Ample completion of OMs and CUOMs

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joint work with Victor CHEPOI, and Kolja KNAUER
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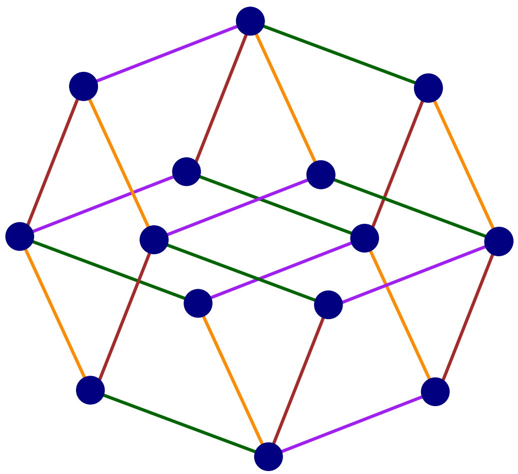


Journées Graphes et Algorithmes
16-18 november 2020

Context

<u>Conjecture</u> [Floyd and Warmuth, 1995] :
Every set family of VC-dimension d has a sample compression scheme of size $O(d)$.

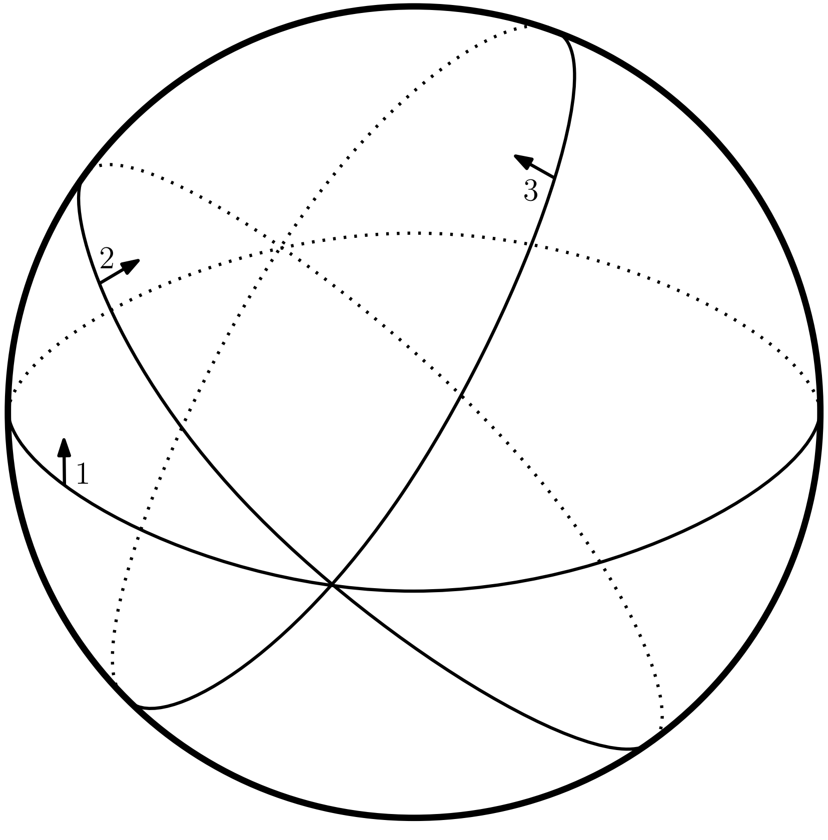
<u>Theorem</u> [Moran and Warmuth, 2016] :
Every ample set family of VC-dimension d has a labeled sample compression scheme of size d.

<u>Question</u> :
Can any set family of VC-dimension d be completed to an ample set family of VC-dimension $O(d)$?

<u>Our result</u> :
Every OM and CUOM of VC-dimension d can be completed to an ample set family of VC-dimension d.

Complexes of oriented matroids

$U = \{1, \ldots, m\}$ and $\mathcal{L} = \{-1, 0, +1\}^m$



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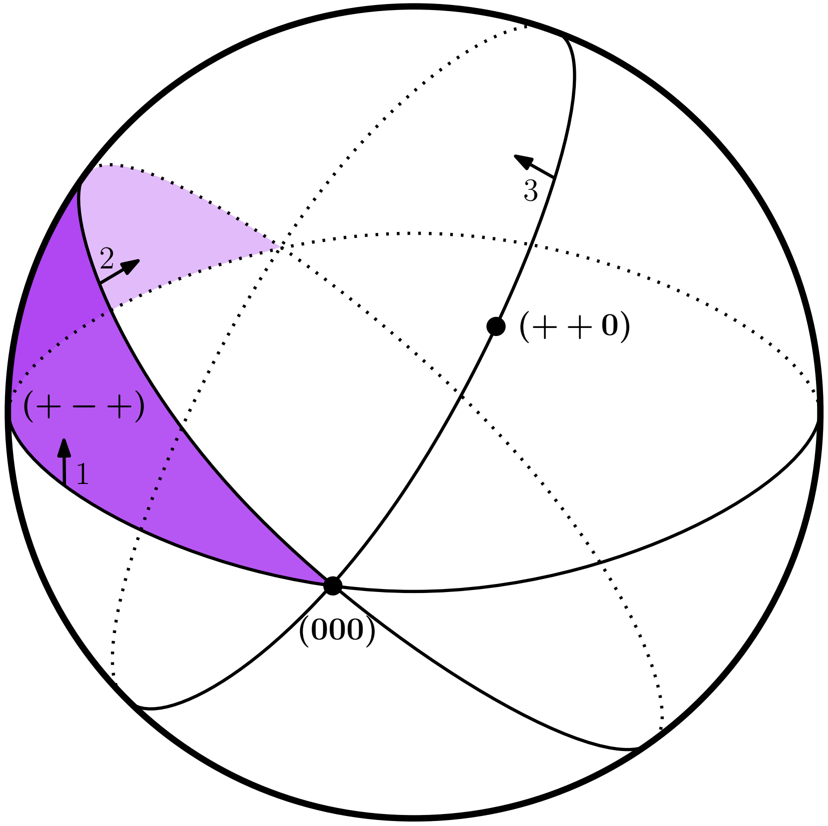


Complexes of oriented matroids

$U = \{1, \ldots, m\}$ and $\mathcal{L} = \{-1, 0, +1\}^m$

$(U, \mathcal{L})$ **OM** iff

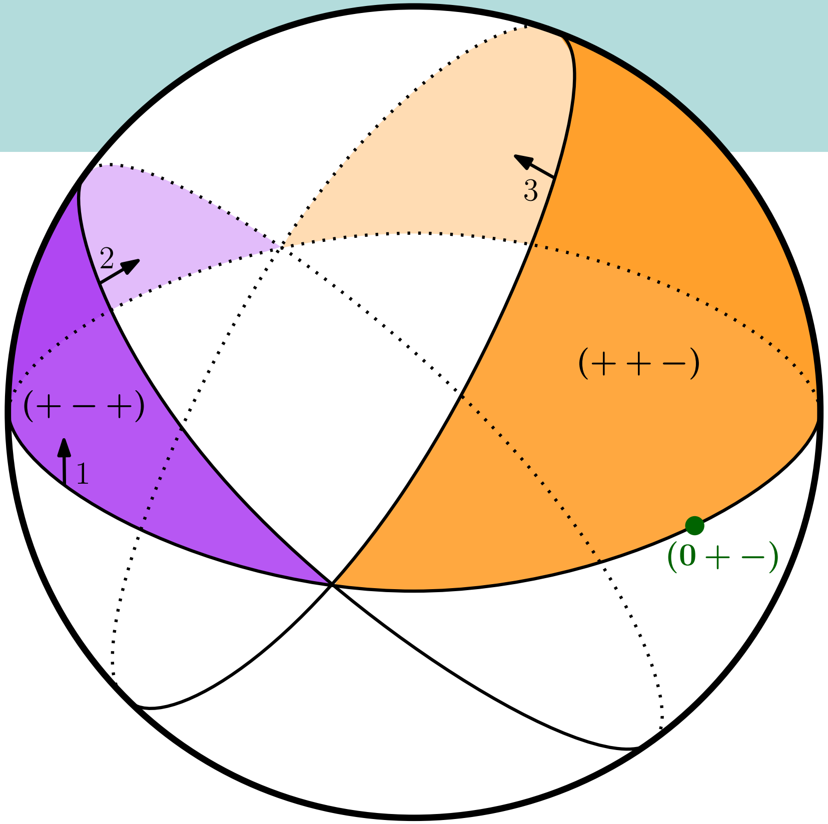
$$(X \circ Y)_i = \begin{cases} X_i & \text{if } X_i \neq 0 \\ Y_i & \text{otherwise.} \end{cases}$$

