Comparison between the metric dimension and the zero-forcing number in graphs

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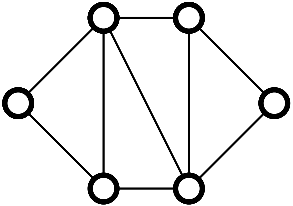
JGA, November 2020

Metric dimension [Harary and Melter, 1975]

Identify a vertex with its distances to some specific vertices.



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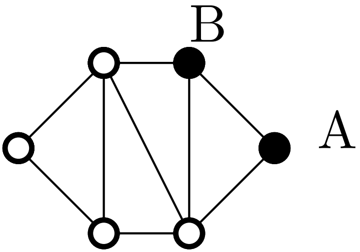

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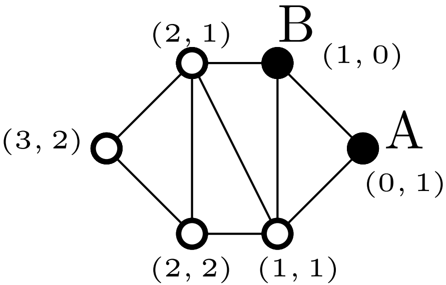

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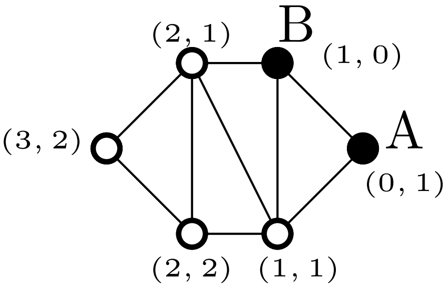
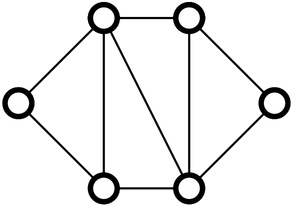


Resolving set: Subset of vertices S such that all the distance vectors to S are different.
Metric dimension: $\dim(G)$ Minimal size of a resolving set.

Zero-forcing number [AIM, 2008]



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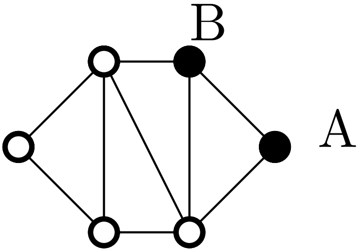

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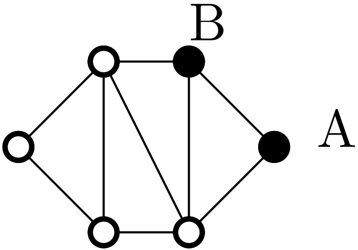
Rule : A black vertex forces its last white neighbour to become black



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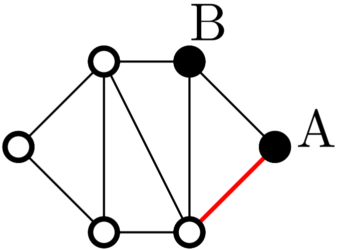

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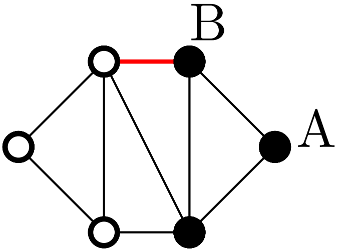

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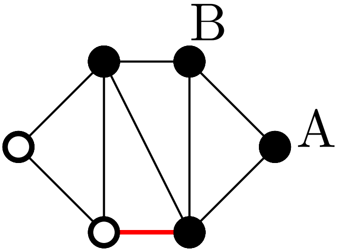

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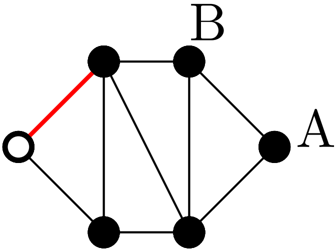

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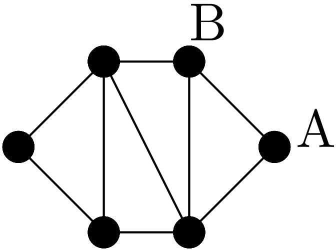
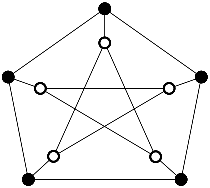


Zero-forcing number: $Z(G)$

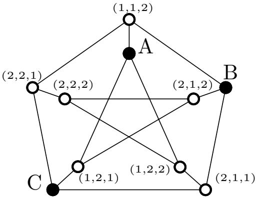
Minimum size of a set coloring the whole graph.

