# Chapter 3

# 3D Rotations matrices

The goal of this chapter is to exemplify on a simple and useful manifold part of the general methods developed previously on Riemannian geometry and Lie groups. This document also provide numerically stable formulas to implement all the basic operations needed to work with rotations in 3D.

## 3.1 Vector rotations of $\mathbb{R}^3$

### 3.1.1 Definition

Let $\{i, j, k\}$ be a right handed orthonormal basis of the vector space $\mathbb{R}^3$ (we do not consider it here as a Euclidean space), and $\mathcal{B} = \{e_1, e_2, e_3\}$ be a set of three vectors. The linear mapping from $\{i, j, k\}$ to $\mathcal{B}$ is given by the matrix

$$R = [e_1, e_2, e_3] = \begin{bmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} \tag{3.1}$$

If we now want $\mathcal{B}$ to be an orthonormal basis, we have that $\langle\, e_i \mid e_j \,\rangle = \delta_{ij}$, which means that the matrix $R$ verifies

$$R.R^{\mathrm{T}} = R^{\mathrm{T}}.R = I_3 \tag{3.2}$$

This equation implies in turn that $\det(R.R^{\mathrm{T}}) = 1$ (i.e. $\det(R) = \pm 1$) and gives rise to the subset $\mathcal{O}_3$ of orthogonal matrices of $\mathcal{M}_{3\times 3}$. If we want $\mathcal{B}$ to be also right handed, we have to impose $\det(R) = 1$.

$$\mathcal{S}O_3 = \{R \in \mathcal{M}_{3\times 3} / R.R^{\mathrm{T}} = \mathrm{I}d \quad \text{and} \quad \det(R) = +1\} \tag{3.3}$$

Such matrices are called (proper) *rotations*. Each right handed orthonormal basis can then be represented by a unique rotation and conversely each rotation maps $\{i, j, k\}$ onto a unique right handed orthonormal basis.

As a subset of matrices, rotations can also be seen as a subset of linear maps of $\mathbb{R}^3$. They correspond in this case to their usual interpretation as transformations of $\mathbb{R}^3$: they are positive isometries (maps conserving orientation and dot product): $\langle\, R.x \mid R.y \,\rangle = \langle\, x \mid y \,\rangle$. In particular, they conserve the length of a vector: $||R.x|| = ||x||$. Seen as a transformation group of $\mathbb{R}^3$ the matrix multiplication as the composition law and $\mathrm{I}d$ as identity, rotations forms a non-commutative group denoted by $\mathcal{S}O_3$ (*3D rotation group*).

The three main operations on rotations are:

- the composition of $R_1$ and $R_2$: $R = R_2.R_1$ (beware of the order),
- the inverse of $R$: $R^{(-1)} = R^{\mathrm{T}}$,
- and the application of $R$ to a vector $x$: $y = R.x$.

The orthogonal group $\mathcal{O}_3$ is defined as a subspace of $\mathbb{R}^{3\times 3}$ by equation (3.2) which give rise to 6 independent scalar equations since $R.R^{\mathrm{T}}$ is symmetric. Hence $\mathcal{O}_3$ is a differential manifold of dimension 3. As the determinant is a continuous function in the vector space of matrices, the constraint $\det(R) = \pm 1$ shows that $\mathcal{O}_3$ as a manifold has two non connected leaves. Taking into account the constraint $\det(R) = 1$ amounts to keep the component of identity, which is calle the *special orthogonal group* $\mathcal{SO}_3$. Since the composition and inversion maps are infinitely differentiable, this is moreover a Lie group. Notice that the other leave of the manifold is not a group as it does not contain the identity and it is not stable by composition (the composition of two improper rotations is a proper rotation).

### 3.1.2 From matrices to rotations

From a computational point of view, a rotation matrix is encoded as a non-constrained matrix, and it is necessary to verify that the orthogonality constraints are satisfied before using it. If this is not the case (or if the numerical error is too large), there is a need to recompute the most likely rotation matrix. This can be done by optimizing for the closest rotation matrix in the sense of the distance in the embedding space.

The Froebenius dot products on matrices is a natural metric on the vector space of matrices:

$$\langle\, X \mid Y \,\rangle_{\mathbb{R}^{n\times n}} = \frac{1}{2}\mathrm{Tr}(X^{\mathrm{T}}.Y) = \frac{1}{2}\mathrm{Tr}(X.Y^{\mathrm{T}}) = \frac{1}{2}\sum_{i,j} X_{ij}.Y_{ij}$$

With this metric, the distance between a rotation matrix $R$ and an unconstrained matrix $K$ is thus $\mathrm{dist}(R,K)^2 = \frac{1}{2}\mathrm{Tr}((R-K)^{\mathrm{T}}.(R-K)) = \frac{1}{2}\mathrm{Tr}(K.K^{\mathrm{T}}) + \frac{3}{2} - \mathrm{Tr}(R.K^{\mathrm{T}})$. Notice incidentally that the extrinsic distance between two rotations matrices is $\mathrm{dist}_{Fro}(R_1,R_2)^2 = 3 - \mathrm{Tr}(R_1.R_2)$.

To find the closest rotation to $K$, we have to maximize $\mathrm{Tr}(R.K^{\mathrm{T}})$ subject to the constraint $R.R^{\mathrm{T}} = \mathrm{I}d$. This is called the *orthogonal Procrustes problem* in statistics, and it relates very closely to the *absolute orientation problem* in photogrammetry, the *pose estimation problem* in computer vision and the *rotational superposition problem* in crystallography. Several closed form solutions have been developed, using unit quaternions [Hor87, Kea89, Aya91, Fau93, HM93], singular value decomposition (SVD) [Sch66, McL72, AHB87, Ume91], Polar decomposition [HHN88] or dual quaternions [WS91]. Additional bibliography includes [Mos39, Joh66, Kab76, Kab78, Hen79, Mac84, Dia88]. We present here the SVD method because it is valid in any dimension.