(C) $\forall X, Y \in \mathcal{L}, X \circ Y \in \mathcal{L}$;

(SE) $\forall X, Y \in \mathcal{L}$ and for each $e \in U$ with $X_e Y_e = -1, \exists Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f \; \forall f \in U$ with $X_f Y_f \neq -1$

(Sym) $\mathcal{L} = -\mathcal{L} := \{-X : X \in \mathcal{L}\}$.

$$\begin{pmatrix} 0 \\ + \\ - \end{pmatrix} \circ \begin{pmatrix} + \\ - \\ + \end{pmatrix} = \begin{pmatrix} + \\ + \\ - \end{pmatrix}$$



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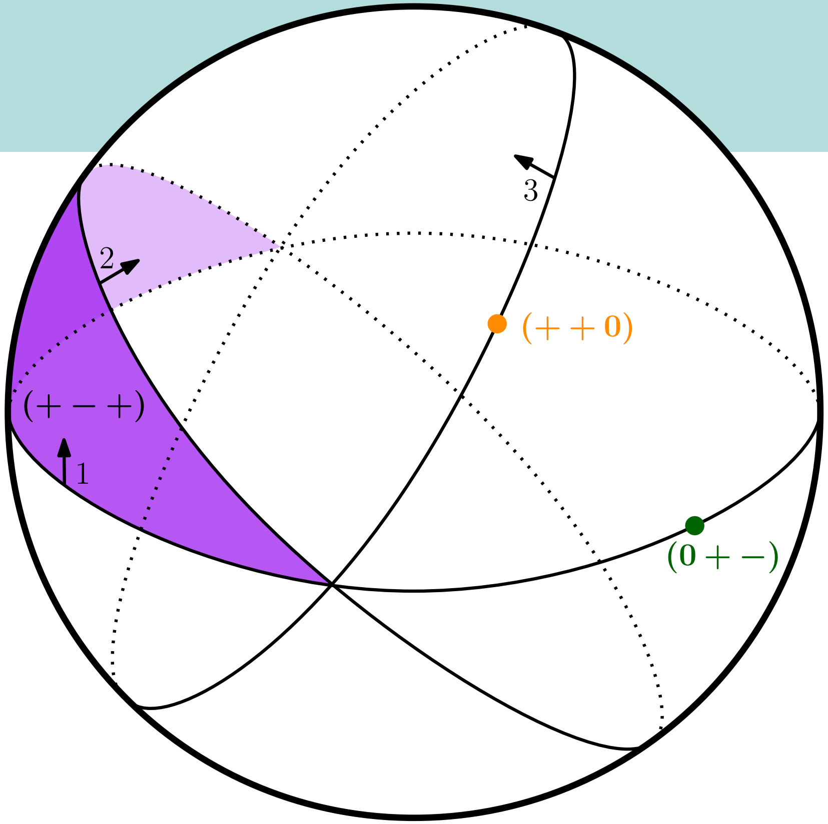

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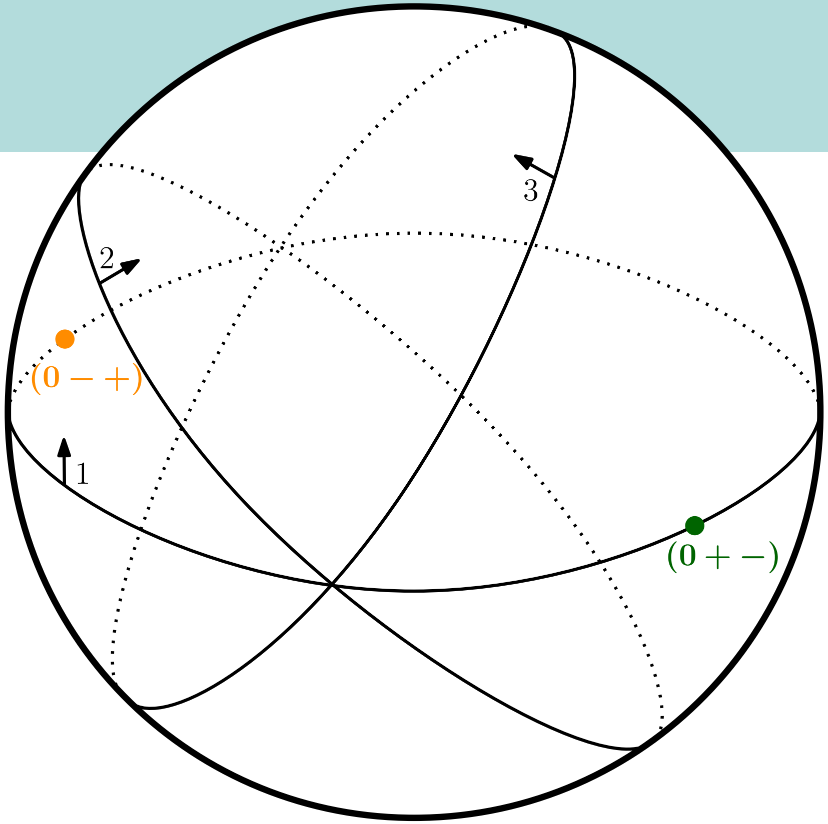

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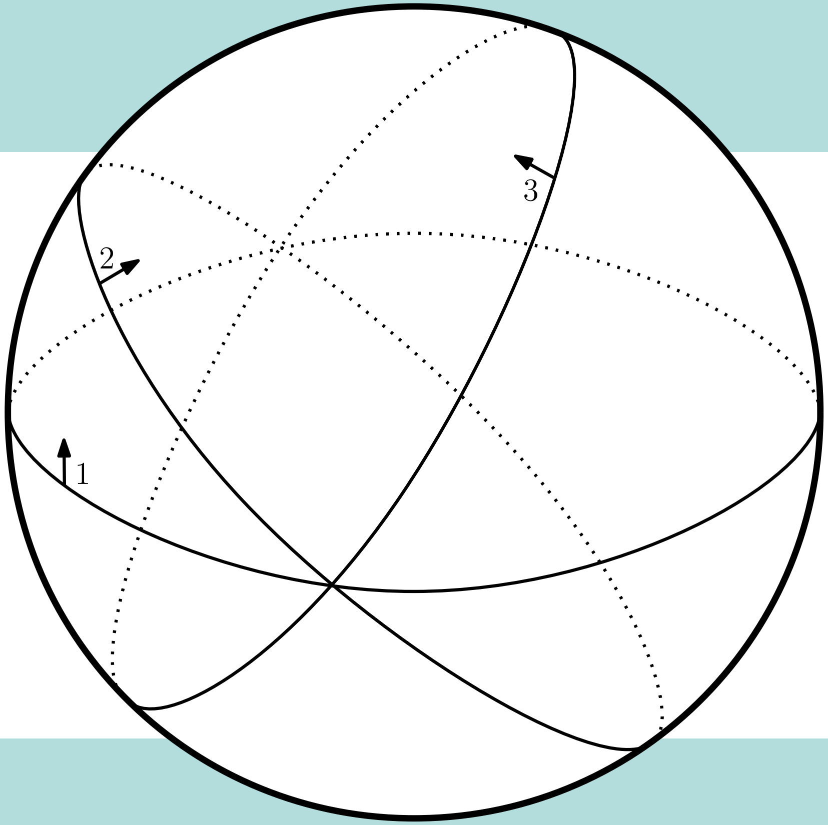


