On the k shortest simple paths : A faster algorithm with low memory consumption

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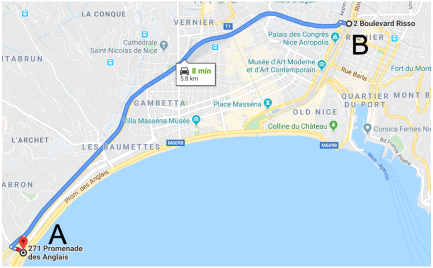
Motivation

A shortest path is not enough!

- A shortest path may be affected
- Some constraints can be added
  - Bounded delay, cost ...
  - A user may prefer the coast road ...
- User likes diversity!



Give the user a set of 'good' choices

Motivation

Sometimes, it is hard to specify constraints that a path should satisfy

Applications:

- bioinformatics: biological sequence alignment
- natural language processing
- list decoding
- parsing
- network routing
- many more ...

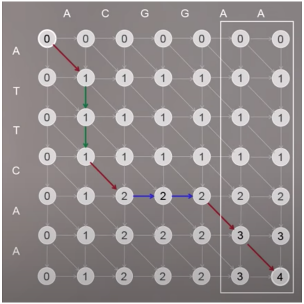


Figure: aligning two DNA sequences

k shortest paths problem

Definition

Input:

- Directed weighted graph $D = (V, A)$ with $w : A \to \mathbb{R}^+$,
- Two terminals s and t and an integer k

Output:

- k paths $P_1, P_2, \ldots, P_k$ from s to t such that $w(P_i) \leqslant w(P_{i+1})$, $1 \leqslant i < k$ and $w(P_k) \leqslant w(Q)$ for all other s-t paths Q

  where $w(P) = \sum_{e \in A(P)} w(e)$

simple vs not simple

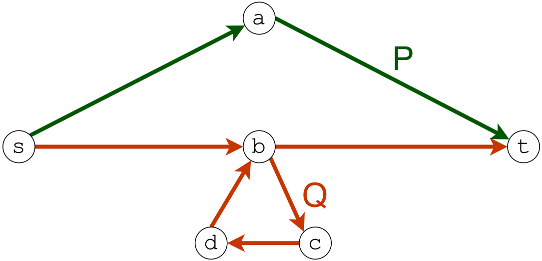


Figure: P is simple, Q is not simple

Definition (simple path)

a path is simple if and only if it has no repeated vertices

Complexity of the problem

Theorem (Eppstein '97)

The problem of finding k shortest paths can be solved in time $O(m + n \log n + k)$

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Theorem (Yen '71)

The problem of finding k shortest simple paths can be solved in time $O(kn(m + n\log n))$

Complexity of the problem

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Theorem (Yen '71)

The problem of finding k shortest simple paths can be solved in time $O(kn(m + n\log n))$

Theorem (Williams and Williams '10)

All-Pairs-Shortest-Paths (APSP) $\prec_{(m,n)}$ *2-SSP ($\Leftrightarrow \tilde{O}(n.m)$ for 2-SSP)*

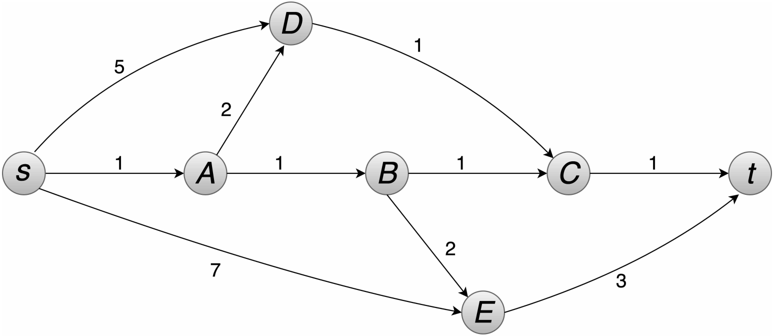
Yen's algorithm (the algorithm)

Yen's idea:

- A second shortest simple path is a shortest simple detour from a shortest path

Complexity: $O(kn \underbrace{(m + nlogn)}_{\text{Complexity of finding one SP}})$

Yen's algorithm (example)



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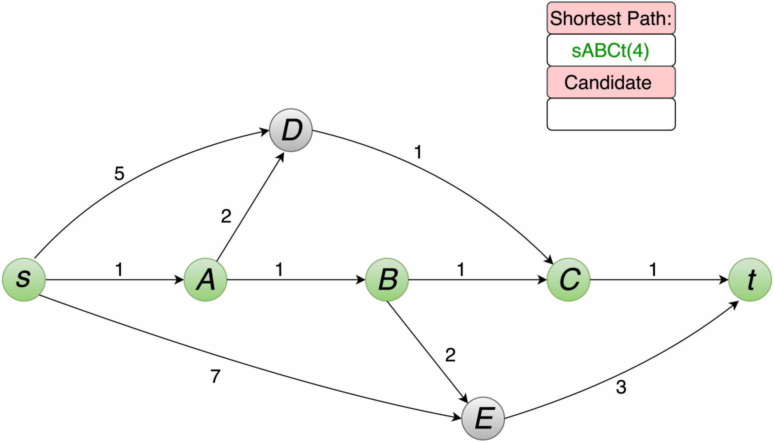