Taking into account the constraints $R.R^{\mathrm{T}} = \mathrm{I}d$ and $\det(R) = +1$, the Lagrangian is

$$\Lambda = \mathrm{Tr}(R.K^{\mathrm{T}}) - \frac{1}{2}\mathrm{Tr}\left(L.(R.R^{\mathrm{T}} - \mathrm{I}d)\right) - g.(\det(R) - 1)$$

where $L$ is a symmetric matrix and $g$ a scalar. According to appendix A.1, we have :

$$\frac{\partial(\mathrm{Tr}(R.K^{\mathrm{T}}))}{\partial R} = K \qquad \frac{\partial(\mathrm{Tr}(L.R.R^{\mathrm{T}}))}{\partial R} = 2.L.R \qquad \frac{\partial \det(R)}{\partial R} = \det(R).R^{(-\mathrm{T})} = R$$

Thus, the optimum is characterized by

$$\frac{\partial\Lambda}{\partial R} = K - R.L - g.R = 0 \qquad \Longleftrightarrow \qquad R(L + g\,\mathrm{Id}) = K \tag{3.4}$$

Let $L' = L + g.\,\mathrm{Id}$ and $K = U.D.V^\mathrm{T}$ be a singular value decomposition of $K$. Remember that $U$ and $V$ are orthogonal matrices (but possibly improper) and $D$ is the diagonal matrix of (positives) singular values. As $L$ (thus $L'$) is symmetric, we have:

$$L'^2 = (L'.R^\mathrm{T})(R.L') = K^\mathrm{T}.K = V.D^2.V^\mathrm{T}$$

The symmetric matrices $L'^2$ and $L'$ trivially commute. Thus, they can be diagonalized in a common basis, which implies that $L' = V.D.S.V^\mathrm{T}$, where $S = \mathrm{DIAG}(s_1, \ldots, s_n)$ and $s_i = \pm 1$. As $S = S^{(-1)}$ and $SD = DS$, we can simplify (3.4) into:

$$R.V.D = U.S.D \qquad \Longleftrightarrow \qquad R = U.S.V^\mathrm{T}$$

The positive determinant constraints gives moreover $\det(S) = \det(U).\det(V)$. Inserting $R = U.S.V^\mathrm{T}$ in the criterions, we find that

$$\mathrm{Tr}(R.K^\mathrm{T}) = \mathrm{Tr}(U.S.D.U^\mathrm{T}) = \mathrm{Tr}(D.S) = \sum_{i=1}^{n} d_i.s_i$$

Assuming that singular values are sorted in decreasing order, the maximum is obtained for $s_1 = \ldots = s_{n-1} = +1$ and $s_n = \det(U).\det(V)$.

We should note that the rotation $R = U.S.V^\mathrm{T}$ is always a optimum of our criterion, but is may not be the only one. This minimum is obviously unique if all the singular values are non zero ($K$ has then maximal rank), and [Ume91] shows that it ios still unique if $\mathrm{rank}(K) \geq n - 1$.

**Theorem 3.1 (Least-squares estimation of a rotation from a matrix K)**
*Let $K = U.D.V^T$ be a singular value decomposition of a given square matrix $K$ with (positives) singular values sorted in decreasing order in $D$ and $S = \mathrm{DIAG}(1, \ldots 1, \det(U)\det(V))$. The optimally closest rotation in the Least Squares (Froebenius) sense is given by*

$$R = U.S.V^T \tag{3.5}$$

*This optimum is unique if rank$(K) \geq n - 1$.*

**Exercise 3.1 Orthonormalization of matrices**

- Implement the SVD orthogonalization procedure Mat2RotMat(M) to compute the proper rotation which is closest to a given unconstrained matrix $M$.

### 3.1.3 Geometric parameters: axis and angle

Let $R$ be a 3-D rotation matrix. It is well known that it is characterized by its axis $n$ (unit vector) and its angle $\theta$. To understand why, let us compute the eigenvalues of the matrix $R$: they are solutions of the system $\det(R - \lambda.\,\mathrm{Id}) = 0$. This is a real polynomial of order 3. Thus, it has one real zero $\lambda_1$ and two complex conjugate zeros $\lambda_{2,3} = \mu.e^{\pm i\theta}$. Let $n$ be the unit eigenvector corresponding to the real eigenvalue: $R.n = \lambda_1.n$. As a rotation conserves the norm, we have $\|R.n\| = |\lambda_1| = \|n\| = 1$, which means that $\lambda_1 = \pm 1$. However, the determinant of $R$ being one, this imposes $\det(R) = \lambda_1.\mu^2 = 1$, from which we conclude that $\lambda_1 = +1$ and $|\mu| = 1$. Thus, every 3D rotation has an invariant axis $n$ which is the eigenvector associated to the eigenvalue 1, and two unitary complex conjugate eigenvalues $\lambda_{2,3} = e^{\pm i\theta}$ which realize a 2D rotation of angle $\theta$ in the 2D plane orthogonal to that axis.