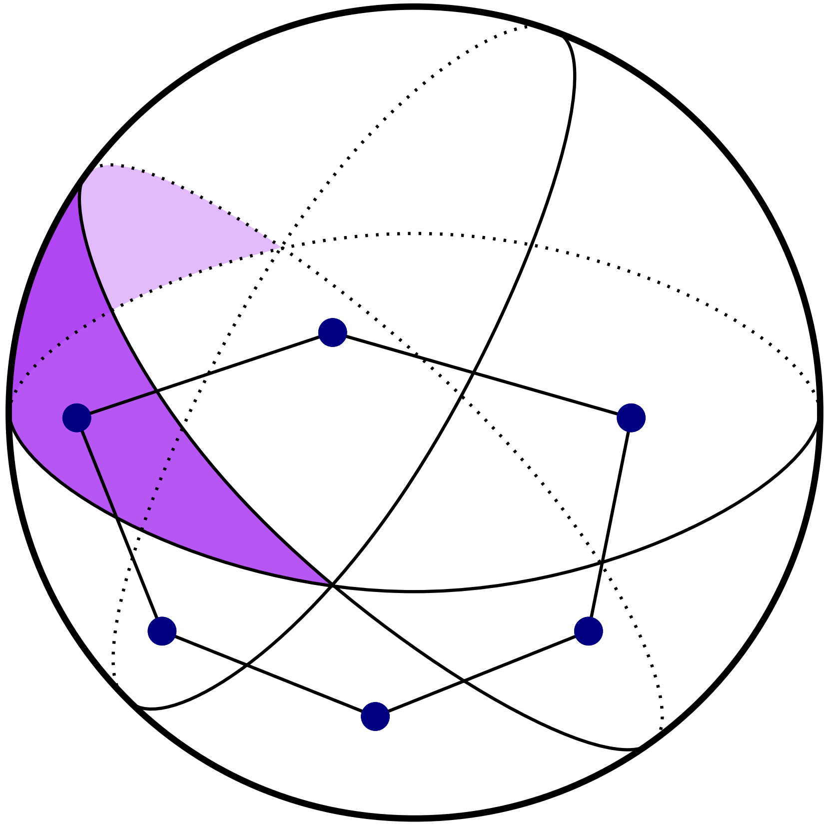
$(U, \mathcal{L})$ **COM** iff

(C), **(SE)** and **(FS)** $\forall X, Y \in \mathcal{L}, X \circ -Y \in \mathcal{L}$.

Tope graphs

Topes of $\mathcal{L}$: covectors without zero entries.

Tope graph of $\mathcal{L}$: subgraph induced by its topes in the hypercube $\{+,-\}^m$.



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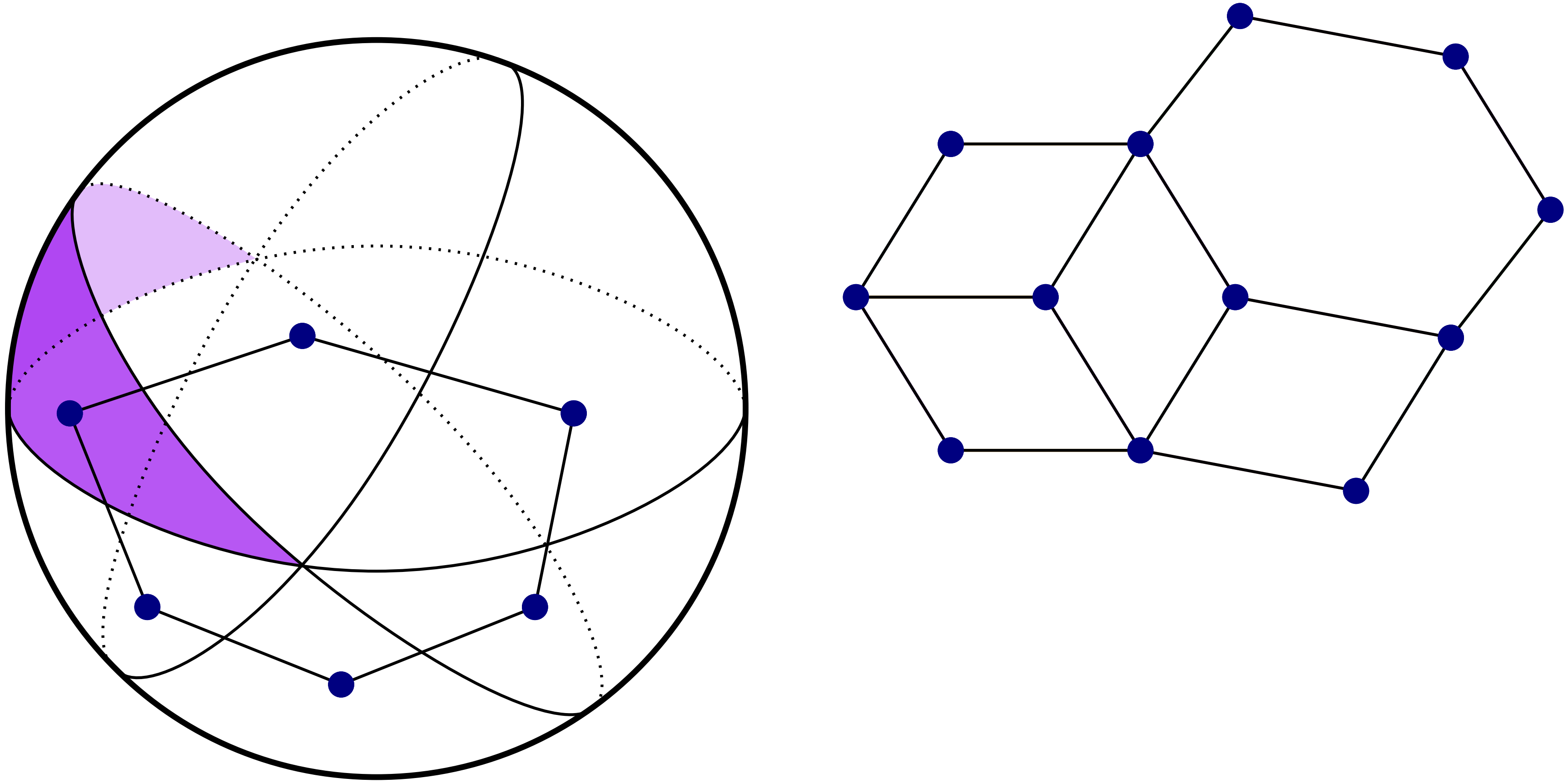

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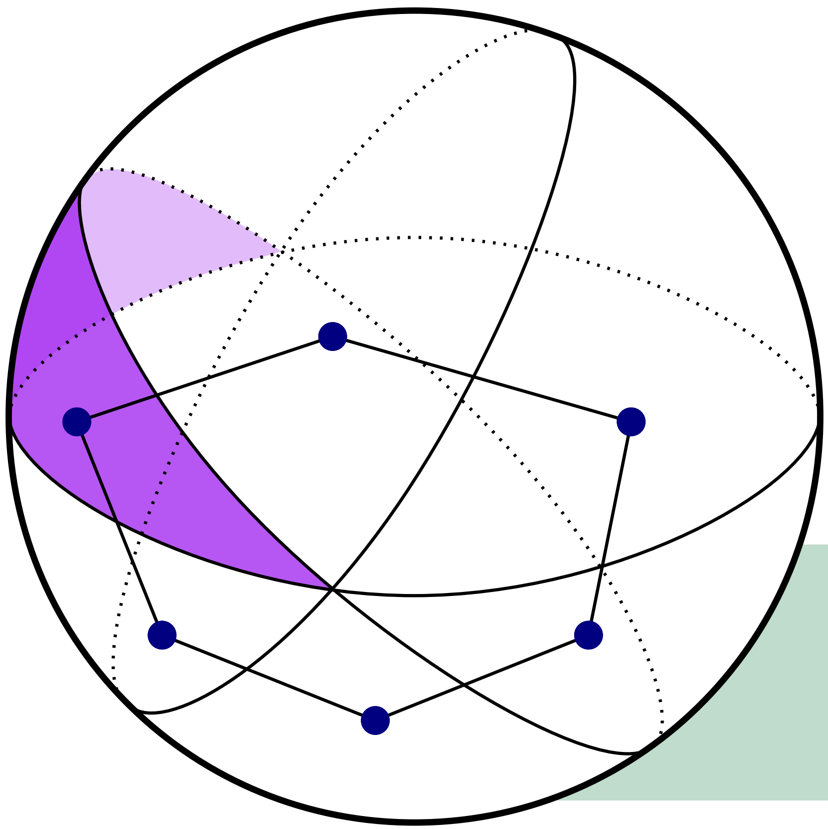
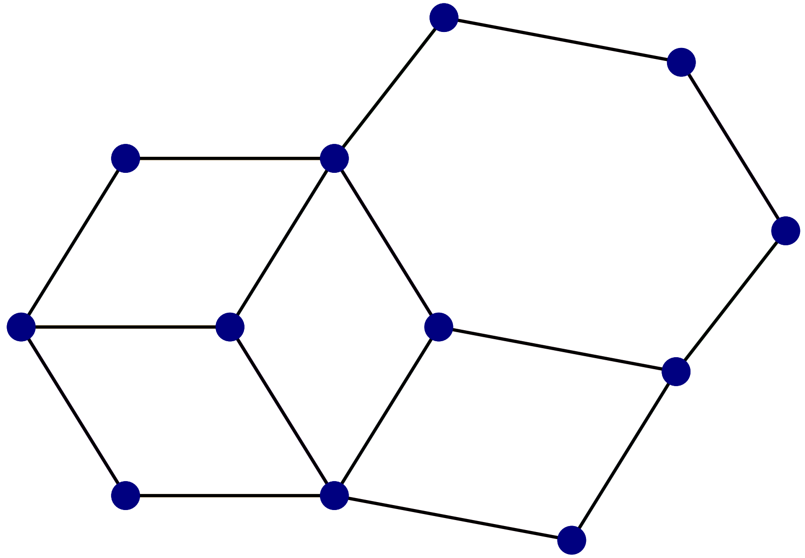


