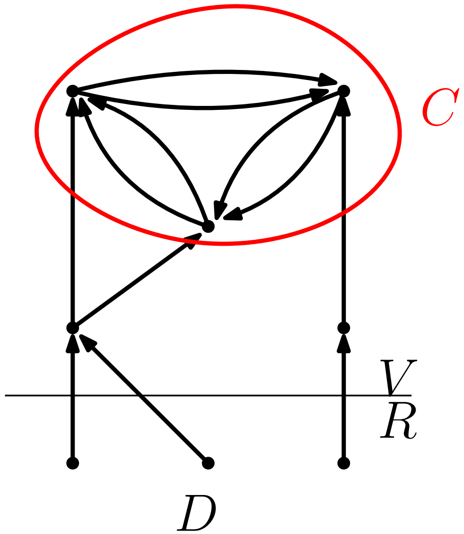

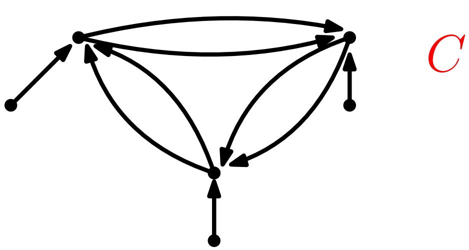

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- We start from $D[C]$.
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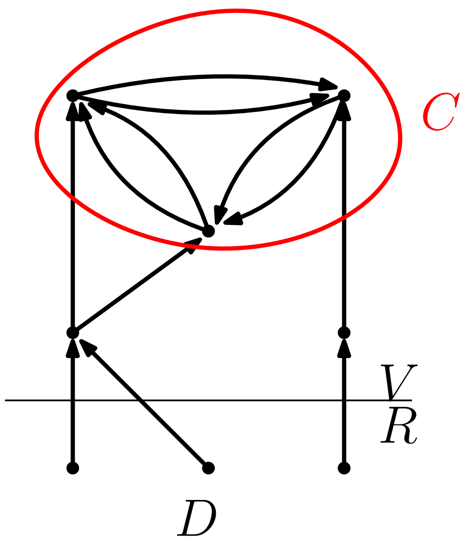

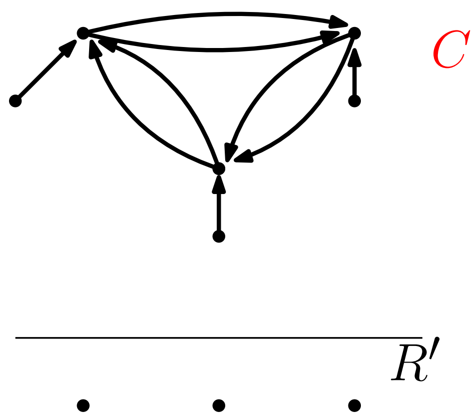

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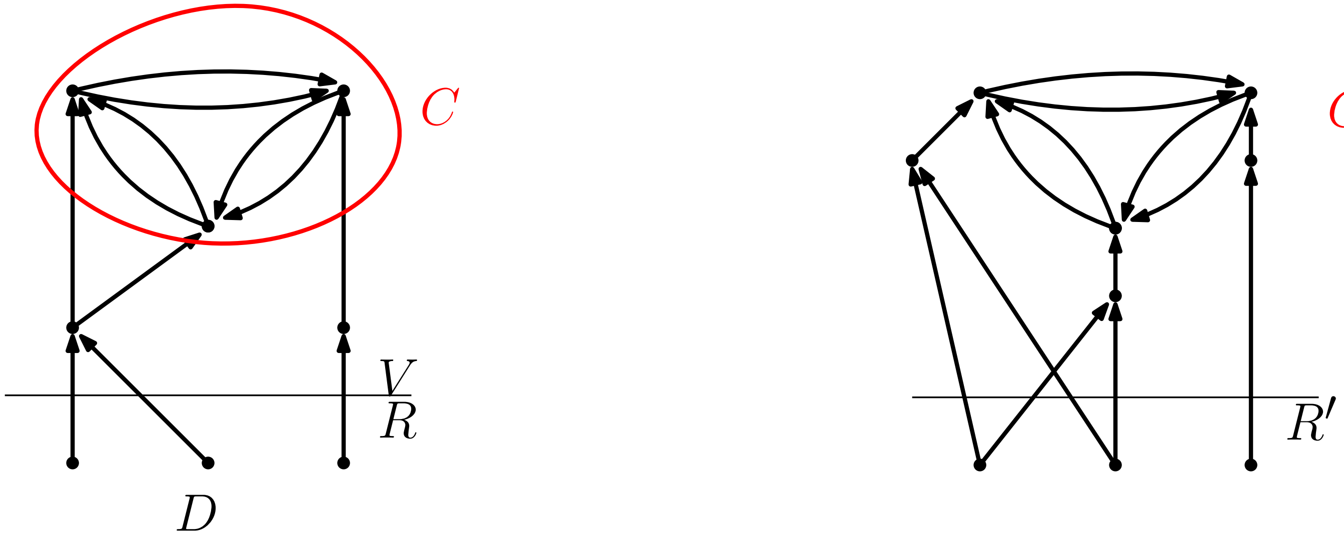