**Rodrigues' formula: computing the rotation matrix $\mathcal{R}(\theta, n)$**

To rotate a vector $x$, we have to conserve the part $x_\parallel = n.n^\mathrm{T}.x$ which is parallel to $n$, and to rotate the orthogonal part $x_\perp = (\mathrm{Id} - n.n^\mathrm{T}).x$ by an angle $\theta$ around $n$. Let us first notice that the rotation of angle $\pi/2$ of $x_\perp$ around the axis $n$ is simply given by $w = n \times x_\perp = n \times x$. Thus the rotation of angle $\theta$ is simply $y_\perp = \cos(\theta).x_\perp + \sin(\theta).w$, and the final rotated vector can be written:

$$y = \mathcal{R}(n, \theta).x = x_\parallel + y_\perp = n.n^\mathrm{T}.x + \cos(\theta).(\,\mathrm{Id} - n.n^\mathrm{T}).x + n \times x$$

Let $S_x = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$ be the linear operator (skew matrix) corresponding to (left) cross product: for all vector $y$ we have $S_x.y = x \times y$. One easily verifies the identity $S_n^2 = n.n^\mathrm{T} - \mathrm{Id}$. Using this matrix, we can rewrite the rotation formula in a matrix form which is valid for all vectors $x$. This expression is called Rodrigues formula:

$$R = \mathrm{Id} + \sin\theta.S_n + (1 - \cos\theta).S_n^2 = \cos\theta.\,\mathrm{Id} + \sin\theta.S_n + (1 - \cos\theta).n.n^\mathrm{T} \tag{3.6}$$

**Computing the angle and axis parameters from the rotation matrix**

Conversely, we can determine the angle and axis of a rotation $R$ using:

$$\theta = \arccos\left(\frac{\mathrm{Tr}(R) - 1}{2}\right) \quad \text{and} \quad S_n = \frac{R - R^\mathrm{T}}{2.\sin\theta} \tag{3.7}$$

The equation for the axis is valid only when $\theta \in ]0; \pi[$. Indeed, for $\theta = 0$ (i.e. identity) the rotation axis $n$ is not determined, and $\sin(\theta) = 0$ for reflections (i.e. when $\theta = \pi$). From a computational point of view, this is creating numerical instabilities around the values $\theta = 0$ and $\theta = \pi$ that need to be adressed.

**$R$ close to identity: $\theta$ is small** Since the axis $n$ is not defined for identity, there is a singularity and a numerical instability around it. However, we can compute the rotation vector with a Taylor expansion:

$$S_r = \theta\, S_n = \frac{\theta}{2\sin\theta}.(R - R^\mathrm{T}) = \frac{1}{2}.\left(1 + \frac{\theta^2}{6}\right).(R - R^\mathrm{T}) + O\left(\theta^4\right)$$

**$R$ close to a reflection: $\pi - \theta$ is small** The axis is this time well defined, but we have to use another equation. From Rodrigues formula, we get $R + R^\mathrm{T} - 2.\,\mathrm{Id} = 2.(1 - \cos\theta).S_n^2$, and since $S_n^2 = n.n^\mathrm{T} - \mathrm{Id}$, we have

$$n.n^\mathrm{T} = \mathrm{Id} + \frac{1}{2.(1 - \cos\theta)}.\,(R + R^\mathrm{T} - 2.\,\mathrm{Id})$$

Let $\varrho = 1/(1 - \cos\theta)$; taking diagonal terms gives

$$n_i^2 = 1 + \varrho.(R_{i,i} - 1) \qquad \Rightarrow \qquad n_i = \varepsilon_i \sqrt{1 + \varrho.(R_{i,i} - 1)}$$

The off diagonal terms are used to determine the signs $\varepsilon_i$: considering that the sign of $n_1$ is $\varepsilon \in \{-1; +1\}$, we can compute that

$$\mathrm{sign}(n_k) = \varepsilon.\mathrm{sign}(R_{1,k} + R_{k,1})$$

If we have an exact reflection, the sign $\varepsilon$ does not matter since rotating clockwise or counter-clockwise gives the same result, but for a quasi-reflection, this sign is important. In this case, the vector $w = 2.\sin\theta.n$ is very small but not identically null: it can be computed without numerical instabilities with $S_w = R - R^{\mathrm{T}}$. Since $\theta < \pi$, the largest component $w_k$ in absolute value of this vector must have the same sign as the corresponding component $n_k$ of vector $n$.

**The rotation vector $r = \theta.n$**

As a conclusion, the rotation angle and axis are not always numerically stable (in particular around $\theta = 0$ and $\theta = \pi$), but we can always compute a stable compound version: the rotation vector $r = \theta.n$.

From an algorithmic point of view, the rotation vector is thus a representation of choice for 3D rotations. This representation of rotations is studied since a long time (see for instance [Stu64]), but it takes a particular importance in robotics [Pau82, Lat91] and computer vision [Kan93]. Although the geometric angle and axis parameters were well known, their combination into the rotation vector was not practically used before [Aya89, chap. 12].

As this angle is defined up to $2\pi$, all the vectors $r_k = (\theta + 2.k.\pi).n$ ($k \in \mathbb{Z}$) represent the same rotation $R = \mathcal{R}(\theta, n)$. Thus, a first chart is rotations vectors $r = \theta.n$ from the open ball $\mathcal{D} = \mathcal{B}(0, \pi)$. To define an atlas, we should define at least three other charts. Following [Aya89], one could keep the same representation $r = \theta.n$ with different definition domains, for instance the half open ball $\mathcal{B}_1 = \{r \in \mathcal{B}(0, 2\pi)/r_1 > 0\}$ (respectively $\mathcal{B}_2$ and $\mathcal{B}_3$) covering the rotations having a non null component of the axis of rotation along $e_1$ (respectively $e_2$ and $e_3$).

Rodrigues' formula allow us to compute directly the rotation matrix from the rotation vector:

$$R = \mathrm{Id} + \frac{\sin\theta}{\theta}.S_r + \frac{(1-\cos\theta)}{\theta^2}.S_r^2 \qquad \text{with} \qquad \theta = \|r\|$$

To get avoid numerical instabilities around $\theta = 0$, one can use the following Taylor expansions:

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} + O\left(\theta^4\right) \qquad \text{et} \qquad \frac{(1-\cos\theta)}{\theta^2} = \frac{1}{2} - \frac{\theta^2}{24} + O\left(\theta^4\right)$$

**Exercise 3.2 Rotation matrix to rotation vector**

- Implement the operator $S_n$ (SkewSymMat(r)).
- Verify that it corresponds to the left cross product.
- Implement a function RotVect2RotMat(r) converting a rotation vector to a rotation matrices and its inverse RotMat2RotVect(r).

