New Algorithms for Mixed Dominating Set

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Summary

Mixed Dominating Set

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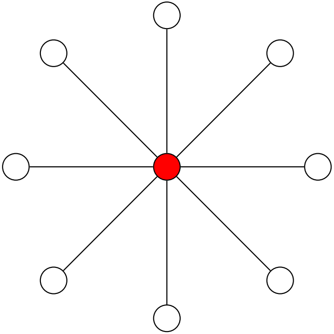
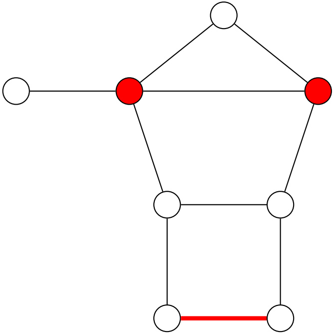
- Graph $G = (V, E)$.
- A vertex $u \in V$ dominates itself, its incident edges and its neighbors.
- An edge $e \in E$ dominates itself, its two endpoints, and its adjacent edges.
- *Mixed dominating set* : Set of vertices $D \subseteq V$ and set of edges $M \subseteq E$ which dominates all vertices and edges of the graph G.
- Goal : A mixed dominating set (*mds*) of minimum size.

Mixed Dominating Set





State of the Art

- **NP**-complete problem (Majumbar, 1992).
- 2-approximation in polynomial time (Hatami, 2010).
- $O^*(2^n)$ exact algorithm (and exponential space) (Madathil et al., 2019).
- FPT algorithm parameterized by the solution size in $O^*(4.172^k)$ (Xiao, Sheng, 2019).
- FPT algorithm parameterized by the treewidth in $O^*(6^{tw})$ and by the pathwidth in $O^*(5^{pw})$ (Jain et al., 2017).

Our Results

- $O^*(2^n)$ exact algorithm (and exponential space) (Madathil et al., 2019).
  - $O^*(1.912^n)$ and polynomial space.
- FPT algorithm parameterized by the solution size in $O^*(4.172^k)$ (Xiao, Sheng, 2019).
  - $O^*(3.510^k)$ and polynomial space.
- FPT algorithm parameterized by the treewidth in $O^*(6^{tw})$ and by the pathwidth in $O^*(5^{pw})$ (Jain et al., 2017).
  - $O^*(5^{tw})$ parameterized by the treewidth.
  - Under SETH, for any $\varepsilon > 0$, no algorithm in time $O^*((5-\varepsilon)^{pw})$.

Accepted Paper

- New Algorithms for Mixed Dominating Set, Louis Dublois, Michail Lampis, Vangelis Th. Paschos.
- 15th International Symposium on Parmeterized and Exact Computation (IPEC 2020).

Nice mds

Definition of nice mds

Definition

A *nice* mixed dominating set of a graph $G = (V, E)$ is a mixed dominating set $D \cup M$ which satisfies the following :

(i) $D \cap V(M) = \emptyset$;

(ii) M is a matching ;

(iii) For all $u \in D$ there exists at least two private neighbors of u, that is, two vertices $v_1, v_2 \in V \setminus (D \cup V(M))$ with $N(v_1) \cap D = N(v_2) \cap D = \{u\}$.

Existence of nice mds

Lemma

For any graph $G = (V, E)$ without isolated vertices, G has an mds $D \cup M$ of size at most k if and only if G admits a nice mds $D' \cup M'$ of size at most k.

Existence of nice mds

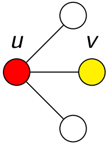
- Let $G = (V, E)$ be a graph without isolated vertices, and $D \cup M$ an mds of G.
- By a result of (Madathil et al., 2019), we know :
  - If a graph has an mds of size k, then it also has an mds that satisfies the first two properties (i.e. (i) $D \cap V(M) = \emptyset$ and (ii) M is a matching).
- From this mds $D \cup M$, we will edit it to obtain the third property.
- Let $I = V \setminus (D \cup V(M))$.

