Connected tree-width of a series-parallel graphs

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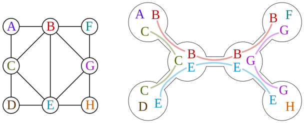
Treewidth

Definition

A tree decomposition T of a graph G = (V,E) is a tree where nodes are subset of V_G. Each vertex and each edge appear in T and $\forall x \in$V, the set of nodes containing x has to induce a connected sub-tree of T.



[1]

Treewidth

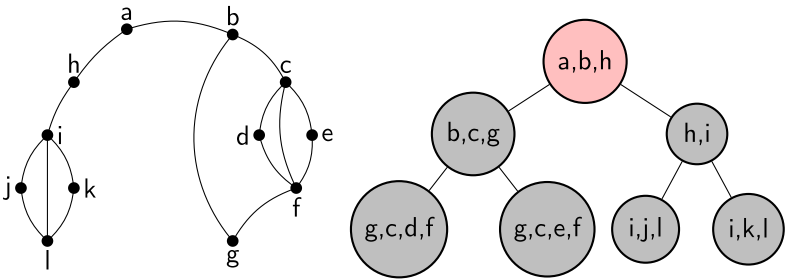
Let G be a graph. The width of a tree decomposition T of G is width(T) $= \max\{|X| - 1 \mid X \in$ T$\}$.
The treewidth of the graph G is tw(G) $= \min_{\text{T}}\{$width(T) with T a tree decomposition of G$\}$.

[1] Source: Wikipedia

Connected treewidth

Connected treewidth

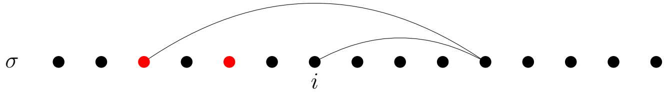
A tree decomposition T is connected if there exists r such that for every path p from r, the subgraph of G induces by the vertices in p is a connected subgrah of G. The connected treewidth of a graph G is $\mathrm{ctw(G)} = \min_{\mathrm{T}}\{\mathrm{width(T)}$ with T a connected tree decomposition of G$\}$.



Layout

Layout

Let G be a graph. A layout σ is a permutation of the vertices of G.



Support set

$\forall i \in [1, n]$, we define $S_\sigma(i) = \{x \in V_G \mid \sigma(x) < \sigma(i) \wedge \exists$ a path p from i to x with internal vertices in $\sigma_{>i}\}$.

Support set

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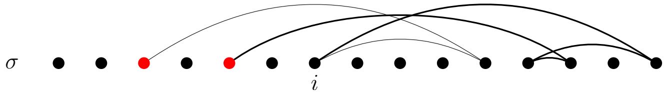


Support set

$\forall i \in [1, n]$, we define $S_\sigma(i) = \{x \in V_G \mid \sigma(x) < \sigma(i) \wedge \exists$ a path p from i to x with internal vertices in $\sigma_{>i}\}$.

Tree vertex separation number

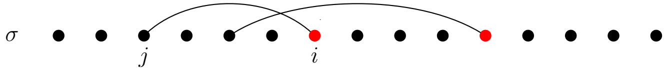
We denote $tvs(G) = \min_\sigma \max_{i \in [1,n]} |S_\sigma(i)|$.

Connected layout

Connected layout

Let G be a connected graph. A layout σ is a connected layout if one of the two following equivalent proprieties are satisfied:

- $\forall i \in \mathrm{V_G}, \mathrm{G}[\sigma_{\leq i}]$ induces a connected subgraph of G.
- $\forall i \in \mathrm{V_G}, \exists j$ such that σ_j is a neighbour of σ_i with $j < i$.



Connected tree vertex separation number

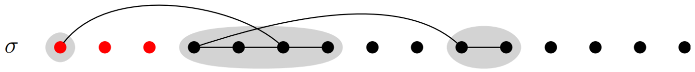
We denote $\mathrm{ctvs(G)} = \min_\sigma \max_{i\in[1,n]} |\mathrm{S}_\sigma(i)|$ with σ a connected layout.