### 3.1.4 Uniform random rotation

For testing our computational framework on rotations, we will need to generate some random rotations. Although any distribution covering the whole space would be sufficient for that, it is more satisfying to have a distribution that covers uniformly the manifold. In the context of a groups, uniformity is defined in terms of invariance by left or right translation: the measure $\mu$ is left invariant if $\mu(R.S) = \mu(R)$ for any rotation $R$ and set $S \subset \mathcal{SO}_n$.

For any locally compact topological groups, the left (resp. right) Haar's measure $\mu_l$ (resp $\mu_r$) is the unique (up to a multiplicative constant) left-invariant Borel measure on the group. Left and right Haar's measure are generally different (we have $\mu_r(S) = \mu_l(S^{(-1)})$) , unless the group is unimodular which is the case of rotations. Moreover, as our Riemannian metric is left and right invariant, the induced Riemannian measure is exactly the Haar measure.

Algorithms to generate random uniform orthogonal matrices from square matrices with Gaussian entries have been developed based on the QR recomposition (the orthogonal matrix Q is uniform as long as the diagonal of R contains only positive entries) [Ste80] or more efficiently on the subgroup algorithm [DS87], which iteratively builds an $(n+1)\times(n+1)$ orthogonal matrix from an $nn$ one and a uniformly distributed unit vector of dimension $n+1$ by applying a Householder reflection from the vector to the smaller matrix (embedded in $\mathcal{M}_{(n+1)\times(n+1)}$ with 1 in the bottom corner).

In our case, as the efficiency is not our main concern, it is more consistent to continue with the SVD. Let us assume a rotationally symmetric (spherical) distribution on the embedding space of square matrices (considered as a vector space with the standard Froebenius norm) [FH84]. Rotational invariance means that the pdf of the matrix $K$ does only depend on its norm: $p(K) = \varphi(\mathrm{Tr}(K.K^{\mathrm{T}})$. Let $K = U.S.V^{\mathrm{T}}$ be a SVD decomposition; this can be further simplified into $p(K) = \varphi(\sum_i s_i)$. Thus, the orthogonalization procedure $K = U.S.V^{\mathrm{T}} \to R = U.V^{\mathrm{T}}$ amounts to marginalize with respect to the singular values, and we have that $p(R) = \int \varphi(\sum_i s_i).\prod_i ds_i = Cte$ with respect to the restriction of the embedding measure to the manifold. A zero mean and unit vraiance Gaussien distribution on each matrix being a rotationally symmetric distribution, its orthogonalization using the SVD method is thus a uniform distribution on $\mathcal{SO}_n$.

**Exercise 3.3 Random uniform rotations**

- Implement a uniform random rotation matrix generator UnifRndRotMat().
- Verify that the distribution of the rotation angle is proportional to $\sin(\theta/2)^2$.
- Verify the orthogonality of the SVD orthogonalization and the consistency of the transformation between rotation matrices and rotation vectors on large number of random rotations.

### 3.1.5 Differential properties

#### Tangent spaces

Let $R(t) = R + t.\dot{R} + O(t^2)$ be a curve drawn on $\mathcal{SO}_3$, considered as an embedded manifold into the set of matrices $\mathcal{M}_{3\times3}$. The constraint (3.2) is differentiated into

$$\dot{R}.R^{\mathrm{T}} + \left(\dot{R}.R^{\mathrm{T}}\right)^{\mathrm{T}} = 0 \qquad \text{or} \qquad R^{\mathrm{T}}.\dot{R} + \left(R^{\mathrm{T}}.\dot{R}\right)^{\mathrm{T}} = 0$$

which means that $\dot{R}.R^{\mathrm{T}}$ and $R^{\mathrm{T}}.\dot{R}$ are skew-symmetric matrices (these two conditions are equivalent). Since $3\times3$ skew-symmetric matrices have 3 free components, we have determined all the tangent vectors of the 3-dimensional tangent spaces:

**Theorem 3.2** *the tangent space $T_{\mathrm{Id}}\mathcal{SO}_3$ of $\mathcal{SO}_3$ at identity is the vector space of skew matrices. The tangent space $T_R\mathcal{SO}_3$ at $R \in \mathcal{SO}_3$ is given by*

$$T_R\mathcal{SO}_3 = \{X \in \mathcal{M}_{3\times3} / X.R = -(X.R)^T\} = \{X \in \mathcal{M}_{3\times3} / R.X = -(R.X)^T\}$$

### Left and right translations

Two important and canonical maps on a Lie group are very useful for studying the tangent space: these are the left and right translations. In $\mathcal{S}O_3$ this is nothing else than the left and right composition by a fixed rotation $R_0$

$$\begin{array}{rccl} & \mathcal{S}O_3 & \longrightarrow & \mathcal{S}O_3 \\ \mathcal{L}_{R_0}: & R & \longmapsto & \mathcal{L}_{R_0}(R) = R_0.R \\ \mathcal{R}_{R_0}: & R & \longmapsto & \mathcal{R}_{R_0}(R) = R.R_0 \end{array} \tag{3.8}$$