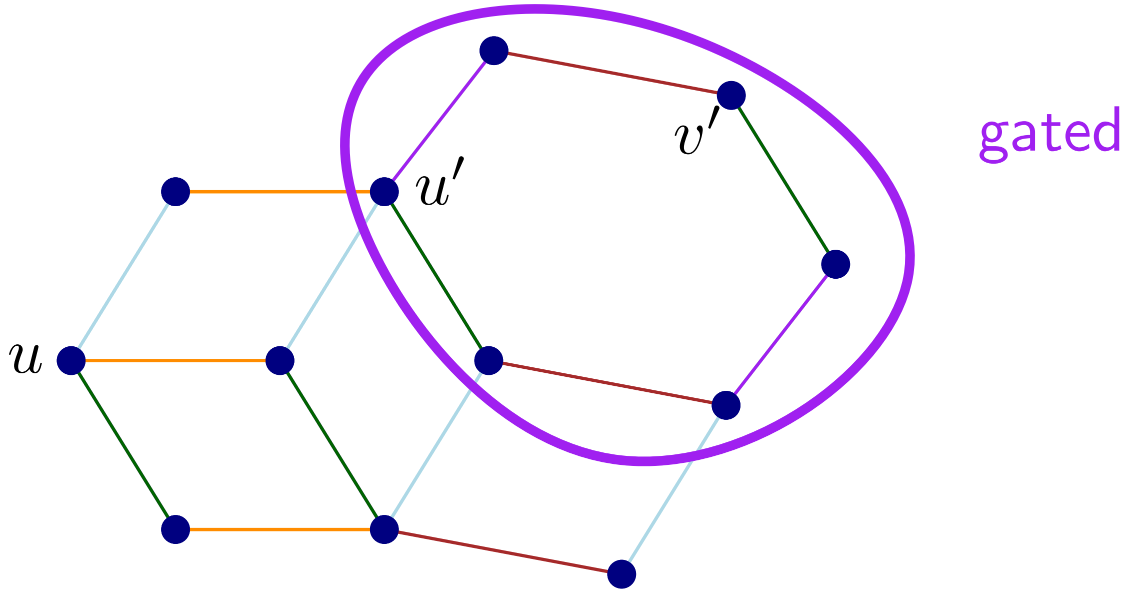


Proposition [Bandelt et al., 2018]: COMs are uniquely determined by their topes.

Remark : tope graphs G of COMs are **isometric** subgraphs of hypercube Q, i.e., $\forall u, v \in V(G), d_G(u,v) = d_Q(u,v)$.

Gated and antipodal subgraphs

$G' \subseteq G$ **gated** if $\forall u \in V(G)\ \exists u' \in V(G')$ s.t. $\forall v' \in V(G')$ there is a shortest (u, v')-path through u'.



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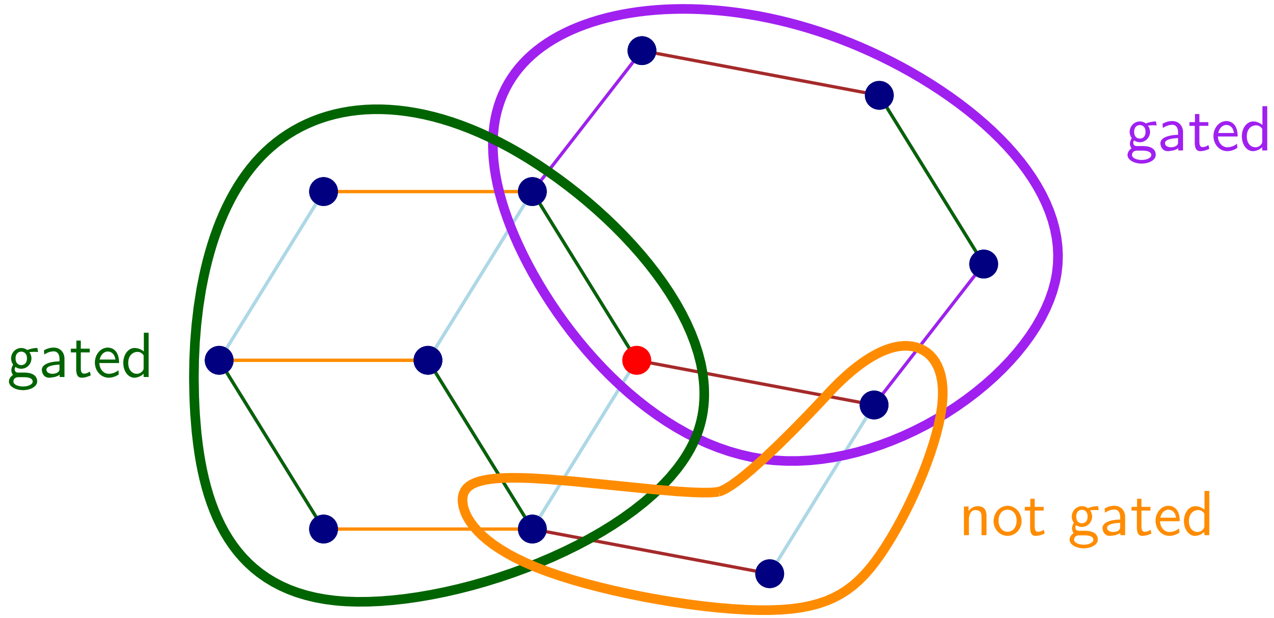