By definition, we have $\mathrm{tvs(G)} \leq \mathrm{ctvs(G)}$.

Connected rooted layout

Rooted layout

Let (G, R) a rooted graph where $\mathrm{R} \subset V_{\mathrm{G}}$. A rooted layout σ on (G, R) is a layout where R is a prefix of σ.



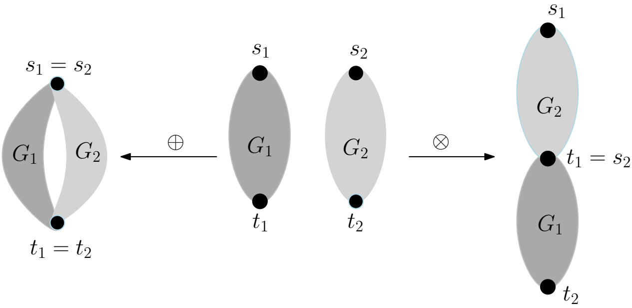
Connected rooted layout

A rooted layout σ is connected if, $\forall i \in [1, n]$, every connected component of the subgraph $\mathrm{G}[\sigma_{\leq i}]$ contains a root vertex.

Series-parallel graphs

Definition

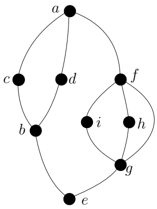
A graph G is a series-parallel graph if it is an edge $\{x, y\}$ or it can be built from two other series-parallel graphs G_1 and G_2 by the series composition $\bigotimes$ or by the parallel composition $\bigoplus$.

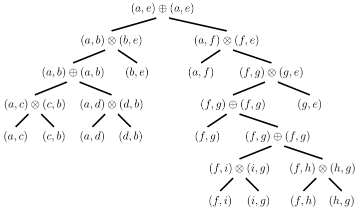


Series-parallel tree

Series-parallel tree

A series-parallel graph G can be represented by a series-parallel tree where each node is $\bigotimes$ or $\bigoplus$ composition.





Theorem

If G is a bi-connected series-parallel graph, then:
$\forall (x,y) \in E_G, (G,(x,y)) = (G_1,(x,y)) \bigoplus (G_2,(x,y))$ with $G_2 = K_2$.

Results

Theorem

- [Dendris N., Kirousis L., Thilikos D. TCS'97]: $\mathrm{tw}(G) = \mathrm{tvs}(G)$.
- [Adler I., Paul C., Thilikos D. FST-TCS'19]: $\mathrm{ctw}(G) = \mathrm{ctvs}(G)$.
- Price of connectivity: $\forall k \in \mathbb{N}$, there exists G such that

$$\mathrm{ctw}(G) - \mathrm{tw}(G) \geq k.$$

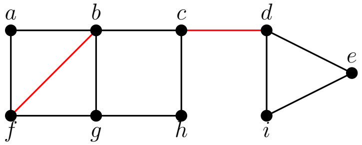
Our result

[Mescoff G., Paul C., Thilikos D.]: We can compute the connected treewidth of series-parallel graphs in $\mathcal{O}(n^2 \cdot \log n)$ time where n is the number of vertices of G.

Extended graph

Extended graph

Let G = (V,E) a graph and F a set of edges disjoint from E_G. We denote G^{+F} the extended graph G with with fictive edges from F. We said that G is the solid graph of G^{+F}.



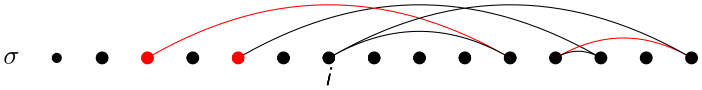
Connexity of G^{+F}

Fictive edges do not increase connexity of G^{+F}. Connected components of G^{+F} are exactly the connected components of the solid graph G.

Extended Path

Extended path

An extended path is a path of G^{+F} containing fictive edges.