Their differentials realize canonical isomorphisms between the tangent spaces of $\mathcal{S}O_3$ at different points. Let $X$ be a vector of $T_R\mathcal{S}O_3$ and $R_X(t) = R + t.X + O(t^2)$ be a curve on $\mathcal{S}O_3$ going through $R$ with tangent vector $\frac{dR_X(t)}{dt} = X$. The left translation of $R_X$ is the curve $R_0.R_X(t) = R_0.R + t.R_0.X + O(t^2)$ and its tangent vector at 0 is $Y = R_0.X$. The differential $D\mathcal{L}_{R_0}$ of $L_{R_0}$ is then for any $R$ in $\mathcal{S}O_3$

$$\begin{array}{rccc} & T_R\mathcal{S}O_3 & \longrightarrow & T_{R_0.R}\mathcal{S}O_3 \\ D\mathcal{L}_{R_0}: & X & \longmapsto & D\mathcal{L}_{R_0}(X) = R_0.X \end{array}$$

Notice that $\mathcal{L}_R$ and its differential $D\mathcal{L}_R$ take the same form *in the embedding space of matrices* although they act on different subspaces. The differential $D\mathcal{R}_{R_0}(X) = X.R_0$ is defined similarly.

For $R = \mathrm{I}d$, this gives two canonical isomorphisms between the tangent space at identity $T\mathcal{S}O_3$ and the tangent space $T_R\mathcal{S}O_3$ at any point $R$ (since the formulations of $D\mathcal{L}_{R_0}$ and $D\mathcal{R}_{R_0}$ and independent of $R$, we denote their restriction to $R = \mathrm{I}d$ by the same names).

$$\begin{array}{rccc} & T_{\mathrm{Id}}\mathcal{S}O_3 & \longrightarrow & T_R\mathcal{S}O_3 \\ D\mathcal{L}_R: & X & \longmapsto & R.X \\ D\mathcal{R}_R: & X & \longmapsto & X.R \end{array} \tag{3.9}$$

Differential properties of rotations are then reducible to differential properties around identity in $T_{\mathrm{Id}}\mathcal{S}O_3$.

### Bases of tangent spaces

In order to be able to compute in tangent spaces, we need to have coordinates and thus to define bases. Any skew-symmetric $X$ matrix verifies $X_{ij} = -X_{ji}$ so that only off diagonal components are non-null. Moreover, the lower triangular part is completely determined by the upper triangle one. Thus, a basis of $T\mathcal{S}O_3$ is given by

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The projection in this basis determines an isomorphism between the tangent space at identity and the vector space $\mathbb{R}^3$:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mapsto S_x = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in T_{\mathrm{Id}}\mathcal{S}O_3 \tag{3.10}$$

In order to minimize the number of notations, the basis has been chosen so that the matrix $S_n$ is the skew matrix corresponding to (left) cross product, as defined before.

To obtain bases of all other tengent spaces, we could find ad-hoc bases from the embedding space. However, we have two canonical ways to transport a structure from the tangent plane at identity to any point: the left and right translations of $E_i$ are $E_i^l = R.E_i$ and $E_i^r = E_r.R$. The coordinates in these two bases give rise to two different canonical isomorphisms between the tangent space at any point and $\mathbb{R}^3$:

$$\text{Given } dR \in T_R\mathcal{SO}_3, \quad \exists\, \omega_r, \omega_l \in \mathbb{R}^3 \qquad \text{such as} \qquad dR = S_{\omega_r}.R = R.S_{\omega_l} \tag{3.11}$$

**Exercise 3.4 Bases and coordinates in tangent spaces**

- Implement functions Ei(i), Ei_l(R,i) and Ei_r(R,i) that gives the above basis vectors of $T_{\mathrm{Id}}\mathcal{SO}_3$, and their left and right translation which constitute bases of $T_R\mathcal{SO}_3$ at any rotation $R$.
- Implement a projection projection TgCoordRotId(dR) : $dR \in T_{\mathrm{Id}}\mathcal{SO}_3 \mapsto dr \in R^3$ which compute the coordinates in the above basis. Remark: compute the projection error because $dR$ might not numerically be exactly skew symmetric.
- Extend that projection by left and right translations to obtain coordinates in the left and right translated bases: TgCoordRot_r(R, dR) and TgCoordRot_r(R, dR).
- Implement the reverse functions InvTgCoordRotId(dr) : $dr \in R^3 \mapsto dR \in T_{\mathrm{Id}}\mathcal{SO}_3$ , InvTgCoordRot_r(R, dr) and InvTgCoordRot_r(R, dr).
- Verify that dR = InvTgCoordRot(TgCoordRot(R, dR)) for random tangent vectors at random rotation matrices.

**The Lie algebra of $\mathcal{SO}_3$**

We already know that $T_{\mathrm{Id}}\mathcal{SO}_3$ is the vector space of skew symmetric matrices, identifiable with $\mathbb{R}^3$. If we now consider that $(\mathbb{R}^3, +, \times)$ is an algebra ($\times$ is the cross product), we can induce an algebra on $T_{\mathrm{Id}}\mathcal{SO}_3$. Let $X = S_x$ and $Y = S_y$ be two vectors of $T\mathcal{SO}_3$, then the vector $Z = S_{(x\times y)}$ belongs to $T\mathcal{SO}_3$ and is called the bracket of $X$ and $Y$:

$$Z = S_{(x\times y)} = S_x.S_y - S_y.S_x = X.Y - Y.X = [X, Y]$$

Hence $(T_{\mathrm{Id}}\mathcal{SO}_3, +, [.,.])$ is an algebra with the standard matrix commutator $[X, Y] = X.Y - Y.X$. This is not by chance, it is in fact the Lie algebra of the group $\mathcal{SO}_3$.

