Identification and function theory

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General purpose

We survey some constructive aspects of

how to recover a function analytic in a plane domain from complete or partial knowledge of its boundary values

in connection with

identification issues for linear dynamical systems, *i.e.* one-dimensional deconvolution.

Our domains of analyticity will be either the unit disk or the half-plane, as encountered in this context. Many things carry over with additional complications to smooth simply connected domains, but irregular boundaries or multiple connectedness may cause difficulties
.

$\boxed{\text{For a comparison...}}$

Sometimes in system theory one looks for a rational function, in which case we aim at rational approximation but the poles should lie outside the domain.

In previous lectures we saw Padé approximation and Borel resummation used to try to construct analytic continuation of a power series expansion to some domain.

Here, we prescribe a domain of analyticity and our data consist of boundary values or partial estimation of them.

We need to define what "boundary value" means to us. It means nontangential limit, *i.e.* the limit of the function when the argument tends to a boundary point within a cone, whose aperture turns out to be irrelevant.

$\boxed{\text{...nontangential limits}}$
$\boxed{\text{also define the function.}}$

In fact, if an analytic function is bounded on cones of fixed aperture over a (measurable) subset E of the boudary, it has non-tangential limit at almost every point of E. And if the function has nontangential limit 0 on a set of positive (arclength) measure of the boundary, then it vanishes identically [Privalov].

Why worry about refined notions of boundary values and not just work with continuous functions on the closure of the domain?

Because we approach the recovery issue as an extremal problem in Banach spaces of analytic functions that need not have continuous boundary values. This can be seen as a regularization technique for an inverse problem which is known to be ill-posed since Hadamard (Cauchy problem for the Laplace equation).

Hardy Spaces

Let $\mathbb{T}$ be the unit circle and $\mathbb{D}$ the unit disk in the complex plane. Denote by $L^p = L^p(\mathbb{T})$ the familiar Lebesgue spaces.

For $1 \leq p \leq \infty$, the Hardy space H^p of the unit disk is the closed subspace of L^p consisting of functions whose Fourier coefficients of strictly negative index do vanish.

Alternatively, these are the nontangential limits of functions g analytic in $\mathbb{D}$ having uniformly bounded L^p means over all circles centered at 0 of radius less than 1 :

$$\sup_{0<r<1} \|g(re^{i\theta})\|_p < \infty. \tag{1}$$

Poisson and Cauchy

This correspondence between an analytic function in $\mathbb{D}$ satisfying (1) and its boundary values is one-to-one. Identifying them, we may regard members of H^p as holomorphic functions in the variable $z \in \mathbb{D}$.

The extension to $\mathbb{D}$ is obtained from the values on $\mathbb{T}$ through a Cauchy as well as a Poisson integral, namely if $g \in H^p$ then :

$$g(z) = \frac{1}{2\,i\,\pi} \int_{\mathbb{T}} \frac{g(\xi)}{\xi - z}\, d\xi\,, \qquad z \in \mathbb{D}, \tag{2}$$

and also

$$g(z) = \frac{1}{2\,\pi} \int_{\mathbb{T}} \mathrm{Re}\left\{\frac{e^{i\theta} + z}{e^{i\theta} - z}\right\} g(e^{i\theta})\, d\theta\,, \qquad z \in \mathbb{D}. \tag{3}$$

The space H^∞ consists of bounded analytic functions in $\mathbb{D}$, and by Parseval's theorem we also get that

$$g(z) \in H^2 \quad \text{iff} \quad f(z) = \sum_{j=0}^{\infty} a_k z^k,$$

$$\text{with} \quad \sum_{j=0}^{\infty} |a_j|^2 < \infty.$$

But if $p \neq 2$ it is uneasy to characterize H^p functions from their Fourier-Taylor coefficients. Very good expositions on Hardy spaces are to be found in the books of [Duren, Garnett, Koosis], and we will mention only a few facts here. Actually, we shall work only with $p = 2$ and $p = \infty$, but nothing would be gained at this stage from such a restriction.

Inner-Outer Factorization

A nonzero $g \in H^p$ can be uniquely factored as $g = jw$, where :

$$w(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|\,d\theta\right\} \quad (4)$$

belongs to H^p and is the *outer factor* of g, while $j \in H^\infty$ has modulus 1 a.e. on $\mathbb{T}$ and is the *inner factor* of g.