Extended cost

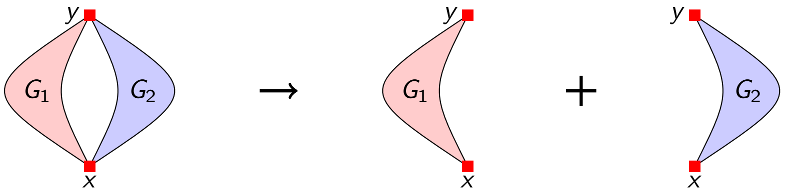
- $\forall i \in V_G$ we define $S_\sigma^{+F}(i) = \{x \in V_G | \sigma(x) < \sigma(i) \wedge \exists$ a extended path p from i to x with internal vertices in $\sigma_{>i}\}$.
- $\mathrm{ectvs}(G) = \min_\sigma \max_{i \in [1,n]} |S_\sigma^{+F}(i)|$ with σ a connected layout.

Parallel composition without fictive edges

Lemma

Let $(\mathrm{G}^{+\emptyset}, \mathrm{R})$ be a rooted extended graph such that $\mathrm{R} = \{x, y\}$ and $\mathrm{G} = \mathrm{G}_1 \bigoplus \mathrm{G}_2$ with $\mathrm{G}_1 = (\mathrm{G}_1, (x, y))$ and $\mathrm{G}_2 = (\mathrm{G}_2, (x, y))$. Then,

$$\mathrm{ectvs}(\mathrm{G}^{+\emptyset}, R) = \max\{\mathrm{ectvs}(\mathrm{G}_1^{+\emptyset}, \mathrm{R}), \mathit{ectvs}(\mathrm{G}_2^{+\emptyset}, \mathrm{R})\}.$$



Parallel composition without fictive edges

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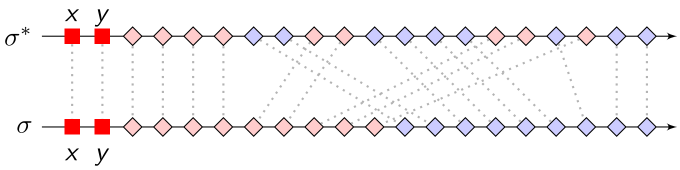
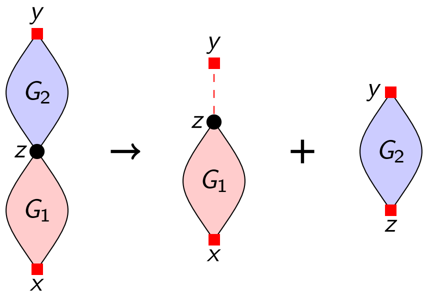


Figure: Rearranging a layout of minimum cost of an extended graph $G = G_1 \oplus G_2$. Red vertices belongs to $V_1 \setminus \{x, y\}$ and blue vertices belong to $V_2 \setminus \{x, y\}$.

Series composition without fictive edges

Lemma

$$\mathrm{ectvs}(G^{+\emptyset}, R) = \min \left\{ \begin{array}{l} \max\left\{\mathrm{ectvs}(\tilde{G}_1^{+\{zy\}}, R), \mathrm{ectvs}(G_2^{+\emptyset}, R_2)\right\} \\ \max\left\{\mathrm{ectvs}(\tilde{G}_2^{+\{zx\}}, R), \mathrm{ectvs}(G_1^{+\emptyset}, R_1)\right\} \end{array} \right\}.$$