### 3.1.6 A bi-invariant metric on $\mathcal{SO}_3$

The next step is to give a metric to the group. Consider the Froebenius dot products on matrices $\langle\, X \mid Y \,\rangle_{\mathbb{R}^{n\times n}} = \frac{1}{2}\mathrm{Tr}(X^{\mathrm{T}}.Y)$. The factor 1/2 is there to compensate the fact that we are counting twice each off diagonal coefficient of the skew-symmetric matrices. This induces on any tangent space $T_R\mathcal{SO}_3$ at $R$ the scalar product

$$\langle\, X \mid Y \,\rangle_{T_R\mathcal{SO}_3} \stackrel{def}{=} \frac{1}{2}.\mathrm{Tr}(X.Y^{\mathrm{T}})$$

This metric is obviously invariant by left and right translation as for any rotation $U$ since we have

$$\langle\, U.X \mid U.Y \,\rangle_{T_{U.R}\mathcal{SO}_3} = \frac{1}{2}.\mathrm{Tr}(U.X.(U.Y)^{\mathrm{T}}) = \frac{1}{2}.\mathrm{Tr}(X.Y^{\mathrm{T}}) = \langle\, X \mid Y \,\rangle_{T_R\mathcal{SO}_3}.$$

and equivalently for the right translation:

$$\langle\, X.U \mid Y.U \,\rangle_{T_{R.U}\mathcal{SO}_3} = \frac{1}{2}.\mathrm{Tr}(X.U.(Y.U)^{\mathrm{T}}) = \frac{1}{2}.\mathrm{Tr}(X.Y^{\mathrm{T}}) = \langle\, X \mid Y \,\rangle_{T_R\mathcal{SO}_3}.$$

Such a metric is called a bi-invariant Riemannian metric.

Notice that the existence of such a metric is not ensured for more general non-commutative and non-compact groups. Moreover, changing the choice of the scalar product of the embedding space (other than by a global scaling) will induce a new metric on all tangent spaces which will not be left nor right invariant.

In order to understand what this scalar product is, let us investigate its expression is a local coordinate system. At the identity, the coordinate vector $x$ of $X$ is such that $X = \sum_i x_i.E_i$. By linearity, the scalar product of two vector $X$ and $Y$ is reduced to the one of two basis vectors : $\langle\, X \mid Y \,\rangle_{T_{\mathrm{Id}}\mathcal{SO}_3} = \sum_{i,j} x_i.y_j.\langle\, E_i \mid E_j \,\rangle_{T_{\mathrm{Id}}\mathcal{SO}_3}$. With the above Froebenius metric and the basis we chose, the expression of the metric tensor is particularly simple:

$$\langle\, E_i \mid E_j \,\rangle_{T_{\mathrm{Id}}\mathcal{SO}_3} = \frac{1}{2}.\mathrm{Tr}(E_i.E_j^{\mathrm{T}}) = \delta_{ij}$$

Thus, our basis is in fact ortho-normal and the Froebenius scalar product corresponds to the canonical dot product of $\mathbb{R}^3$ through our isomorphism:

$$\langle\, S_x \mid S_y \,\rangle_{T_{\mathrm{Id}}\mathcal{SO}_3} = \langle\, x \mid y \,\rangle_{\mathbb{R}^3} = x^{\mathrm{T}}.y$$

In the tangent at rotation $R$, we can use the coordinates $x^l$ of $X$ in the basis $E_i^l = R.E_i$ or the coordinates $x^r$ in the basis $E_i^r = E_i.R$. We can easily see that these two basis are also orthonormal:

$$\langle\, X \mid Y \,\rangle_{T_R\mathcal{SO}_3} = \left\langle\, R.S_{x^l} \mid R.S_{y^l} \,\right\rangle_{T_R\mathcal{SO}_3} = \frac{1}{2}.\mathrm{Tr}(S_{x^l}^{\mathrm{T}}.R^{\mathrm{T}}.R.S_{y^l}) = \frac{1}{2}.\mathrm{Tr}(S_{x^l}^{\mathrm{T}}.S_{y^l}) = x^{l\mathrm{T}}.y^l = x^{r\mathrm{T}}.y^r$$

**Theorem 3.3** *The tangent space of $\mathcal{SO}_3$ at identity $T_{\mathrm{Id}}\mathcal{SO}_3$ is the vector space of skew symmetric matrices. With the Lie bracket $[.,.]$ and the Euclidean metric $\langle\, X \mid Y \,\rangle = \frac{1}{2}.\mathrm{Tr}(X^T.Y)$, it forms a metric algebra which is canonically isomorphic to $(\mathbb{R}^3, +, \times, \langle\, . \mid . \rangle_{\mathbb{R}^3})$.*

*The tangent space $T_R\mathcal{SO}_3$ at $R$ is transported from $T_{\mathrm{Id}}\mathcal{SO}_3$ with the left or right translation by equation (3.9).*

**Exercise 3.5 Bi-invariant scalar product in tangent spaces**

- Implement the scalar product ScalRotId(X, Y) = $\mathrm{Tr}(X.Y^{\mathrm{T}})/2$.
- Verify that ScalRotId(Ei(i),Ei(j)) = $\delta_{ij}$
- Verify that dR = $\langle\, dR \mid E1 \,\rangle$.E1 + $\langle\, dR \mid E2 \,\rangle$.E2 + $\langle\, dR \mid E3 \,\rangle$.E3 for random skew symmetric matrices.
- Implement the scalar product ScalRot(R, X, Y) at any rotation R using left translation.
- Generate Gaussian random vectors in the tangent plane at a random rotation $R$ and verify numerically that their scalr product corresponds to the scalar product of the embedding space.