The inner factor decomposes as $j = bS_\mu$, where :

$$b(z) = e^{i\theta_0} z^k \prod_{z_l \neq 0} \frac{-\bar{z}_l}{|z_l|}\frac{z-z_l}{1-\bar{z}_l z} \quad (5)$$

is a *Blaschke product*,while

$$S_\mu(z) = \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\right\} \quad (6)$$

is a *singular inner factor* associated with $\mu \geq 0$ singular with respect to Lebesgue measure.

The z_l are the zeros of g in $\mathbb{D} \setminus \{0\}$, counted with their multiplicities, while k is the order of the zero at 0. If there are infinitely many zeros, the convergence of the product $b(z)$ in $\mathbb{D}$ is ensured by the condition

$$\sum_l (1 - |z_l|) < \infty$$

which holds automatically when $g \in H^p \setminus \{0\}$. If there are only finitely many zeros, say n counting multiplicities, we say that (5) is a finite Blaschke product of degree n.

That $w(z)$ in (4) is well-defined rests on the fact that $\log |g| \in L^1$ if $f \in H^1 \setminus \{0\}$; this entails that a H^p function cannot vanish on a set of strictly positive Lebesgue measure on $\mathbb{T}$ unless it is identically zero.

Closely related is the Nevanlinna class N^+ consisting of holomorphic functions in $\mathbb{D}$ that can be factored as jE, where j is inner and E an outer function of the form :

$$E(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\rho(e^{i\theta})\,d\theta\right\}, \quad (7)$$

for some $\rho \geq 0$ such that $\log\rho \in L^1(\mathbb{T})$. Such functions again have nontangential boundary values a.e. on $\mathbb{T}$, and $N^+ \cap L^p = H^p$. In fact, (7) defines an H^p-function with modulus ρ a.e. on $\mathbb{T}$ if, and only if, $\rho \in L^p$. A useful consequence is that, whenever $g_1 \in H^{p_1}$ and $g_2 \in H^{p_2}$, we have $g_1 g_2 \in H^{p_3}$ if, and only if, $g_1 g_2 \in L^{p_3}$.

Other Hardy Spaces

- We let $\bar{H}^p$ be the Hardy space of the complement of the disk, consisting of L^p functions whose Fourier coefficients of strictly positive index do vanish; these are, a.e. on $\mathbb{T}$, the complex conjugates of H^p-functions, and they are also nontangential limits of functions analytic in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ having uniformly bounded L^p means over all circles centered at 0 of radius bigger than 1.

- We single out the subspace $\bar{H}^p_0 \subset \bar{H}^p$, consisting of functions vanishing at infinity or, equivalently, having vanishing mean on $\mathbb{T}$. Thus, a function belongs to $\bar{H}^p_0$ if, and only if, it is of the form $e^{-i\theta}\overline{g(e^{i\theta})}$ for some $g \in H^p$.

- The Hardy spaces $\mathcal{H}^p$ of the right half-plane. consists of functions G analytic in $\Pi_+ = \{s;\ \mathrm{Re} s > 0\}$ such that

$$\sup_{x>0} \int_{-\infty}^{+\infty} |G(x+iy)|^p \, dy < \infty,$$

and they have nontangential boundary functions in $L^p(i\mathbb{R})$. Clearly $\mathcal{H}^\infty$ consists of bounded analytic functions in Π_+, and a theorem of Paley-Wiener characterizes $\mathcal{H}^2$ as the space of Fourier transforms of functions in $L^2(\mathbb{R})$ that vanish for negative arguments. The study of $\mathcal{H}^p$ reduces to that of H^p *via* the isometry :

$$g \mapsto (1+s)^{-2/p} g\left(\frac{s-1}{s+1}\right) \tag{8}$$

from H^p onto $\mathcal{H}^p$.

Rational and meromorphic functions

We let $\mathcal{R}_{m,n}$ be the set of rational functions of type (m,n) that can be written p/q where p and q are algebraic polynomials of degree at most m and n respectively. A rational function belongs to some H^p if, and only if, its poles lie outside $\overline{\mathbb{D}}$, in which case it belongs to every H^p.

A rational function belongs to $\bar{H}^p$ if, and only if, it can be written as p/q with deg$p \leq$ deg$q$ where q has roots in $\mathbb{D}$ only. In Systems terminology Such a rational function is called stable and proper, and it belongs to $\bar{H}^p_0$ if, and only if, deg$p <$ deg$q$ in which case it is called strictly proper.