Comparison

Graph	dim	Z
Path	1	1
Cycle	2	2
Star	$n-2$	$n-2$
Petersen	3	5



$Z(G) = 5$

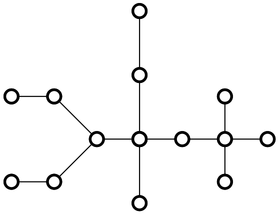


$\dim(G) = 3$

Inequality on trees

Inequality on trees Eroh, Kang, Yi, 2017

For any tree T, $\dim(T) \leq Z(T)$



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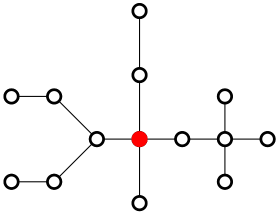

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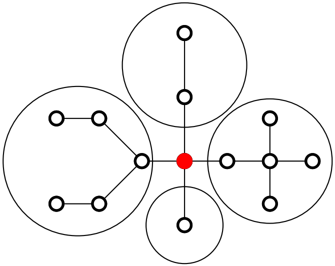

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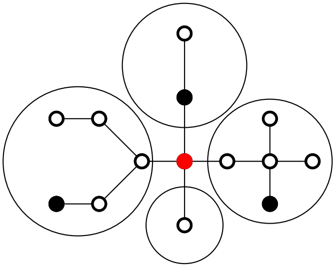

A resolving set or a zero-forcing set should contains one vertex in all but one subset.

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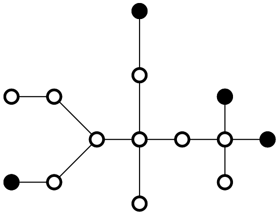

A resolving set or a zero-forcing set should contains one vertex in all but one subset.
The condition is sufficient for a resolving set but not for a zero-forcing set.

Conjecture

Conjecture [Eroh et al., 2017]

$\dim(G) \leq Z(G) + c(G)$ for any graph G.

Cycle-rank : Size of a feedback edge set of G.
Equal to $|E| - |V| + 1$ if G is connected.

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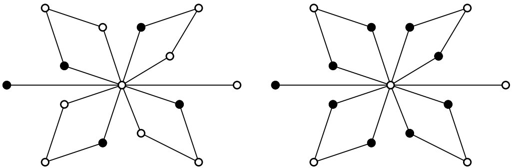

$Z(G) = c + 1$ $\quad$ $\dim(G) = 2c + 1$

Our work

Conjecture [Eroh et al., 2017]

$\dim(G) \leq Z(G) + c(G)$ for any graph G.

Two results from [Eroh et al., 2017] :

- If $c = 1$ then $\dim(G) \leq Z(G) + 1$.
- If G contains only odd cycles then $\dim(G) \leq Z(G) + 2c$.

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- Proof of the conjecture when the cycles are disjoint.

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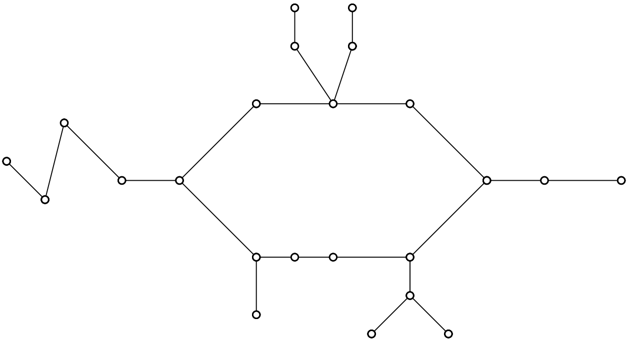
Our results :

- Proof of the conjecture when the cycles are disjoint.
- $\dim(G) \leq Z(G) + 6c$ for any graph G.