Series composition without fictive edges

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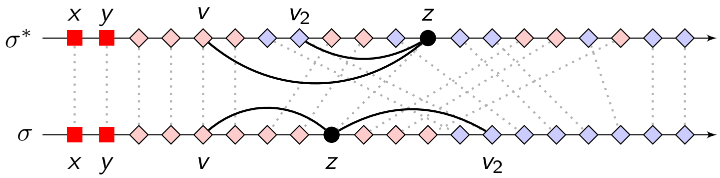
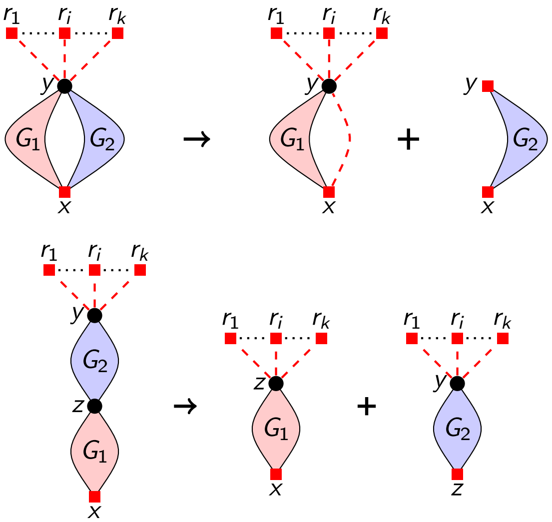
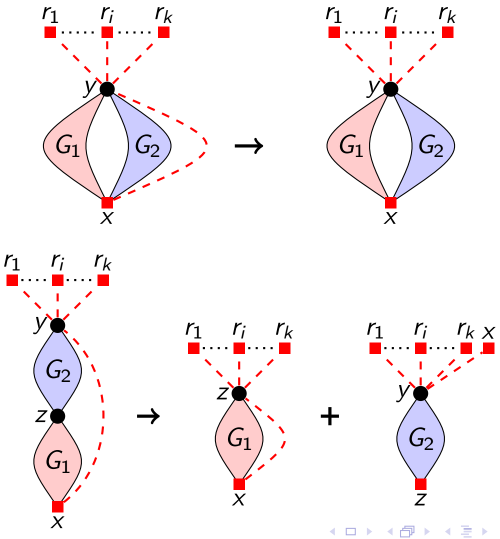


Figure: Rearranging a layout σ^* of $G = G_1 \otimes G_2$ of minimum cost into $\sigma = \langle x, y \rangle \odot \sigma^*[V_1 \setminus \{x\}] \odot \sigma^*[V_2 \setminus \{y, z\}]$.

Parallel and series composition with extended graph



Composition with the fictive edge (x,y)



Complexity

Complexity

Let G a biconnected graph.

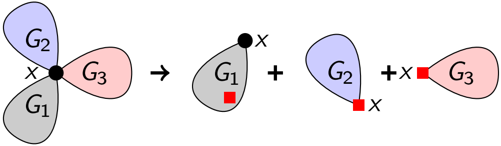
- $\mathrm{ctw}(\mathrm{G}) = \min_{(x,y)\in \mathrm{G}}(\mathrm{G}^{+\emptyset}, \{x, y\}) \leftarrow \mathcal{O}(n)$.
- We have at most $2n$ steps in our algorithm $\leftarrow \mathcal{O}(n)$.
- For each step, we compute at most αn results with some constant $\alpha \leftarrow \mathcal{O}(n)$.

Which gives a total time complexity in $\mathcal{O}(n^3)$. With a better complexity analysis, we can show that the real time complexity is $\mathcal{O}(n^2 \cdot \log n)$.

Generalization

Treewidth at most 2

A graph G has treewidth at most 2 iff its biconnected components are series-parallel graphs. So, G contain a cut vertex or G is a biconnected series-parallel graph.



Complexity

Since the complexity is $\mathcal{O}(n^2 \cdot \log n)$ for every biconnected component and since we try for every starting vertex, the total time complexity is $\mathcal{O}(n^3 \cdot \log n)$.

Conclusion

Conclusion

We see in this presentation how compute the connected treewidth for graph with treewidth at most 2. The complexity of the general problem is still open:

- **Conjecture 1:** Connected treewidth can be computed by an $\mathcal{O}(n^{f(\text{tw}(G))})$-time algorithm.
- **Conjecture 2:** Connected treewidth bounded by k can be decided by an $\mathcal{O}(n \cdot f(k, \text{tw}(G)))$-time algorithm.
- **Conjecture 3:** Connected treewidth bounded by k can be decided by an $\mathcal{O}(n^{f(k)})$-time algorithm.