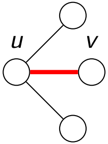
Existence of nice mds

- If there exists $u \in D$ with exactly one private neighbor v : its other neighbors are dominated by $(D \cup M) \setminus \{u\}$.

Existence of nice mds

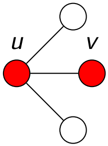
- If there exists $u \in D$ with no private neighbor : its neighborhood is dominated by $(D \cup M) \setminus \{u\}$.
  - If there exists $v \in N(u) \cap I$.

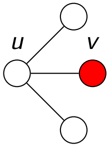




Existence of nice mds

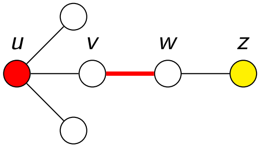
- If there exists $u \in D$ with no private neighbor and $N(u) \cap I = \emptyset$: $N(u) \subseteq D \cup V(M)$.
  - If there exists $v \in N(u) \cap D$.

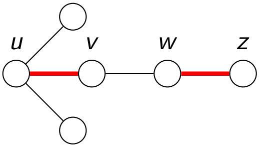




Existence of nice mds

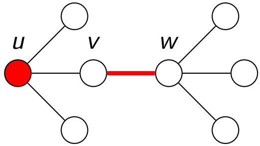
- If there exists $u \in D$ with no private neighbor and $N(u) \cap (D \cup I) = \emptyset$: $N(u) \subseteq V(M)$.
  - If there exists $v \in N(u) \cap V(M)$ with $(v, w) \in M$ such that there exists $z \in N(w) \cap I$.

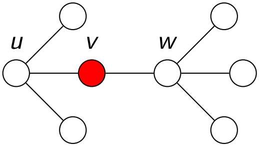




Existence of nice mds

- If there exists $u \in D$ with no private neighbor, $N(u) \subseteq V(M)$, and there does not exist $v \in N(u) \cap V(M)$ with $(v, w) \in M$ such that there exists $z \in N(w) \cap I$: for all $v \in N(u)$ with $(v, w) \in M$, $N(w) \subseteq D \cup V(M)$.





Implications

- Speed-up the branching rules on low-degree vertices (a vertex in D must have two private neighbors).
- Faster FPT and exact branching algorithms.

Ideas of the Theorem

$O^*(5^{tw})$ Algorithm

- Incidence graph $G' = (V', E')$ of $G = (V, E)$:
  - $V' = V \cup E$
  - $E' = E \cup \{(u, e), (e, v) : e = (u, v) \in E\}$

- Mixed Dominating Set on a graph G is equivalent to Distance-2-Dominating Set on the incidence graph of G.

- The incidence graph of G has the same treewidth as G.

- Distance-2-Dominating Set can be solved in time $O^*(5^{tw})$ (Borradaile, Le, 2016).

Theorem

There is an $O^*(5^{tw})$-time algorithm for Mixed Dominating Set *in graphs of treewidth tw.*

Goal

Theorem

Under SETH, for all $\varepsilon > 0$, no algorithm solves MIXED DOMINATING SET *in time $O^*((5-\varepsilon)^{pw})$, where pw is the input graph's pathwidth.*

Method

Definition

A q-CSP-5 instance φ is a Constraint Satisfaction Problem (CSP) instance with n variables $x_1, \ldots, x_n$ taking values over the $\{0, 1, 2, 3, 4\}$, and m constraints $c_0, \ldots, c_{m-1}$, each containing exactly q variables and exactly $C = 5^q - 1$ possible assignments (given as a list) over the q variables, for $j \in \{0, \ldots, m-1\}$.

Lemma (Theorem 2 from (Lampis, 2018))

For any $\varepsilon > 0$, under the SETH, there exists a q such that q-CSP-5 with n variables cannot be solved in time $O^*((5-\varepsilon)^n)$.

Method

- Reduction from an instance φ of q-CSP-5 to an instance $(G = (V, E), k)$ of Mixed Dominating Set such that φ is satisfiable if and only if G admits an mds of size at most k.

- The pathwidth $pw(G)$ of G is upper-bounded by $n + O(1)$.