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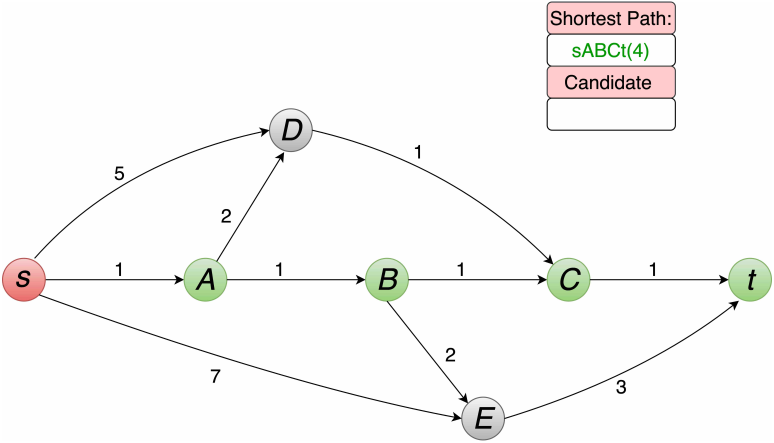

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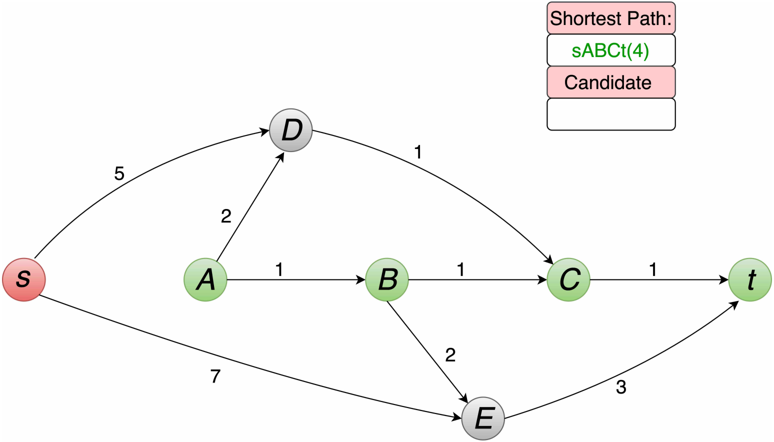

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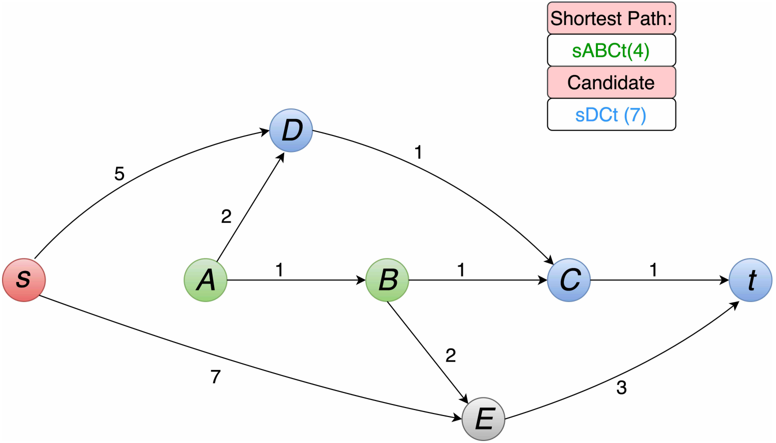

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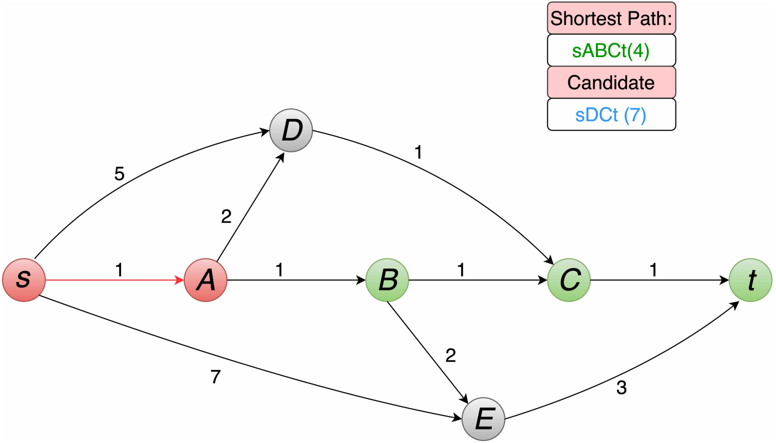

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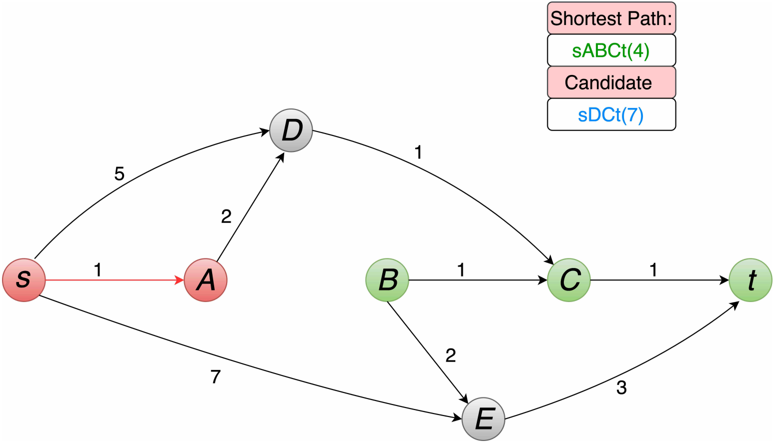