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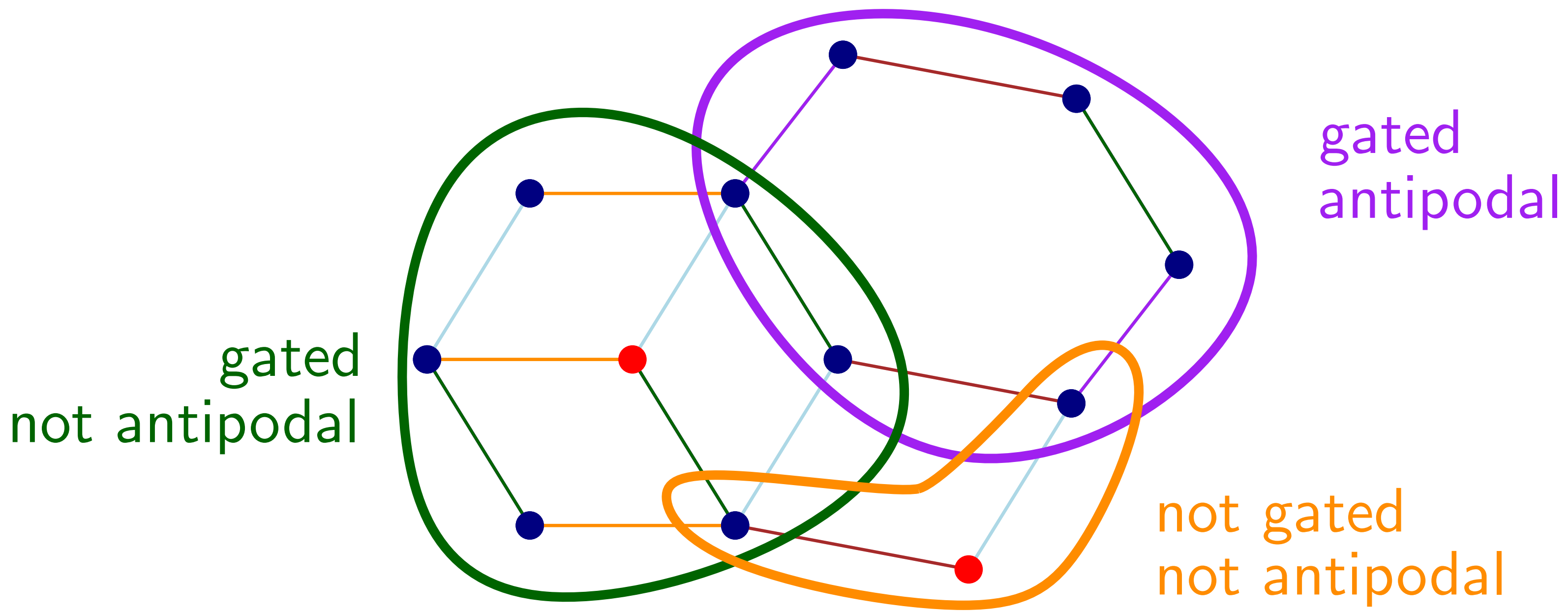

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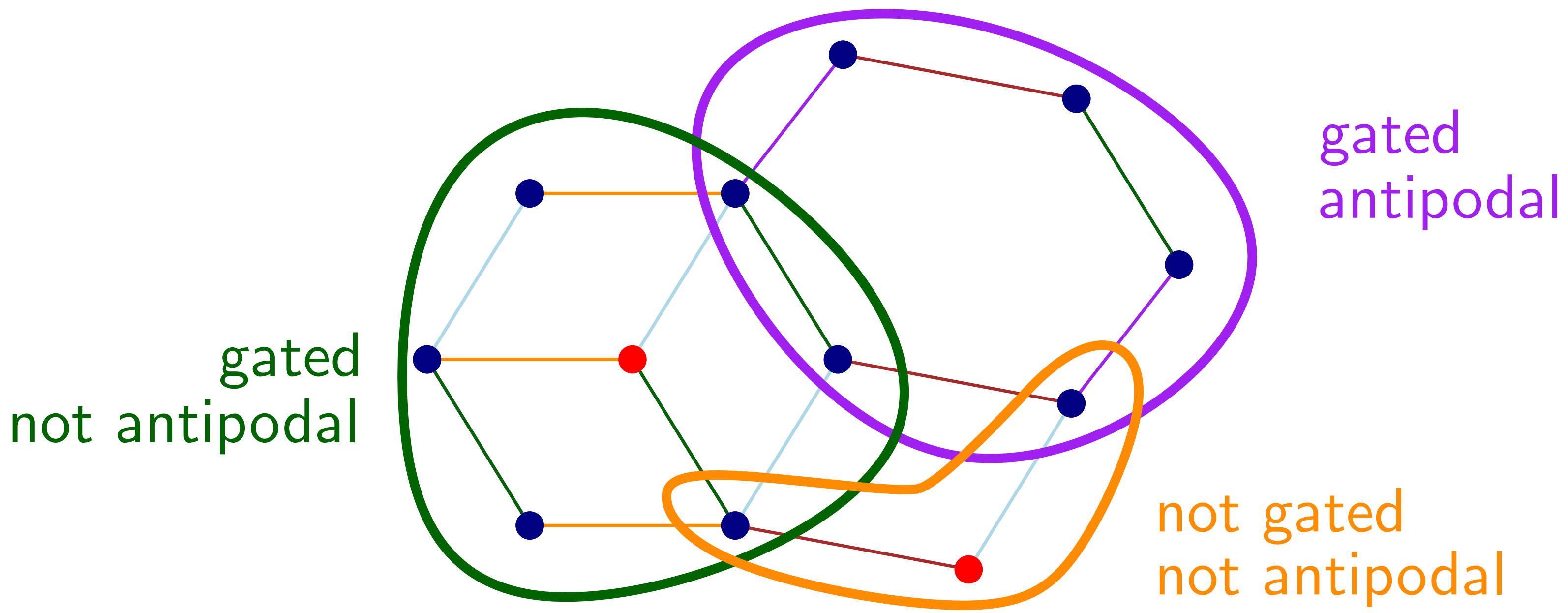

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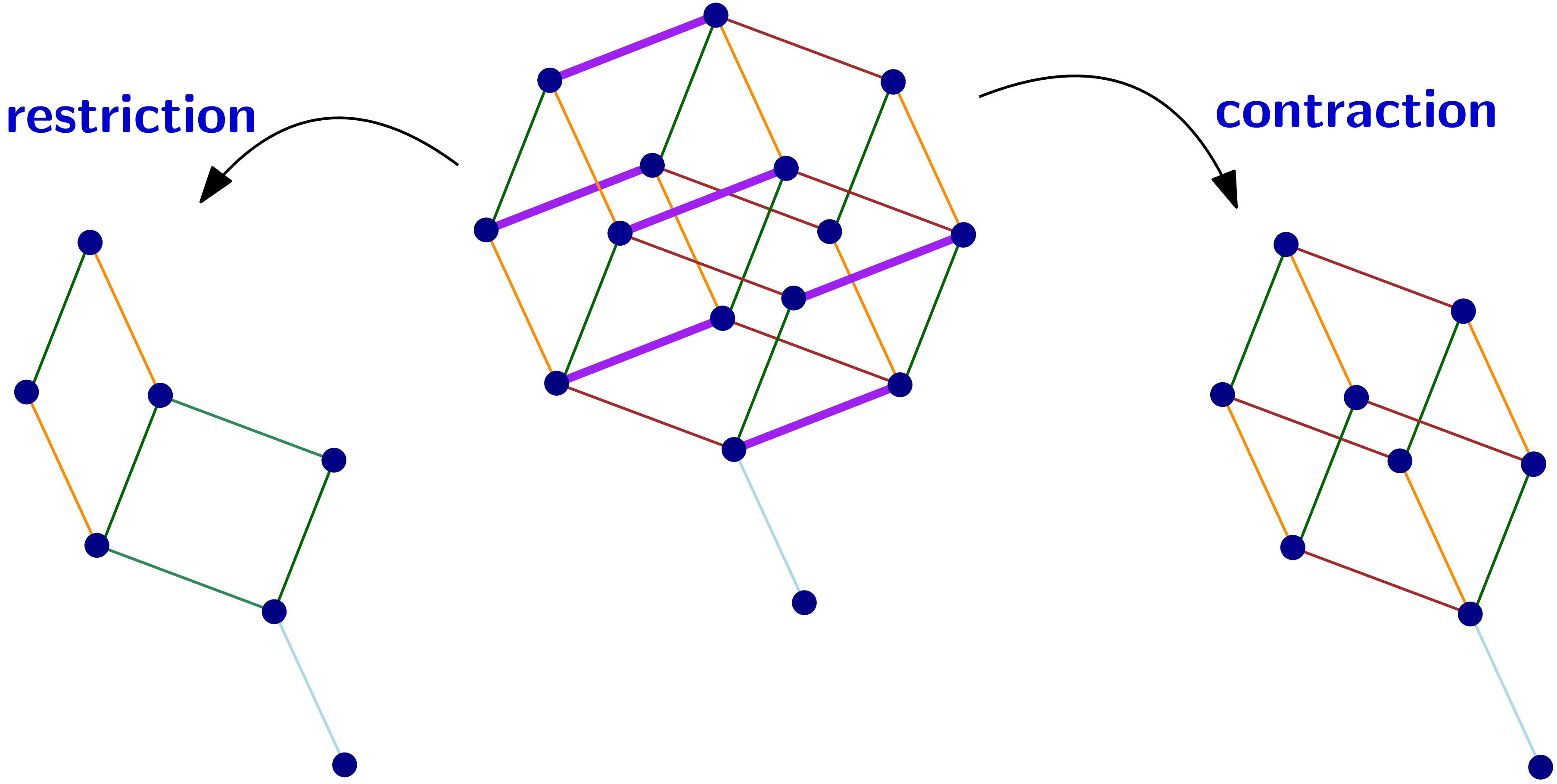
COMs : all antipodal subgraphs are gated.
OMs : antipodal COMs.
AMPs : COMs s.t. all antipodal subgraphs are hypercubes.