H^p_n is the set of meromorphic functions with at most n poles in $\mathbb{D}$, that may be written g/q where $g \in H^p$ and q is a polynomial of degree at most n with roots in $\mathbb{D}$ only.

For applications to system-theory, it is often necessary to consider functions in H^p or $\mathcal{H}^p$ that have the conjugate-symmetry $g(\bar{z}) = \overline{g(z)}$; in the case of H^p this means they have real Fourier coefficients, or in the case of $\mathcal{H}^2$ that they are Fourier transforms of real functions. For rational functions it means that the coefficients of p and q are real in the irreducible form p/q. In the presence of conjugate symmetry, every symbol will be decorated by a subscript or a superscript "$\mathbb{R}$", like in $H^p_{\mathbb{R}}$ or $\mathcal{R}^{\mathbb{R}}_{m,n}$ etc...

Motivations from System Theory

The connections between function theory and linear dynamical system have been largely encountered already in the talks by J. Partington, M. Deistler and G. Turchetti. They rely mainly on two facts :

- the fact that these systems can be described in the so-called frequency domain as a multiplication operator by the transfer function which belongs to certain Hardy classes if the system has certain stability properties (Fourier transform turns convolutions into products);

- the fact that rational functions are precisely transfer functions of systems having finite-dimensional state-space, namely those that can be designed and handled in practice.

Discrete Systems

A *linear causal discrete control system* is a map $u \to y$ where the input $u = (\ldots, u_{k-1}, u_k, u_{k+1}, \ldots)$ is a real-valued function of the discrete time k, generating an output $y = (\ldots, y_{k-1}, y_k, y_{k+1}, \ldots)$ *via*

$$y_k = \sum_{j=0}^{\infty} f_j u_{k-j},$$

with fixed coefficients $f_j \in \mathbb{R}$.

Function theory enters the picture when signals are encoded by their generating functions:

$$u(z) = \sum_{k \in \mathbb{Z}} u_k z^{-k}, \qquad y(z) = \sum_{k \in \mathbb{Z}} y_k z^{-k}.$$

Indeed, if we define the *transfer function* of the linear control system to be:

$$f(z) = \sum_{k=0}^{\infty} f_k z^{-k},$$

the input-output behaviour can be described as $y(z) = f(z)\, u(z)$.

Finite-Dimensional Systems

A linear control system is said to have *finite dimension n* if its evolution can be described in terms of a state variable $x_k \in \mathbb{R}^n$ as :

$$x_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k + Du_k,$$

where A is a real $n \times n$ matrix, B (resp. C) a column (resp. row) vector with n real entries, and D some real number, n being the smallest possible integer for which such an equation holds [Kalman, Rosenbrock]. A linear time-invariant system has dimension n iff its transfer-function is rational of degree n and analytic at infinity. The transfer-function is then :

$$f(z) = D + C\,(zI_n - A)^{-1}\,B.$$

$\boxed{\text{Stability}}$

Stability, namely the fact that the output dies off if the input does, can be quantified using operator norms. For instance :

(i) the system is bounded $l^2 \to l^2$ iff $f \in \bar{H}^\infty_{\mathbb{R}}$, and the operator norm is $\|f\|_\infty$: the system is called (l^2, l^2)-stable;

(ii) the system is bounded $l^2 \to l^\infty$ iff $f \in \bar{H}^2_{\mathbb{R}}$, and the operator norm is $\|f\|_2$: the system is called (l^2, l^∞)-stable.

For finite-dimensional linear systems, any definition of stability amounts to the requirement that the poles of f should lie in $\{|z| < 1\}$ thus to $f \in \bar{H}^p$ for some, and in fact all p.

$\boxed{\text{Stochastic Identification}}$

Consider a discrete time real-valued stationary stochastic process:

$$y = (\ldots, y(k-1), y(k), y(k+1), \ldots).$$

If it is regular we have the Wold decomposition:

$$y(k) = \sum_{j=0}^{\infty} f_j u(k-j)$$

where u is white noise and $f_j \in \mathbb{R}$. By the Parseval identity :

$$\sum_{j=0}^{\infty} f_j^2 = \mathbf{E}\left\{y(k)^2\right\}$$

which is independent on k by stationarity. If we set

$$f(z) = \sum_{k=0}^{\infty} f_k z^{-k} \in \bar{H}^2,$$

we see that a regular process is obtained by feeding white noise to an $l^2 \to l^\infty$ stable linear system.