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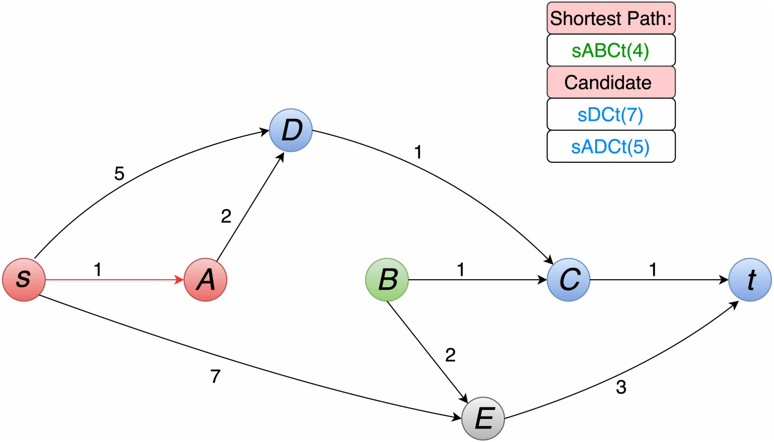

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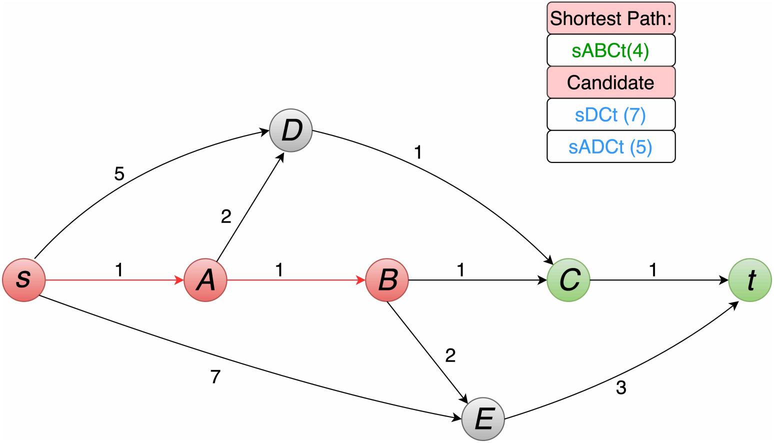

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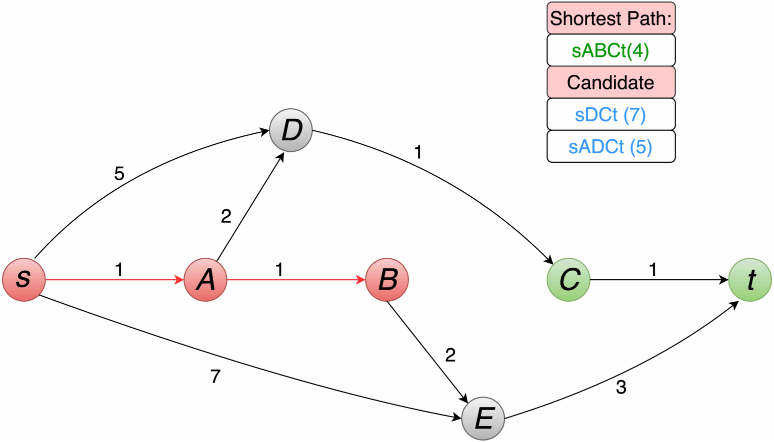

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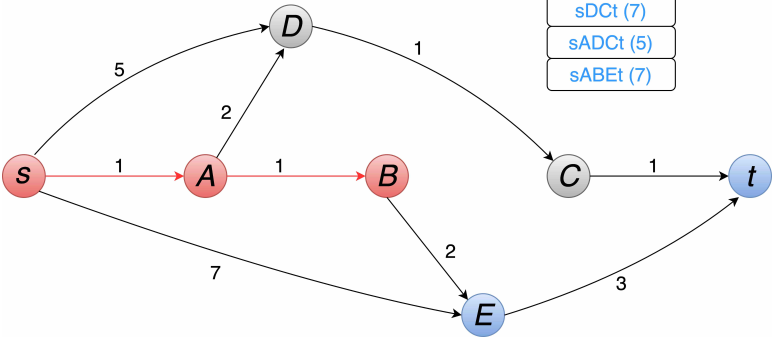
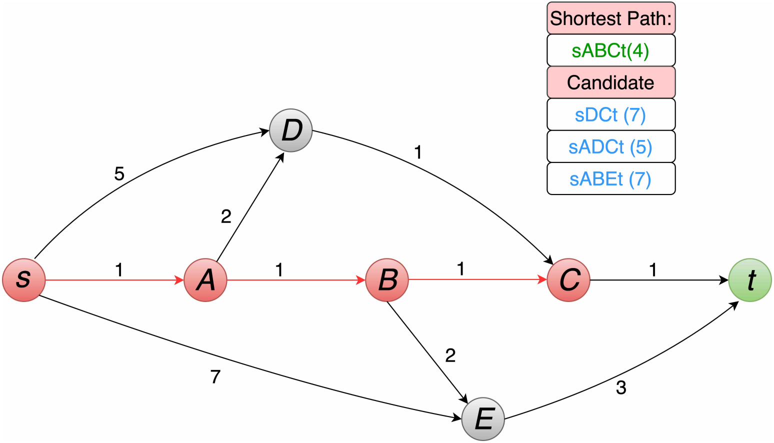


Shortest Path:
sABCt(4)
Candidate
sDCt (7)
sADCt (5)
sABEt (7)

Yen's algorithm (example)