Conjecture when the cycles are disjoint



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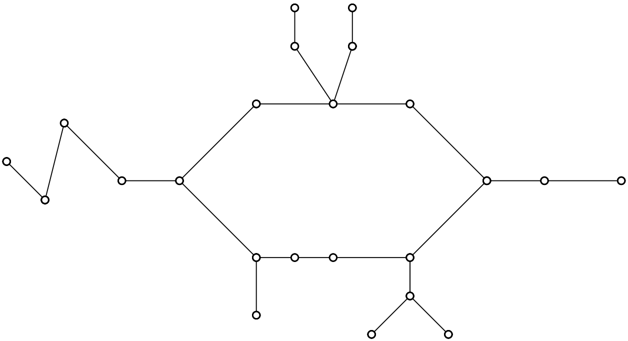
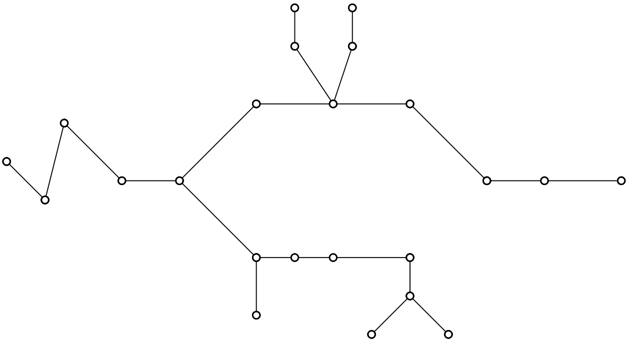


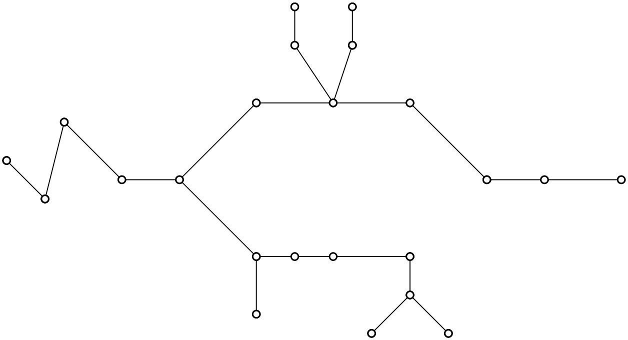
$$\dim(T+e) \leq \dim(T)+1$$

Conjecture when the cycles are disjoint



$$\dim(T + e) \leq \dim(T) + 1 \leq Z(T) + 1$$

Conjecture when the cycles are disjoint



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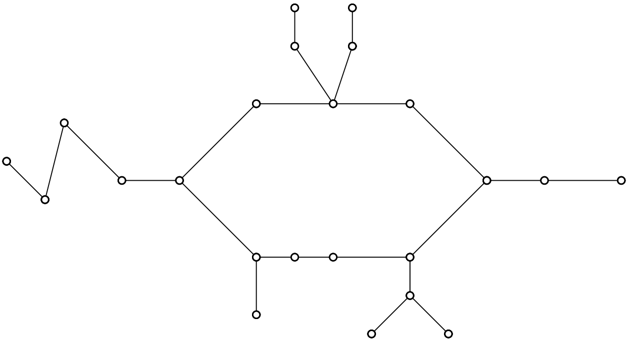

$$\dim(T+e) \leq \dim(T)+1 \leq Z(T)+1 \leq Z(T+e)+1$$

Conjecture when the cycles are disjoint : by induction.

General bound

Theorem

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For any graph G, $\dim(G) \leq Z(G) + \min(5c + |X|, 3c + 5|X|)$.

$|X|$: Size of a feedback vertex set of G.

General bound

Theorem

For any graph G,
$$\dim(G) \leq Z(G) + \min(5c(G) + |X|, 3c(G) + 5|X|).$$

- Find a tree structure of G.
- Create a resolving set S for G based on this tree structure.
- Bound the size of the intermediate sets.
- Compare the size of S and a minimal zero-forcing set of G.

Tree structure

- Create a set of vertices M such that
  - $G \setminus M$ is a forest.
  - Each connected component T is connected to M by at most two edges.

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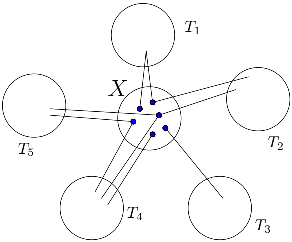

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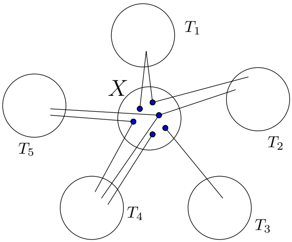

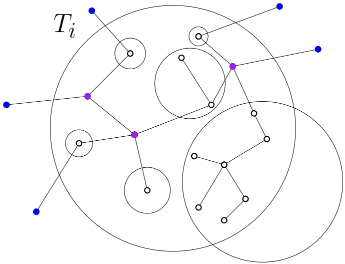

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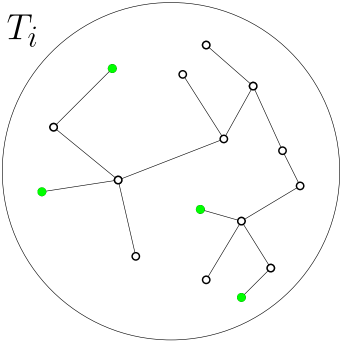
- Create a resolving set S for G containing a resolving set for each component of $G \setminus M$.