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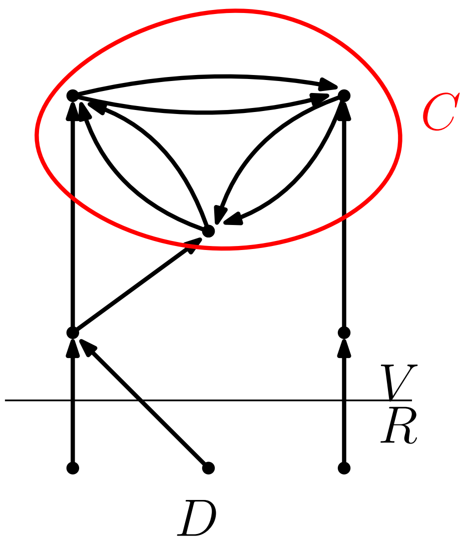
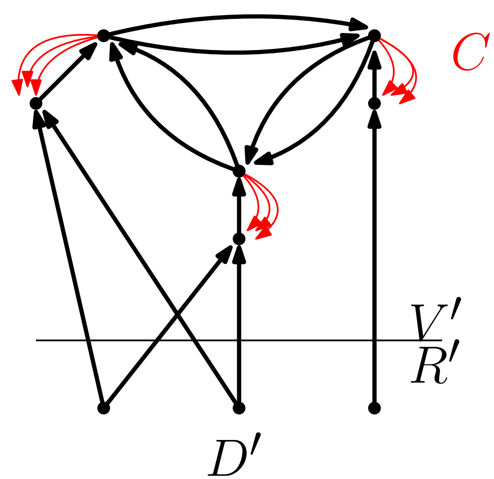
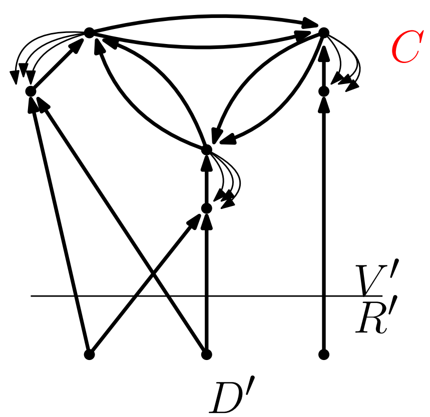
- We add $|R'|$ arcs from the head of a to t_a for every a entering C.





Merging

- We can show that D' satisfies the conditions for Edmonds' strong theorem.



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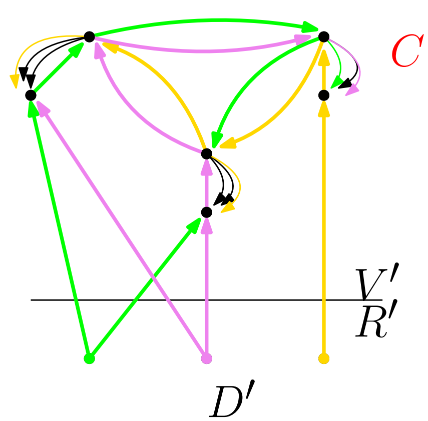

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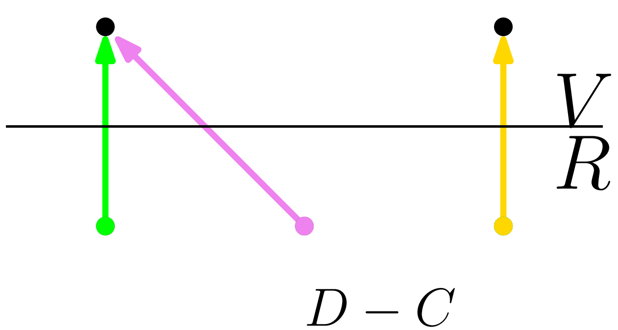