- If, for any $\varepsilon > 0$, Mixed Dominating Set can be solved in time $O^*((5 - \varepsilon)^{pw})$, then q-CSP-5 can be solved in time $O^*((5 - \varepsilon)^n)$, contradicting the Theorem of (Lampis, 2018) and the SETH.

Why 5 ?

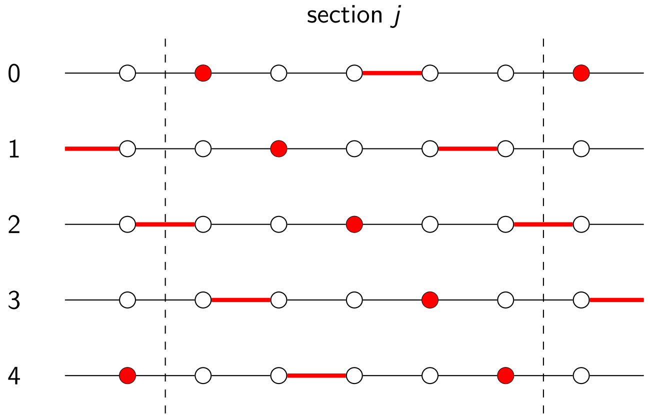
- It corresponds to the base of our target lower bound.
- In our reduction, we will represent the 5 different values a variable can take with a path of 5 vertices in which there is exactly 5 different ways of selecting one vertex and one edge among these 5 vertices.

Construction

Main Part

- The graph G consists of a *main part* of n paths P_i $(1 \leq i \leq n)$ of $5m$ vertices, each divided into m sections :
  - Each path represent a variable.
  - Each section represent a constraint.
- An optimal solution in G will verify, for each path, a specific pattern :
  - For 5 consecutive vertices, there are exactly 5 ways of taking one vertex and one edge to dominate the 5 vertices and the edges between.
- These 5 *configurations* for each path will represent all possible assignments for the variables.

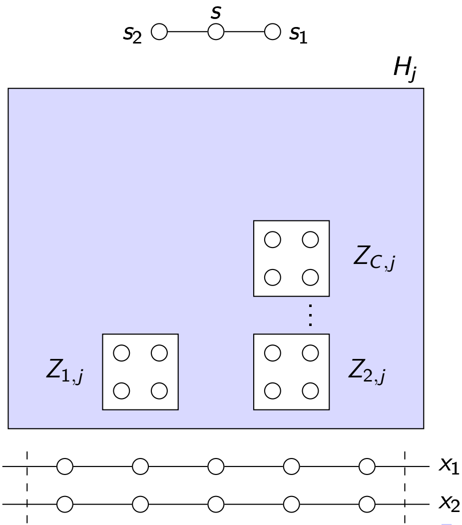
Main Part



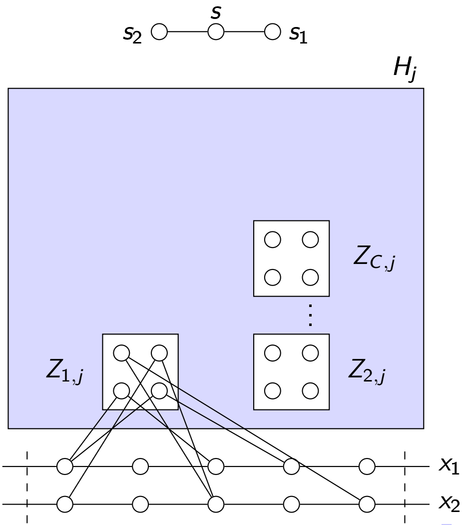
Verification Gadget

- For each $j \in \{0, \ldots, m\}$, we add a *verification gadget* H_j :
  - Only connected to the main part to the 5 vertices of all variables x_i appearing in the constraint c_j.
- An optimal solution in G will verify a specific form in the gadget H_j :
  - The solution has this form in H_j if and only if the constraint is satisfied.