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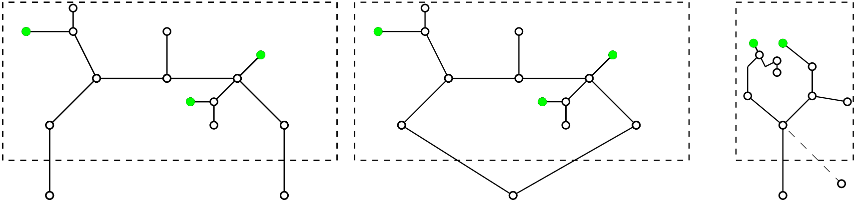

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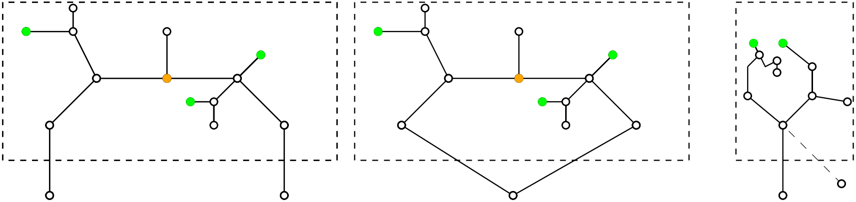

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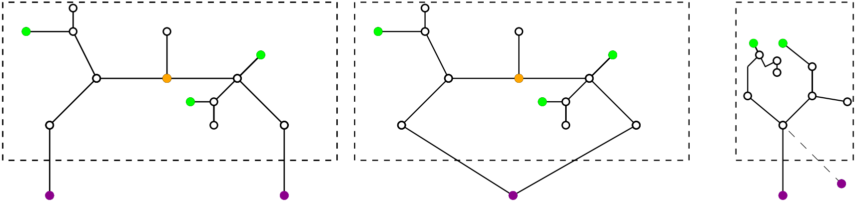
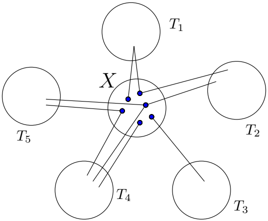


$$S = (\bigcup_{T_i} S_{T_i}) \cup P \cup M$$

Link to Zero-forcing number

- Compare the size of S and a minimal zero-forcing set Z of G.



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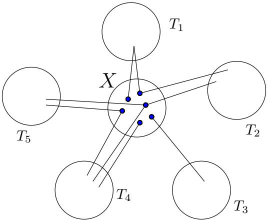

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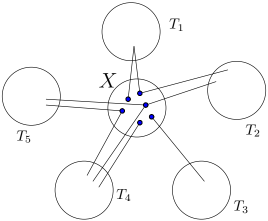

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- $Z(T_i) \geq \dim(T_i)$

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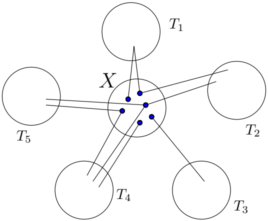

- $|Z| + |X| \geq \sum_{T_i} Z(T_i)$
- $Z(T_i) \geq \dim(T_i)$
- $Z(G) + |X| \geq |S| - |P| - |M|$

Bounds on the sets

- Bound the size of the intermediate sets.
- $|M| \leq \min(c + 2|X|, 2c)$
- $|P| \leq c + |M|$
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$$\dim(G) \leq Z(G) + |X| + |P| + |M|$$

$$\dim(G) \leq Z(G) + \min(5c + |X|, 3c + 5|X|)$$

Conclusion

Conjecture: $\dim(G) \leq Z(G) + c$

Theorem: $\dim(G) \leq Z(G) + \min(5c + |X|, 3c + 5|X|)$

Perspectives

- Improve the theorem
  - Improve the bounds on $|P|$.
  - Modify the set P to reduce its size.
- Get $\dim(G) \leq c + Z(G) + f(|X|)$.
- Look at classes of graphs close to trees such chordal graphs or with bounded tree-width.

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Thank you for your attention

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