### 3.1.7 Group exponential and one-parameter subgroups

The matrix exponential is defined for any matrix $X$ as the limit of the series

$$\exp(X) = \text{ I}d + X/1! + X^2/2! + \ldots = \sum_{k=0}^{+\infty} \frac{X^k}{k!}$$

For any element $X$ of the Lie algebra of $\mathcal{S}O_3$, the skew symmetry implies that $X^3 = -\theta^2.X$ (with $\theta = \|X\|$) so that the series reduces to

$$\exp(X) = \text{ I}d + \frac{\sin\theta}{\theta}.X + \frac{1-\cos\theta}{\theta^2}.X^2 = \text{ I}d + \sin\theta.S_n + (1-\cos\theta).S_n^2$$

We recognize here Rodrigues' formula:t the matrix exponential of the skew symmetric matrix associated to the rotation vector $r = \theta.n$ is the rotation of angle $\theta$ around the unit axis $n$: $\exp(\theta.S_n) = \mathcal{R}(n,\theta)$.

#### One parameter subgroups

Let $R_X(s)$ be a one parameter subgroup of $\mathcal{S}O_3$ (homomorphism from $(\mathbb{R},+)$ to $(\mathcal{S}O_3,.)$. This is a continuous curve which is also a subgroup of $\mathcal{S}O_3$. By definition and since $(\mathbb{R},+)$ is commutative

$$R_X(s+t) = R_X(s).R_X(t) = R_X(t).R_X(s)$$

This means in particular that $R_X(t)$ and $R_X(s)$ commute. Thus, they have the same rotation axis $n$. $R_X(s)$ is thus a rotation of axis $n$ and angle $\theta(s)$. Reporting this in the definition of one parameter subgroups, we find $\theta(s+t) = \theta(s) + \theta(t)$ and hence $\theta(s) = \lambda.s$ with some $\lambda \in \mathbb{R}$. Computing the derivatives, we find

$$\left.\frac{dR_X}{ds}\right|_0 = X = \lambda.S_n \in T\mathcal{S}O_3 \qquad \text{and} \qquad \left.\frac{dR_X}{ds}\right|_s = X.R_X(s) = R_X(s).X \in T_{R_X}\mathcal{S}O_3$$

We established that to each one-parameter subgroup $R_X(s) = \mathcal{R}(n,\lambda.s) = \exp(\lambda.s.S_n)$ of $\mathcal{S}O_3$ corresponds a unique vector $X = \lambda.S_n$ of $T\mathcal{S}O_3$. The converse is also true and $R_X(s)$ is called the integral curve of $X$. It is to be noted that $X$ and $\lambda.X$ generate the same integral curve with proportional parameterizations.

This is a particular case of a more general theorem for Lie groups which state that there is a one to one correspondence between one parameter subgroups of the Lie group and one dimensional sub-algebras of its Lie algebra.

#### Group exponential map

Let $X = \theta.S_n$ with $\|n\| = 1$ be a vector of $T_{\text{Id}}\mathcal{S}O_3$ and $R_X(s) = \mathcal{R}(n,\theta.s) = \exp(\lambda.s.S_n)$ the integral curve of $X$. The group exponential map is defined as the map from to Lie algebra (the tangent plane at identity) to the group $\mathcal{S}O_3$ which assigns $R_X(1)$ to $X$. Hence

$$X \in T_{\text{Id}}\mathcal{S}O_3 \mapsto \exp(X) = \exp(\theta.S_n) = \mathcal{R}(n,\theta) \in \mathcal{S}O_3$$

The term exponential maps comes from fact that one-parameter subgroups of matrix groups can be computed using the matrix exponential. Using the isomorphism between $\mathbb{R}^3$ and $T_{\text{Id}}\mathcal{S}O_3$ we can also define the exponential of the **rotation vector** $r = \theta.n$:

$$\mathcal{R}(r) = \exp(S_r) = \mathcal{R}(n,\theta) \qquad \text{with} \qquad \theta = \|r\| \qquad \text{and} \qquad n = \frac{r}{\theta}$$

The exponential map is a sort of "development" of $\mathcal{SO}_3$ onto its tangent space $T\mathcal{SO}_3$: each one-dimensional subspace $\mathbb{R}.S_n$ is mapped on its integral curve $\mathcal{R}(n, \mathbb{R})$ and we will see that the length along these curves are conserved.

### 3.1.8 Metric properties

#### Definitions

Let $R : s \in [a, b] \subset \mathbb{R} \mapsto R(s) \in \mathcal{SO}_3$ be a piecewise curve on $\mathcal{SO}_3$ and $\dot{R}$ the tangent vector of $R$ at $s$. The *length* of the curve $R$ is defined by

$$\mathcal{L}(R) = \int_a^b \sqrt{\left\langle \dot{R}(s) \,\middle|\, \dot{R}(s) \right\rangle_{R(s)}}.ds \tag{3.12}$$

Let now $\Gamma$ be the set of curves joining rotations $R_1$ and $R_2$. The map

$$\rho : \begin{array}{ccc} \mathcal{SO}_3 \times \mathcal{SO}_3 & \longrightarrow & \mathbb{R} \\ (R_1, R_2) & \longmapsto & \inf_{R \in \Gamma} \mathcal{L}(R) \end{array} \tag{3.13}$$

is the canonical metric on $\mathcal{SO}_3$. The curves $R$ minimizing the criterion $\mathcal{L}(R)$ are called geodesics.