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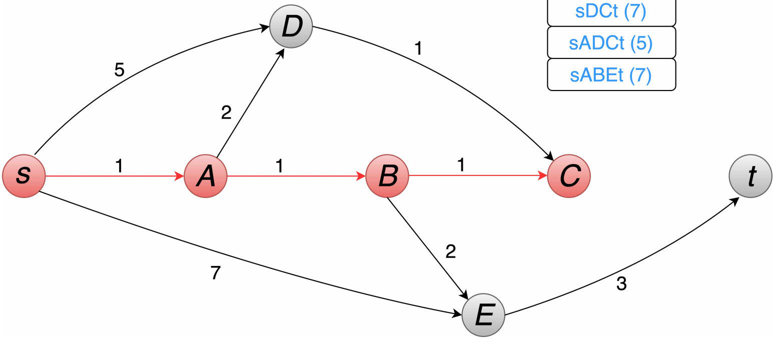

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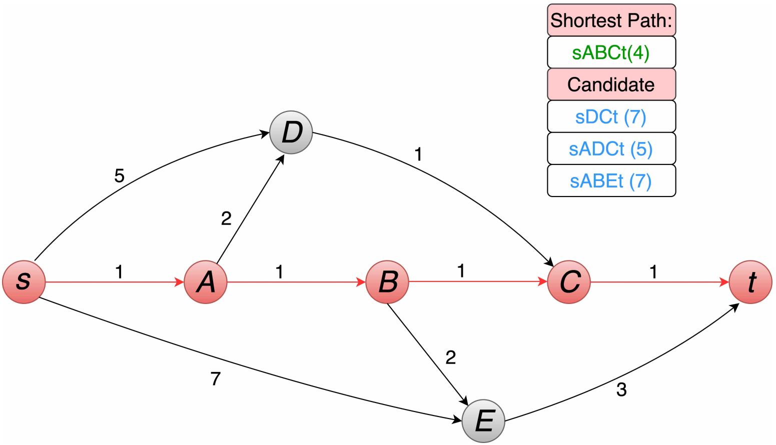

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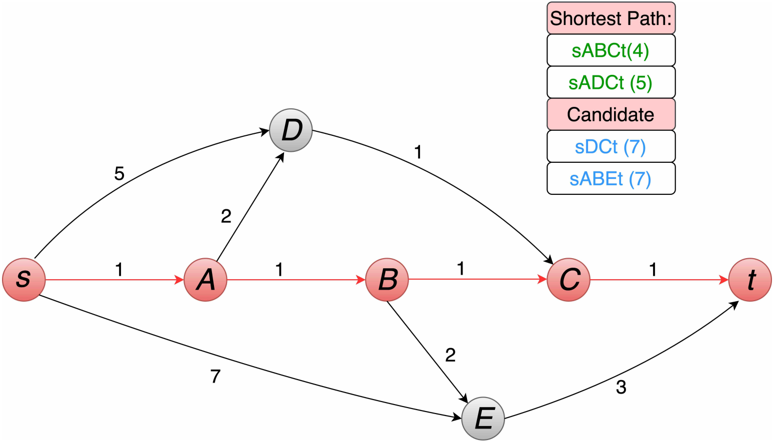

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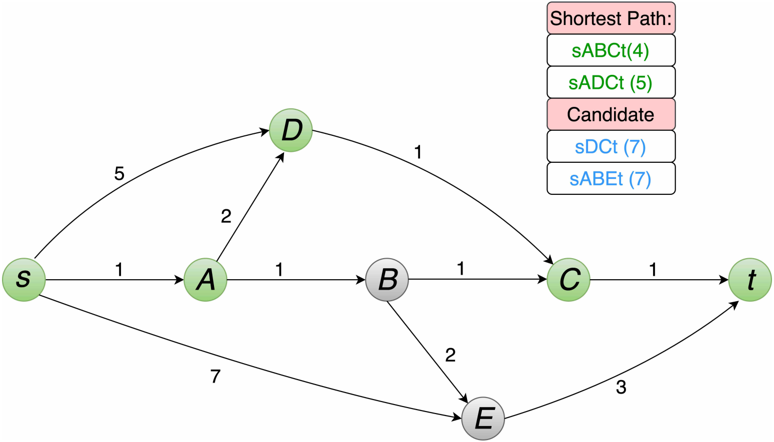

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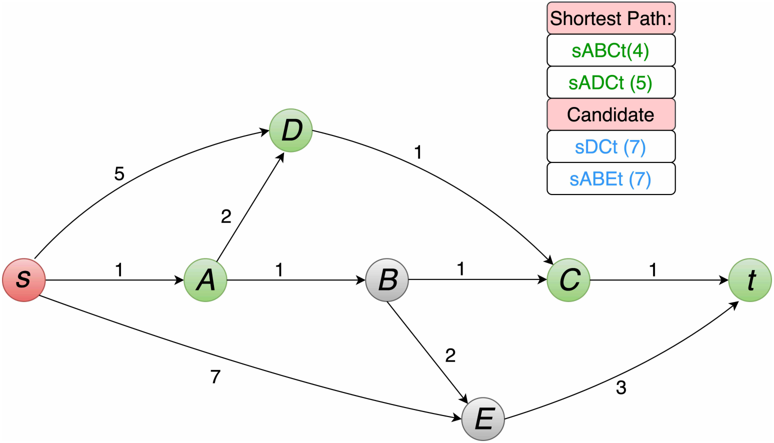