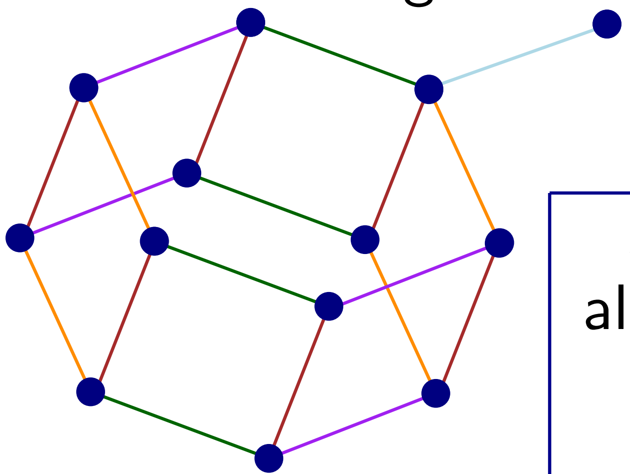
pc-minors and VC-dimension



$\mathrm{VC\text{-}dim}(G) = \max\{d : Q_d$ is a pc-minor of $G\}$.

Classes of partial cubes

COMs = all antipodal subgraphs are gated



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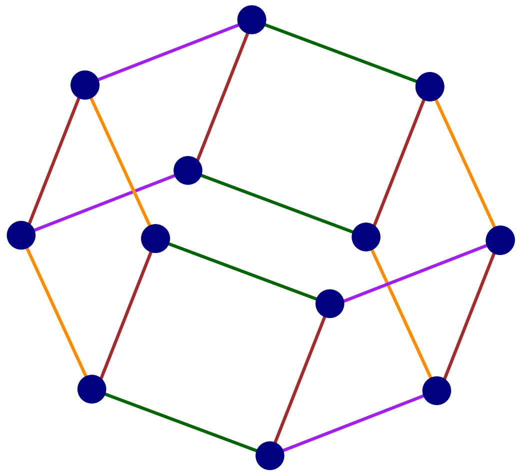

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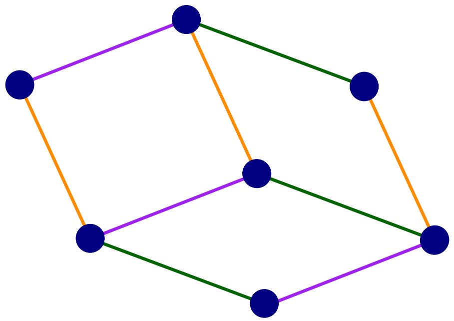

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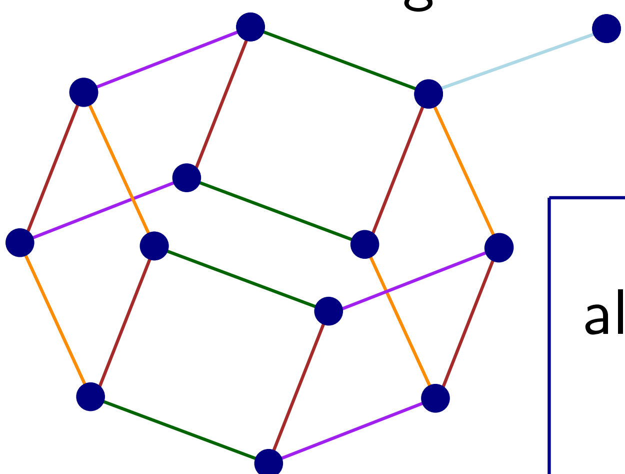

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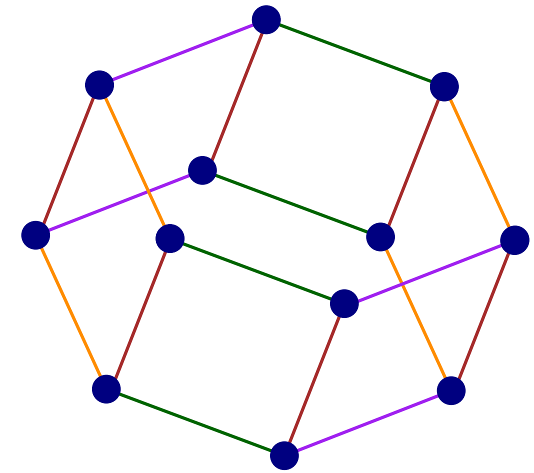

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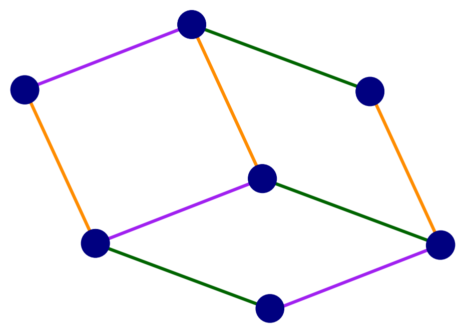
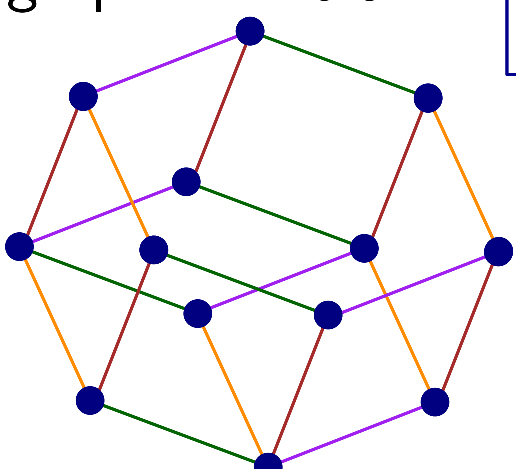
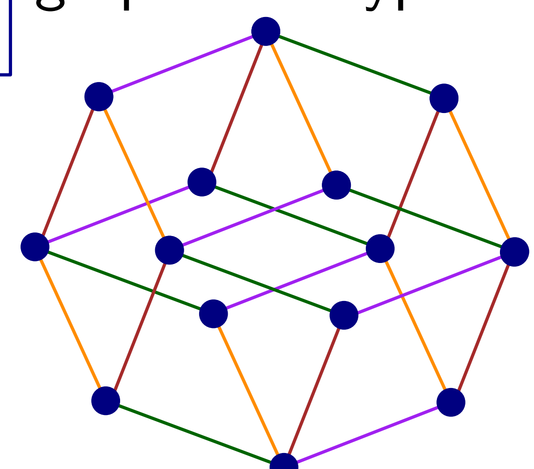


CUOMs = COMs s.t. all proper antipodal subgraphs are UOMs



UOMs = OMs s.t. all proper antipodal subgraphs are hypercubes



Our result

Theorem 1 [Chepoi, Knauer, and P., 2020]:
Any OM of VC-dimension d can be completed to an ample of the same VC-dimension.

Theorem 2 [Chepoi, Knauer and P., 2020]:
Any CUOM of VC-dimension d can be completed to an ample of the same VC-dimension.

Proof of Theorem 1

Any OM of VC-dimension d can be completed to an ample of the same VC-dimension.