#### Geodesics and metric on $\mathcal{SO}_3$

Since the metric is bi-invariant on $\mathcal{SO}_3$, the length criterion (3.12) is invariant by left and right translations and finding geodesics amounts to find geodesics starting from identity. Moreover, for a Lie group with a bi-invariant Riemannian metric, it turns out that geodesics starting from identity, one-parameter subgroups and integral curves (starting also from identity) are three different approaches for the same curves [Spi79, chap.10]. Let $R_X(s) = \exp(\lambda.s.S_n)$ be such a curve. Its derivative at identity is $\dot{R}_X(0) = X = \lambda.S_n$ and $\dot{R}_X(s) = X.R_X(s) = R_X(s).X$ elsewhere. Hence

$$\left\langle \dot{R}(s) \,\middle|\, \dot{R}(s) \right\rangle = \langle X \mid X \rangle = \lambda^2. \langle n \mid n \rangle = \lambda^2$$

and the distance from identity to $\mathcal{R}(n, \theta) = R_X(\theta/\lambda)$ is

$$\rho(I_3, \mathcal{R}(n, \theta)) = \int_0^{\theta/\lambda} \sqrt{\lambda^2}.ds = \theta$$

With an arc-length parameterization, we obtain $R_{S_n}(\theta) = \mathcal{R}(n, \theta) = \exp(\theta.S_n)$. These curves are $2\pi$-periodic and $R_{S_n}(\theta) = R_{(-S_n)}(-\theta)$. Hence shortest paths (or minimizing geodesics) are uniquely defined for $\theta < \pi$ and doubly defined for $\theta = \pi$.

**Theorem 3.4** *The canonical metric on $\mathcal{SO}_3$ is given by*

$$\rho(R_1, R_2) = \rho(I_3, R_1^T.R_2) = \theta(R_1^T.R_2) = \arccos\left(\frac{(\mathrm{Tr}(R_1^T.R_2) - 1)}{2}\right)$$

*Geodesics of $\mathcal{SO}_3$ starting from identity are the curves $\theta \mapsto \mathcal{R}(n, \theta) = \exp(\theta.S_n)$. Shortest paths from identity to the non reflection rotation $R = \exp(\theta.S_n)$ $(0 \leq \theta < \pi)$ are given by*

$$t \in [0, \theta] \mapsto \exp(t.S_n)$$

*Shortest paths from identity to reflection $R = \exp(\pi S_n)$ are doubly defined by the above formula $(\theta = \pi)$ with $n$ and $-n$ as unit vectors. Other geodesics are obtained by left or right translation.*

**Riemannian Exp and Log maps**

Let $X = S_x$ be a vector of $T_{\mathrm{Id}}\mathcal{S}O_3$. From the above results, we know that $\gamma_{(\,\mathrm{Id},X)}(t) = \exp(t.X)$ the unique geodesic starting at $\mathrm{I}d$ with tangent vector $X$. The Riemannian exponential map at the identity for the bi-invariant metric is then defined by:

$$X \in T_{\mathrm{Id}}\mathcal{S}O_3 \mapsto \mathrm{Exp}_{\,\mathrm{I}d}(X) = \gamma_{(\,\mathrm{I}d,X)}(1) = \exp(X) \in \mathcal{S}O_3$$

According to the above theorem, this function is a diffeomorphism for $\|X\| = \theta < \pi$, and the reverse is given by the

$$R \in \mathcal{S}O_3 \mapsto \mathrm{Log}_{\,\mathrm{I}d}(R) = S_r \in T_{\mathrm{Id}}\mathcal{S}O_3 \qquad \text{where} \qquad r = \theta.n \text{ is the rotation vector of } R = \mathcal{R}(n, \theta).$$

Geodesics starting at any other points are defined by left (or right) translation. For instance, $\gamma_{(R,R.X)}(t) = R.\exp(t.X)$ is the unique geodesic starting at $R$ with tangent vector $Y = R.X$, which can be rewritten $\gamma_{(R,Y)}(t) = R.\exp(t.R^{\mathrm{T}}.Y)$. This corresponds to the following reasonning: to find the geodesic starting at $R$ with tangent vector $Y$, we first translate $(R, Y)$ on the left by $\mathbb{R}^t rp$, wich give $(\,\mathrm{I}d, R^{\mathrm{T}}.X)$, take the geodesic at that point, and left translate back its result by $R$. Since the metric is by invariant, the same mechanism can be implemented with right translation. The formula for the exponential map at any point is thus:

$$X \in T_R\mathcal{S}O_3 \mapsto \mathrm{Exp}_R(X) = \gamma_{(R,X)}(1) = R.\mathrm{Exp}_{\,\mathrm{I}d}(R^{\mathrm{T}}.X) = R.\exp(R^{\mathrm{T}}.X) = \exp(X.R^{\mathrm{T}}).R \in \mathcal{S}O_3$$

Likewise, to compute the log map of rotation U at rotation R, we first left translate both rotations by $R^{\mathrm{T}}$, take the log map of $R^{\mathrm{T}}.U$ at $\mathrm{I}d$, and left translate back to result by $R$:

$$U \in \mathcal{S}O_3 \mapsto \mathrm{Log}_R(U) = R.\mathrm{Log}_{\,\mathrm{I}d}(R^{\mathrm{T}}.U) = R.S_x \in T_R\mathcal{S}O_3 \qquad \text{where} \qquad x = \theta.n \text{ is the rotation vector of } R^{\mathrm{T}}.U.$$

A similar expression is obtained using the right translation, and both are equal thanks to bi-invariance.

**Exercise 3.6 Exp and log maps**

- Compute Exp map at Id: ExpRotId(dR)
- Verify that the exponential of any tangent vector at identity corresponds to the matrix exponential.
- Compute Log Map at Id LogRotId(dR)
- Verify that the log at identity of a random rotation corresponds to the matrix logarithm.
- Compute Exp and Log maps at any rotation R using the left invariance
- Verify that the construction using the right invariance give the same result and the consistency of Exp
- Verify the consistency of Exp(R, Log(R,U)) for random rotations and the one of Log(R, Exp(R, X)) for tangent vectors X that are within the cut locus.

## 3.2 Additional literature

There is a large literature on 3D rotations. From a mathematical point of view, [Alt86], [Kan90, chap. 3 & 6] and [Fau93] give quite exhaustive syntheses. Most of the algorithmic parts of this chapter are extracted from [Pen96, PT97].