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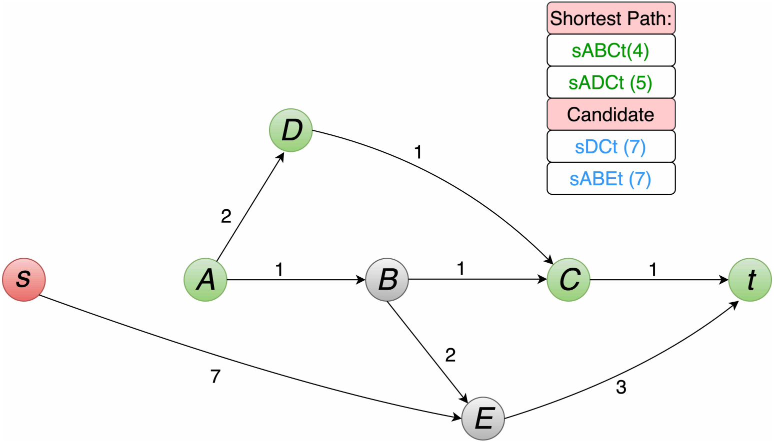

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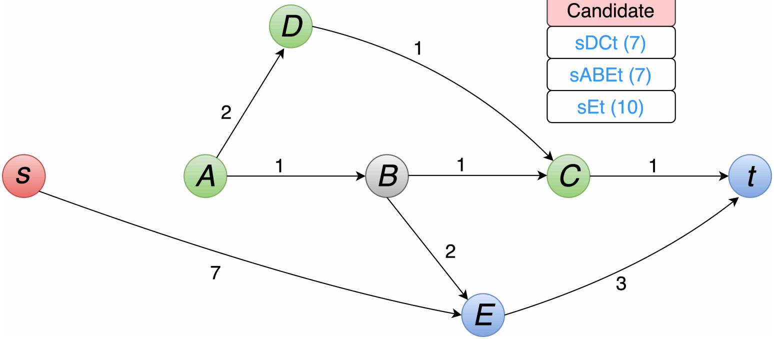

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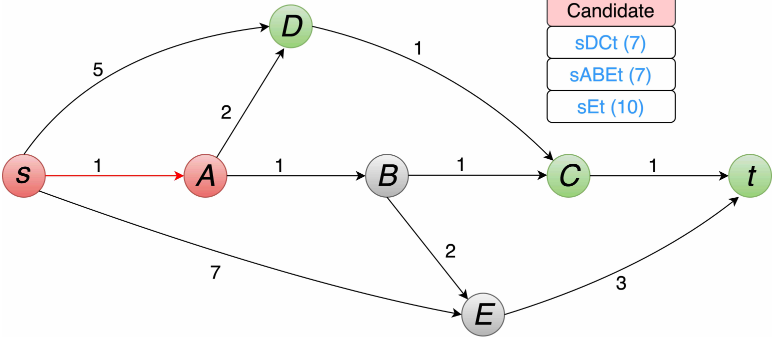

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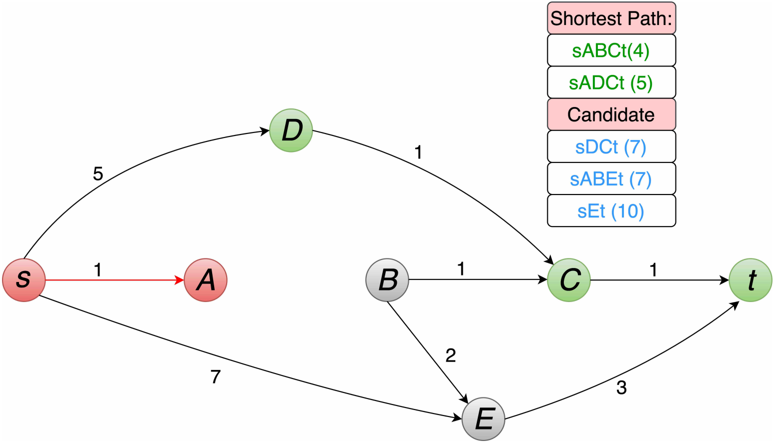

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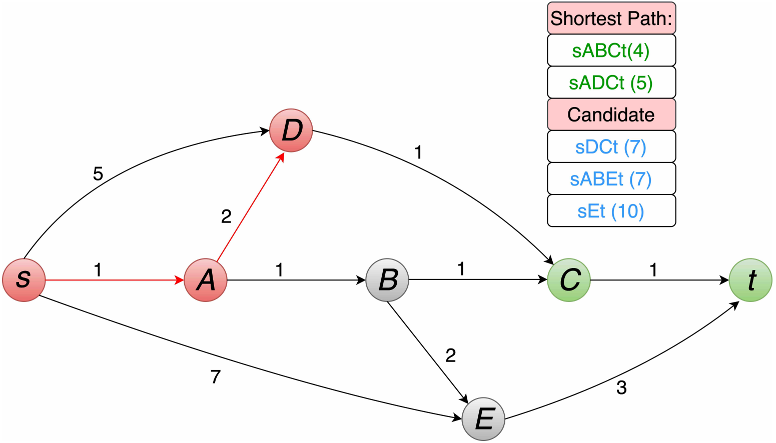

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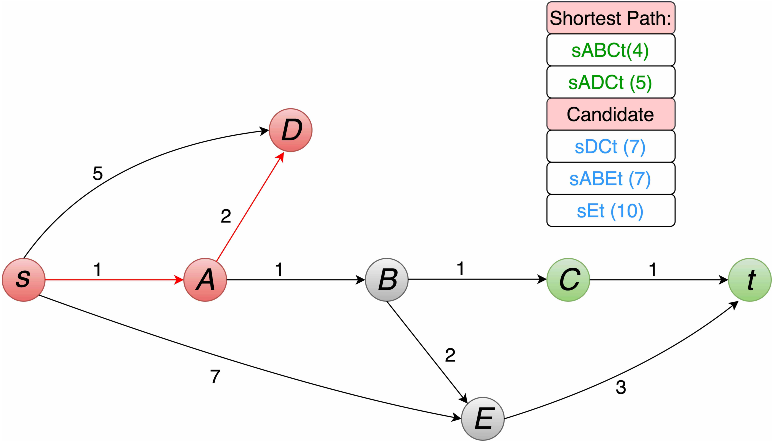