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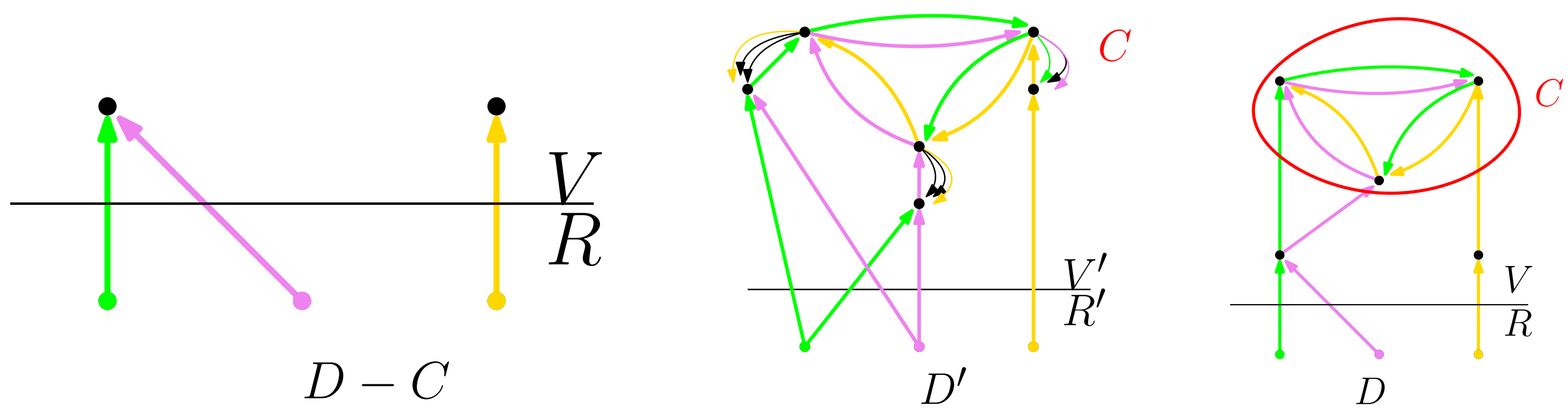

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- Recall the packing $\{B_r^1 : r \in R_1\}$ in D_1.
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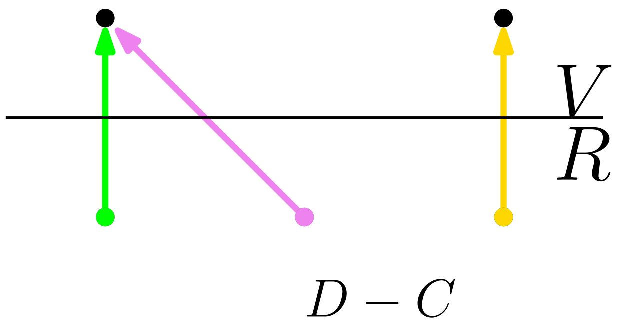
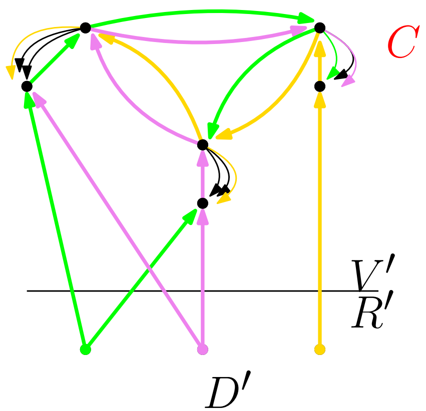
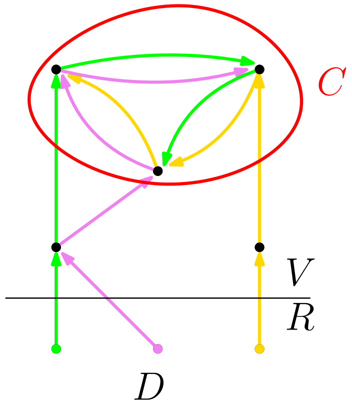
- This is a desired packing of reachability r-arborescences.







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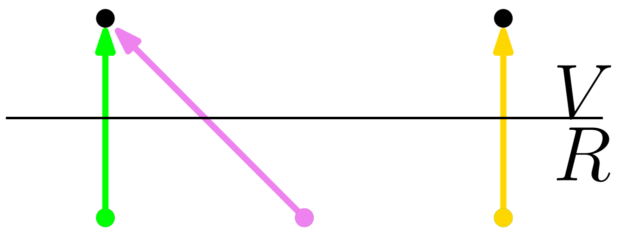
- This finishes the proof.



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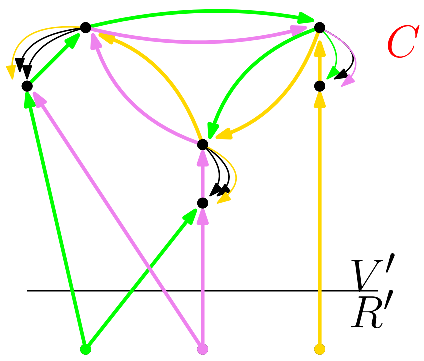

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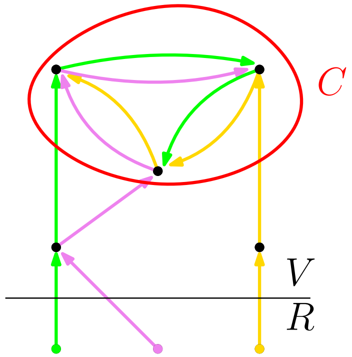

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- The objects considered can be generalized from digraphs to mixed hypergraphs.
- We prove a theorem combining all these generalizations.