Proposition [Björner et al., 1993] :
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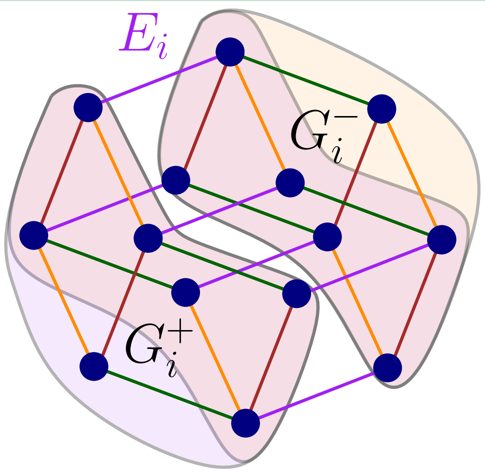

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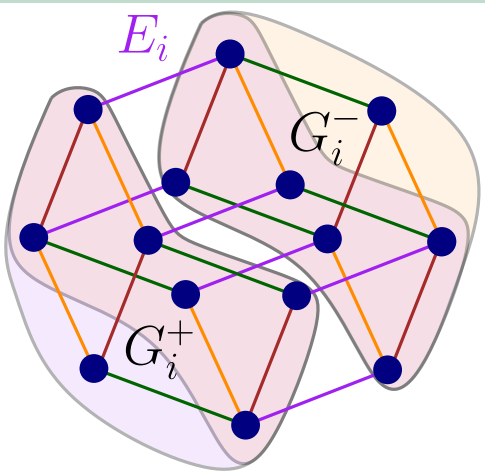

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G' has an ample
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contraction
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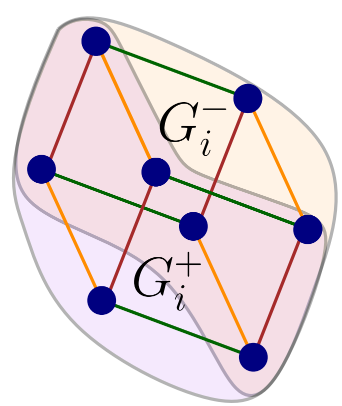

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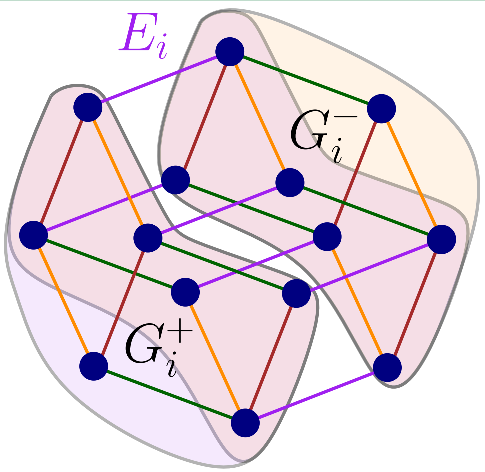

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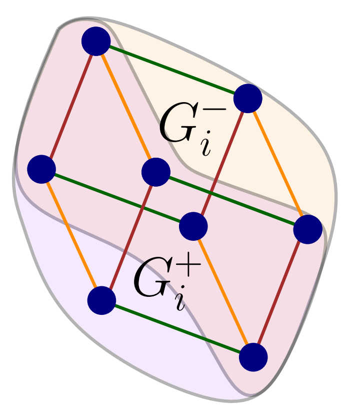
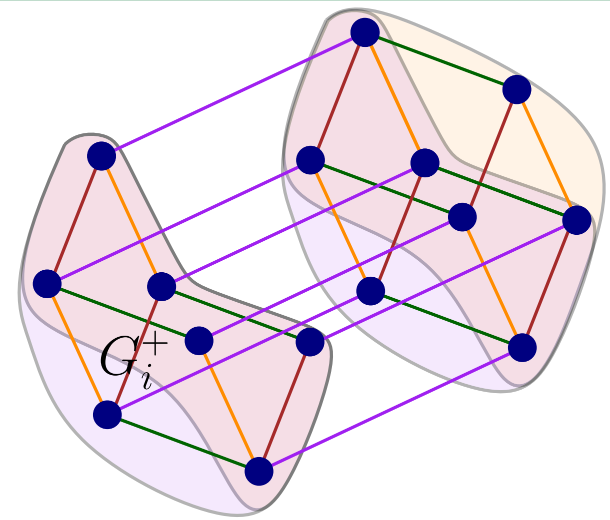


expansion on G_i^+



H is an ample completion of G of VC-dimension d

Proof of Theorem 2 (1/2)

Any CUOM of VC-dimension d can be completed to an ample of the same VC-dimension.

Idea : 1) Complete independently each facet of G to an ample;
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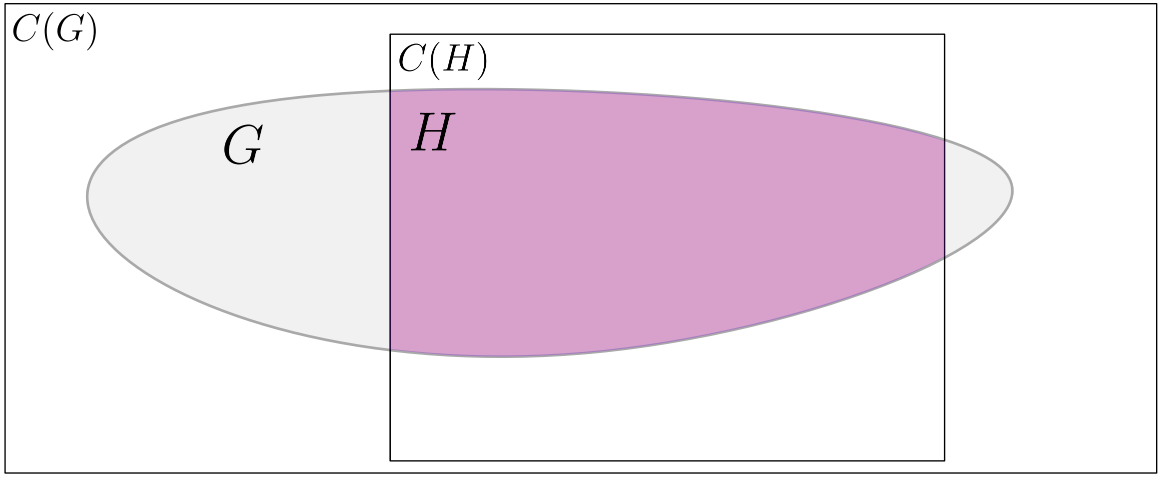
<u>Lemma 1</u> :
G partial cube, $H \subseteq G$ gated and H' partial cube s.t. $H \subseteq H' \subseteq C(H)$
(i) G' partial cube;
(ii) $H' \subseteq G'$ gated;
(iii) $\text{VC-dim}(G') = \max\{\text{VC-dim}(G), \text{VC-dim}(H')\}$.



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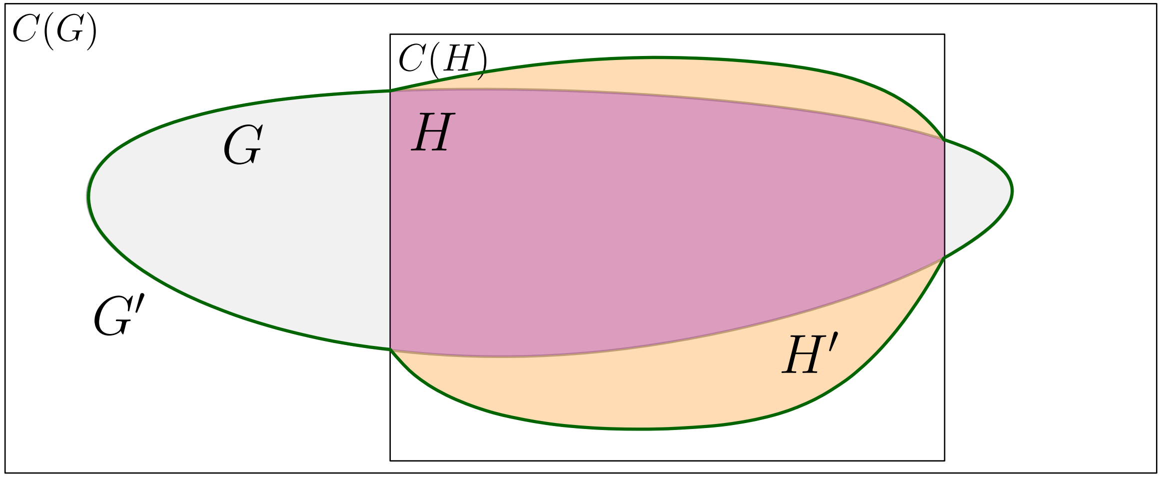