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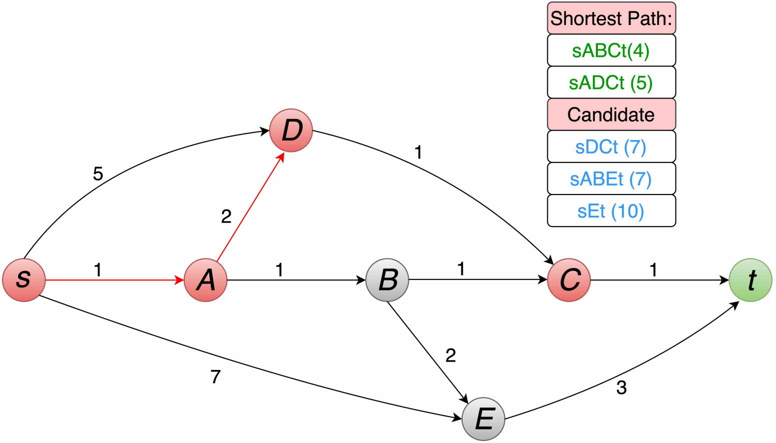

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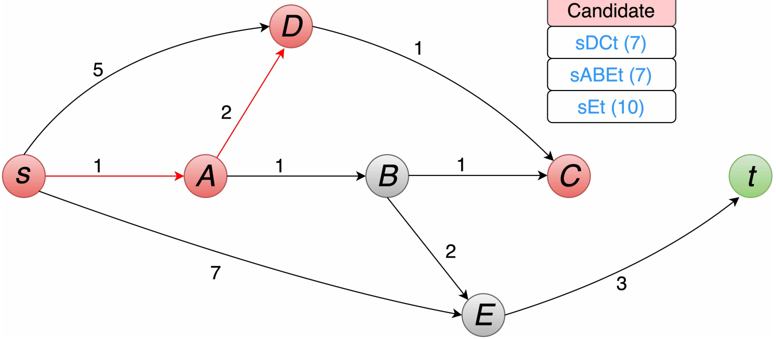

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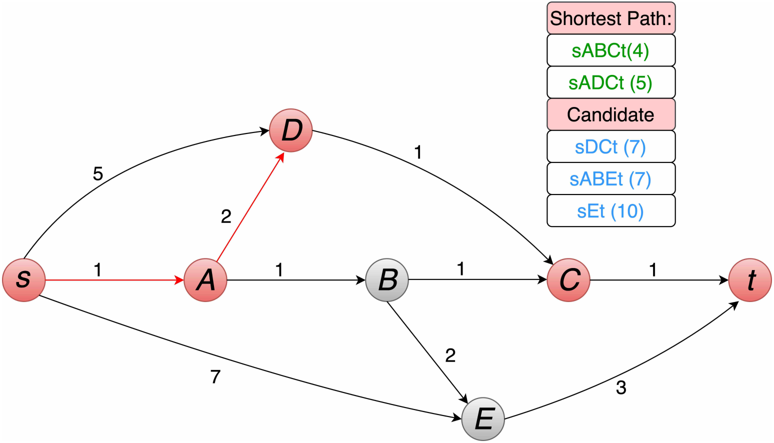

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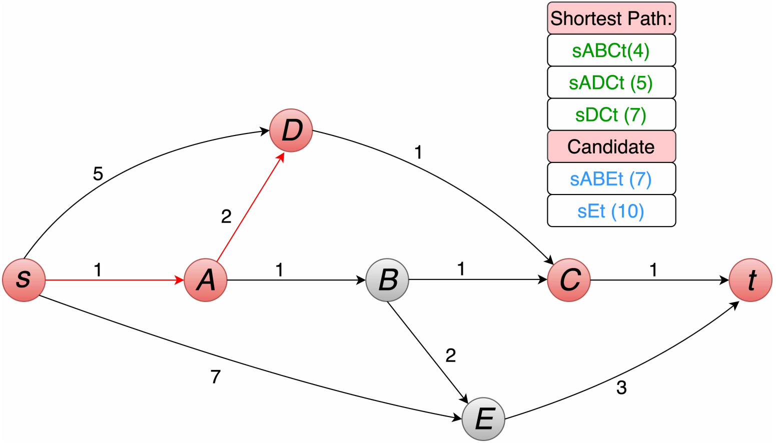


Algorithm engineering

On Road networks:

- 9th DIMAC'S implementation challenge followed by a set of improvements
- (NC) by Feng 2014 (speed up the detours computation)
- (SB*) by Kurz and Mutzel 2016, followed by Al Zoobi et al. 2019 (larger memory consumption)

Algorithm	time (ms)
Yen	11,316
NC	823
SB* (large memory)	117

Table: DC network ($n \approx 10,000$; $m \approx 15,000$ and $k = 1,000$)

- We proposed the PNC
  - The fastest algorithm with low memory consumption

Postponed Node Classification (PNC) aglorithm

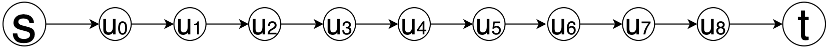
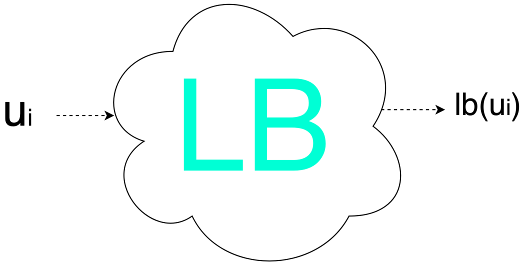
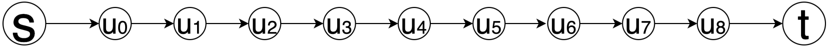


Figure: A shortest path P_0 from s to t

- So many calls of SP algorithm
- Can we skip some ?

