Inverse problems for a Beltrami equation and extrapolation of magnetic quantities in a tokamak

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During the thermonuclear fusion, a model for plasma equilibrium in a tokamak is obtained from magneto-hydrodynamic equations in an axi-symmetrical context. Indeed, the device is seen as a toroidal volume in which the plasma is contained.

In a meridian section of the tokamak which is modelled by a disk not containing the origin, the poloidal magnetic flux u(x,y) fulfills equation [Bl]:

(1) $\mathrm{div}\,(\sigma\ \mathrm{grad}\ u) = 0$ with $\sigma(x,y) = 1/x$,

in the annular domain sitting between the plasma surface and the exterior of the chamber.

Two inverse problems set by the physicians of the research institute of fusion by magnetic containment (CEA Cadarache) whose solution is necessary for the development of ITER, are described below:

(i) From the values of u and its normal deritative $\partial u/\partial n$ on the assumed to be known on the plasma boundary, to extrapolate these quantities on a given curve in an annular domain. It is a Cauchy problem.
(ii) From the values of u and $\partial u/\partial n$ on the exterior side of the annular domain, to find the plasma limit set by the equipotential through a known point. A weak form of this question deals with (i). The strong form is a Bernoulli type issue.

These two problems are ill-posed, since measurements errors, however small, on the Neumann data (i.e. the magnetic field) lead to an inconsistency of the boundary data with the equation. Moreover, a solution of the equation which might be close to these data could give an equipotential very different from the one we are looking for.

The current techniques (based for example on the Taylor approximation) do not give a precise answer as required by the CEA designers and engineers for the next generation of tokamak (ITER). Some stable computational methods are heavily needed in order to determine and control the equilibrium surface of the contained plasma [BBJ].

The present approach will extend the one used for very similar problems for harmonic functions (σ=1) within the APICS research team. It formulates and solves these questions through constrained approximation problems under constraints (bounded extremal problems) in spaces of analytic functions (Hardy classes) [BL, BLP1, LMP].

The first step of this study will consider the equation (1) as the compatibility condition of the following conjugated Beltrami equation, in the complex variable $z = x+iy$ [AP, BN]:

$$(2) \qquad \partial f / \partial \bar{z} = \nu \, \partial \bar{f} / \partial \bar{z} \quad \text{where } \mathrm{Re}\, f = u \text{ and } \nu = (1-\sigma)/(1+\sigma).$$

Inverse problems of mixed Cauchy ones for (1) lead to some inverse Dirichlet (extrapolation) problems for (2) since the u and $\partial u/\partial n$ values determine f on the part of the boundary where they are available (generalized Cauchy-Riemann equations).

The second step of this study will regularize these problems, known as being ill-posed. A suitable approach will constrain u on the boundary part where its values are not provided and identify f (under this constraint) to be as close to the data on the other side of the boundary as possible. This is a bounded extremal problem whose solution in the analytic framework ($\nu=0$) has been examined in [BL, BLP1, BLP2].

The thesis subject deals with the study and the constructive solving of this bounded extremal problem for a non constant conductivity σ, like the one described above in the case of a tokamak. Existence and uniqueness of a solution in some generalised Hardy classes are set in [BLRR] in the case of a disk. To generalise these proprieties to the case of an annulus and to characterise the solution will be the aim of this thesis. This will lead to design a constructive tool to evaluate the magnetic quantities seeked in question (i), the regularisation parameter being the constraint imposed to u on the outer boundary. This will also allow one to handle the weak form of question (ii).

In a third step, we will consider bounded extremal problems of mixed type, with a pointwise constraint and an integral criterion, that seem particularly appropriated for (ii) and the determination of the plasma boundary. Their solution when $\sigma=1$ relies on the multiplicative structure of analytic functions. An analogous study for variable conductivities σ remains to be done [BN]. The initial global geometrical problem would then be localized thanks to pointwise informations given by the constraint value. This would lead to significant improvements of the precision of a descent algorithm.

Algorithmic and software developments are part of the thesis, and tests will be done with the tokamak Tore Supra, running in Cadarache, in collaboration with the physicists of the CEA. Observe that resolution algorithms for bounded extremal problems are delicate, already in situations where $\sigma=1$. Their extension to the case $\sigma=1/x$ will first requires to find suitable bases for the computation of solutions to (1). This equation admits Bessel functions as particular solutions, which could allow us to build such a basis [BLPQS]. A possibility would be to use the representation theorem of [BN] in order to set up – though in an implicit way – an equivalent family of extremal problems in Hardy spaces of analytic functions.

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