Verification Gadget



Verification Gadget



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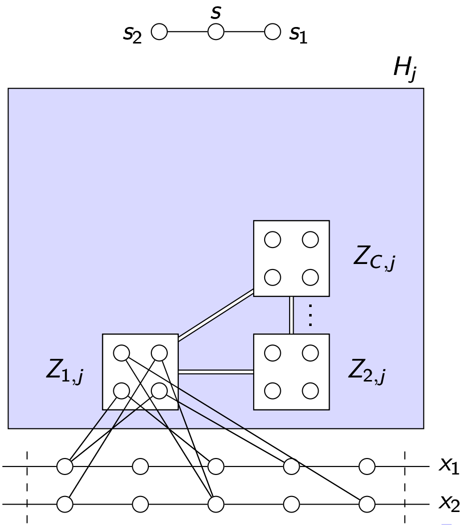
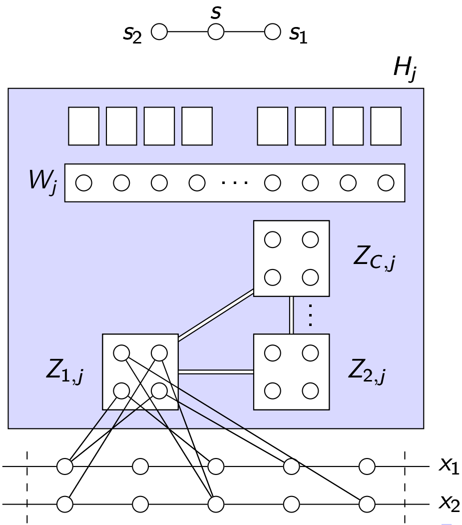


Verification Gadget



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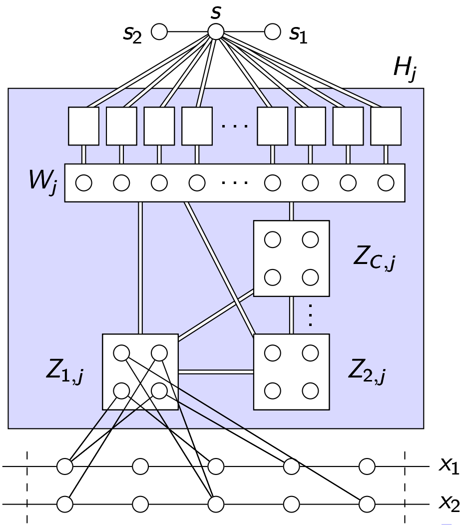
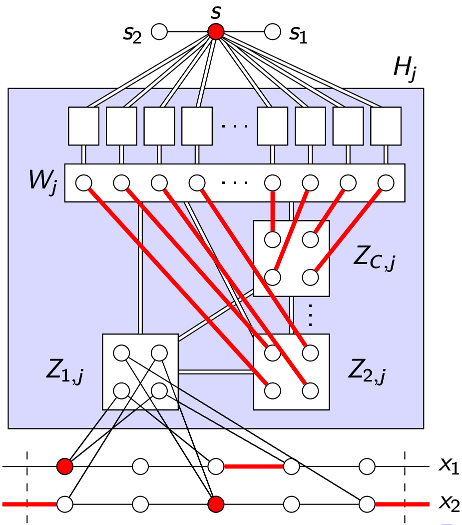


Verification Gadget



Details

- Add *consistency* gadgets connected to each path and each section in order to force an optimal solution to follow one of the five configurations for each path.
- Make $F = (3n+1)(2n+1)$ copies of G and glue them together one after the other.
- $k = 8AFmn + 2Fmn + 2Fmq(C-1) + n + 1$.
- $pw(G) \leq n + O(q5^q)$.

Theorem

Lemma

φ is satisfiable if and only if there exists an mds in G of size at most k.

Lemma

The pathwidth of G is at most $n + O(1)$.

Theorem

Under SETH, for all $\varepsilon > 0$, no algorithm solves Mixed Dominating Set *in time $O^*((5-\varepsilon)^{pw})$, where pw is the input graph's pathwidth.*

Merci