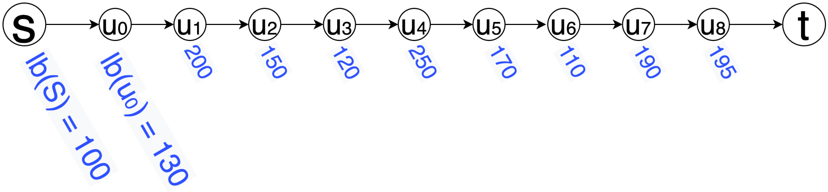
PNC: skipping SP calls





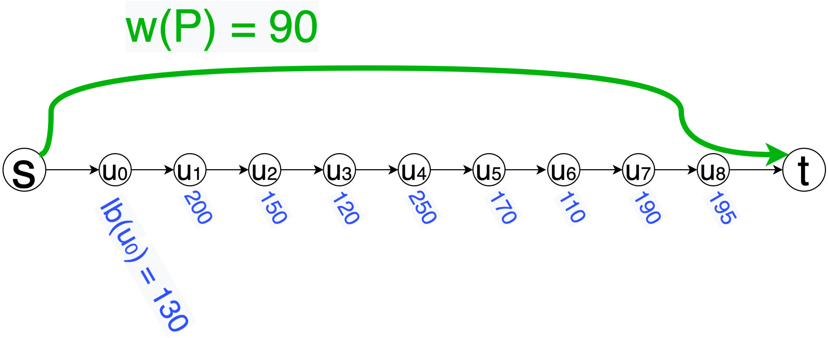
- $lb(u_i) = c$ s.t. a shortest simple detour of P_0 at u_i is bigger than c
- LB answers in a pivot step

PNC: skipping SP calls



- After applying LB on each vertex of P_0

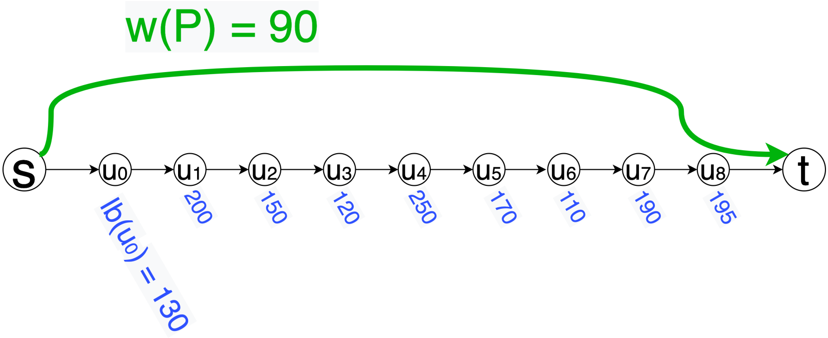
PNC: skipping SP calls



If a shortest simple detour P of P_0 has length less $lb(u_i)$ for each $u_i \in P_0$

- What can we deduce?

PNC: skipping SP calls



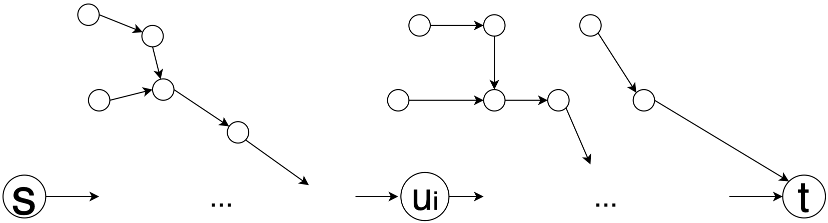
If $w(P) \leqslant lb(u_i)$ for each $u_i \in P_0$

- Then P is a second shortest path

Otherwise

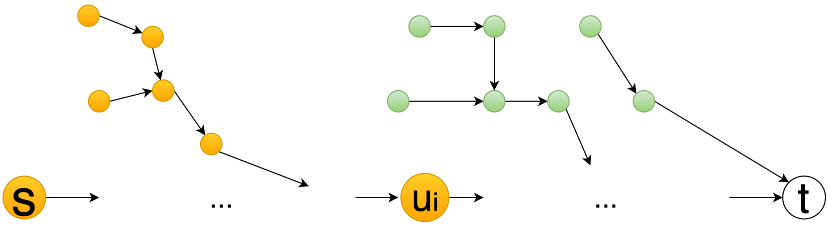
- continue until P' (with $w(P') \leqslant lb(u_i)$ for each $u_i \in P_0$) is found

PNC: describing *LB*



While computing P_0, the reversed shortest path tree T rooted at t is kept

PNC : describing LB

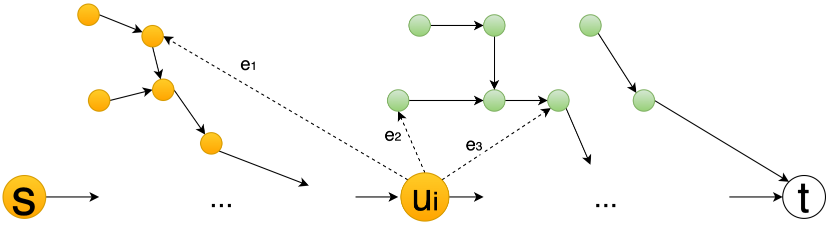


For each vertex u_i, each vertex v in T is colored:

- Yellow if the path from v to t in T cross u_i
- Green Otherwise

Classification of Feng 2014

PNC : describing *LB*



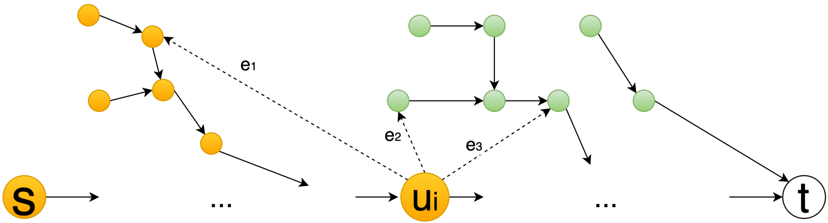
For each arc $e_j = (u_i, v)$ tailing at u_i:

- $\delta(e_i) = w(s, \cdots, u_i) + w(e_j) + w(P^T_{v-t})$: the cost of the shortest detour at e_j

Let $e_{min} = (u_i, v_{min})$ be the arc with minimum δ (δ_{min})

inspired by Kurz and Mutzel (2016)

PNC : describing LB

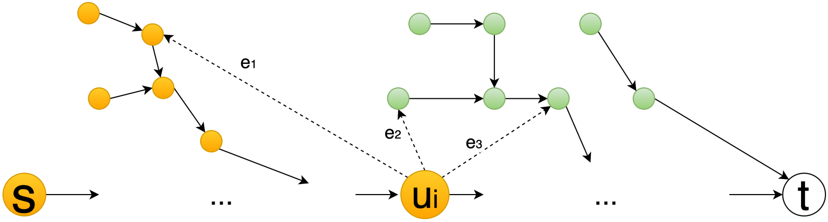


Let $e_{min} = (u_i, v_{min})$ be the arc with minimum δ (δ_{min})

Claim: $\delta_{min} = lb(u_i)$

δ_{min} is the length of a shortest detour (not necessarily simple) at u_i

PNC : describing *LB*



NC algorithm calls an SP algorithm at u_i

- (PNC) postpone such call so it may be skipped

PNC : The algorithm

Algorithm 1 $PNC(G, s, t)$

$C \leftarrow \{P_0\}$
while C is not empty **do**
    $P \leftarrow extractMin(C)$
    if P is simple **then**
        add P to the output
        using the LB procedure, add the shortest detours of P to C
    else
        repair P into a simple path and add it to C

PNC : The algorithm

Algorithm 2 $PNC(G, s, t)$

```
C ← {P_0}
while C is not empty do
    P ← extractMin(C)
    if P is simple then
        add P to the output
        using the LB procedure, add the shortest detours of P to C
    else
        repair P into a simple path and add it to C
```

Remarque:

- Low memory consumption: only one shortest path is kept in the memroy (+ candidate paths)

PNC : Evalutation

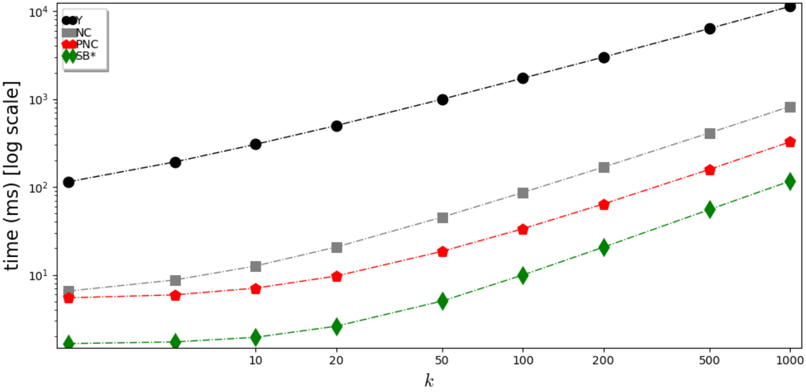


Figure: The average running time of the *kSSP* algorithms with respect to k on DC road networks ($n \approx 10,000$; $m \approx 15,000$)

PNC : Evalutation

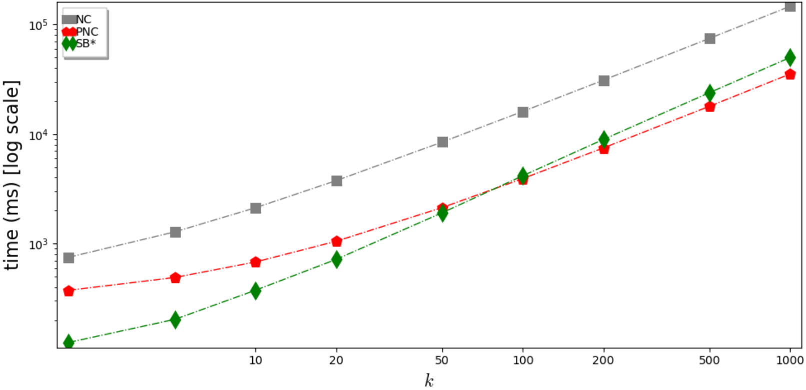


Figure: The average running time of the *kSSP* algorithms with respect to k on COL road networks ($n \approx 500,000$; $m \approx 1,000,000$)

Ongoing - future work

- Evaluate these algorithms on complex networks
- Study the impact of these improvements on others problems
  - Path with resource constraints, bioinformatics problems ...
- Study the problem when the arcs may have negative weights

Questions ?