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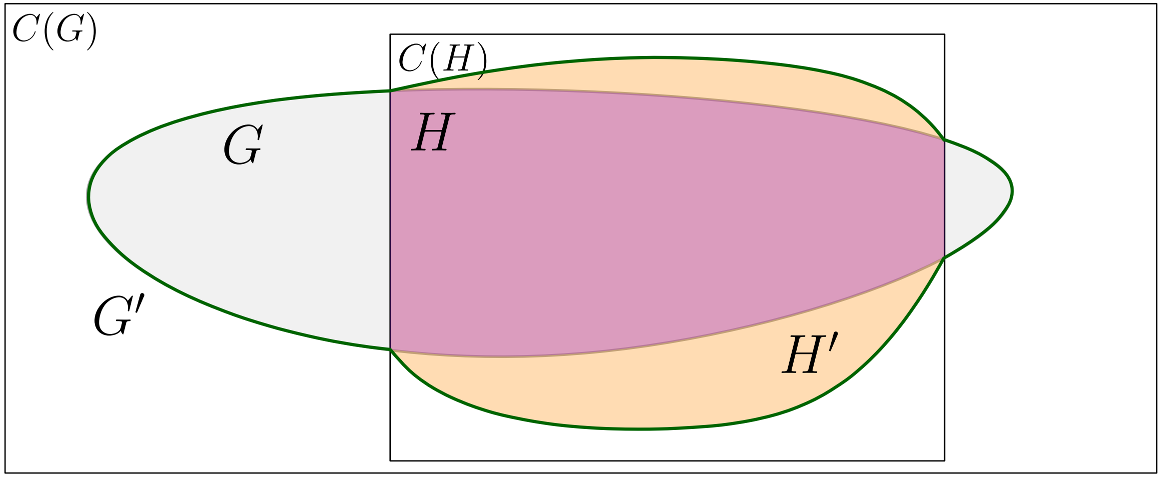
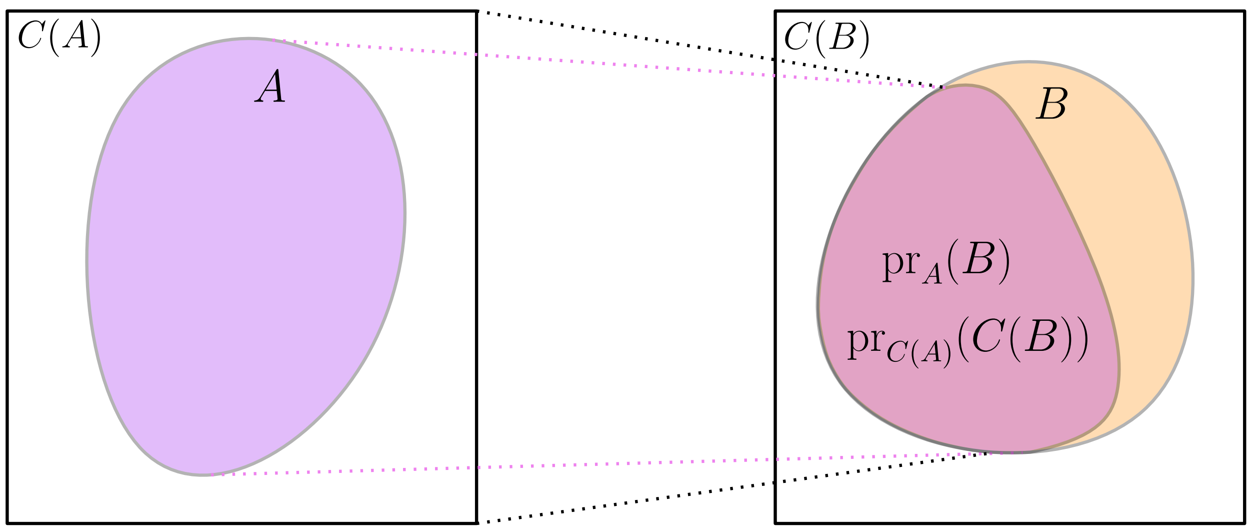


Proof of Theorem 2 (2/2)

Any CUOM of VC-dimension d can be completed to an ample of the same VC-dimension.

distance $d(A, B) := \min\{d(a, b) : a \in A, b \in B\}$.
mutual projection $\mathrm{pr}_B(A) := \{a \in A : d(a, B) = d(A, B)\}$.

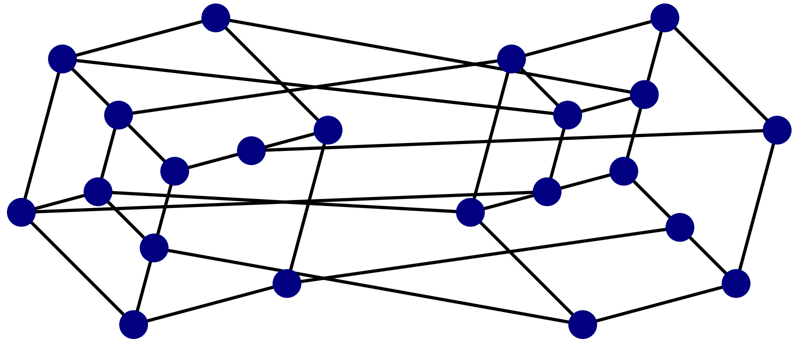
Lemma 2 :
A, B facets of a CUOM $G \Rightarrow \mathrm{pr}_B(A) = \mathrm{pr}_{C(B)}(C(A))$ and $\mathrm{pr}_A(B) = \mathrm{pr}_{C(A)}(C(B))$.



Conclusion

Can any set family of VC-dimension d be completed to an ample set family of VC-dimension $O(d)$?

- Any partial cubes of VC-dimension 2 can be completed to an ample of VC-dimension 2;



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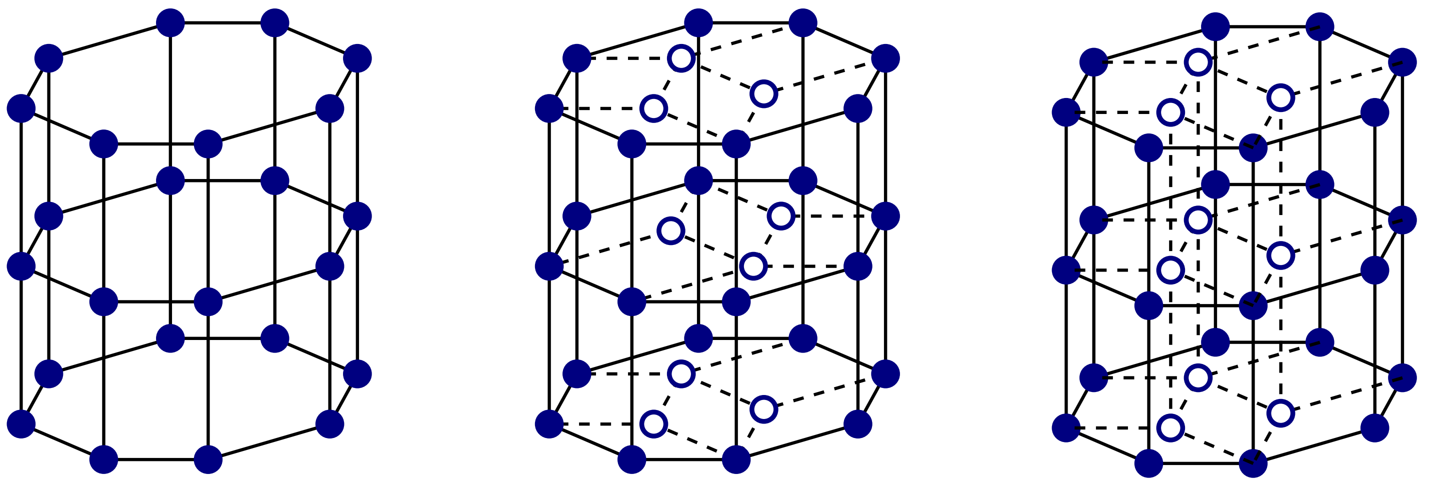

Thank you for your attention !