ARMA models

When f is rational, y is called an Auto-Regressive Moving Average process, which is popular because it lends itself to efficient computations. When trying to fit such a model, say of order n to y, a standard goal is to minimize the variance of the error between the true output and the prediction of the model :

$$\min_{g \in \mathcal{R}_{n,n}^{\mathbb{R}} \cap \bar{H}^2} \|f - g\|_2. \tag{9}$$

This principle can be used to identify a linear system from observed stochastic inputs, although computing the f_j is difficult because it requires spectral factorization. In practice, one would rather use time averages of the observed sample path of y already in the optimization *criterium*, but this is asymptotic to the previous problem *cf* [Hannan and Deistler]. Note that (9) aims at a rational version of the Szegö theory.

Continuous time systems

These are convolution operators $u(t) \to y(t)$ of the form:

$$y(t) = \int_0^t h(t-\tau)u(\tau)\, d\tau.$$

The function $h : [0, \infty) \to \mathbb{R}$ is called the *impulse response* of the system, as it formally corresponds to the output generated by a Dirac delta. If h and u have exponential growth, so does y and the one-sided Laplace transforms $Y(s)$, $U(s)$ and $H(s)$ are defined on some common half-plane $\{\mathrm{Re} z > \sigma\}$. The system operates in this frequency domain as multiplication by the transfer-function H:

$$Y(s) = H(s)U(s).$$

This time, rational transfer-functions of degree n correspond to linear differential operators of order n forced by the input u.

Stability in continuous time.

The Hardy spaces involved are now those of the right half-plane, and their relation to stability is :

(i) the system is bounded $L^2[0,\infty) \to L^2[0,\infty)$ iff $H \in \mathcal{H}^\infty_\mathbb{R}$;

(ii) the system is bounded $L^2[0,\infty) \to L^\infty[0,\infty)$ iff $H \in \mathcal{H}^2_\mathbb{R}$;

(iii) the system is bounded $L^\infty[0,\infty) \to L^\infty[0,\infty)$ iff $H \in \mathcal{W}_\mathbb{R}$, the Wiener algebra of the right half-plane consisting of Laplace transforms of summable functions $[0,\infty) \to \mathbb{R}$.

Harmonic identification

One of the most effective methods to identify a $L^\infty \to L^\infty$ stable system is to plug in a periodic input $u = e^{i\omega t}$ and to observe the asymptotic steady-state output :

$$y(t) = \lambda e^{i\phi} e^{i\omega t},$$

where λ and ϕ are respectively the modulus and the argument of $H(i\omega)$.

In this way, one can estimate the transfer function on the imaginary axis and he often wants to rationally approximate the experimental data he has got. In practice, the situation is more complicated, because experiments are not available on the whole imaginary axis and the system will not behave linearly at high frequencies anyway.

In fact, if Ω designates the bandwidth of the system on the imaginary axis where experiments are performed, one can usually get a fairly precise estimate of $H_{|_\Omega}$, but all he has on $i\mathbb{R}\setminus\Omega$ are qualitative features of the system. It seems natural to seek

$$\min_{G\in\mathcal{R}^{\mathbb{R}}_{n,n}\cap\mathcal{H}^2}\|H-G\|_{L^2(\Omega)},$$

or

$$\min_{G\in\mathcal{R}^{\mathbb{R}}_{n,n}\cap\mathcal{H}^\infty}\|H-G\|_{L^\infty(\Omega)},$$

but such a problem is often poorly behaved because the optimum may not exist (spurious poles!!) and even if it does it may lead to a wild behaviour off Ω. One way out, which is advocated here, is to extrapolate a complete model in $\mathcal{H}^p$ from the knowledge of $H_{|_\Omega}$ before going for further approximation.

Bounded extremal problems

For $I \subset \mathbb{T}$ and $J = \mathbb{T} \setminus I$, if h_1 is a function on E and h_2 a function on J, we denote by $h_1 \vee h_2$ the concatenated function defined on the whole of $\mathbb{T}$. The $L^2(I)/L^2(J)$ analytic bounded extremal problem is :

$ABEP\Big(L^2(I), L^2(J)\Big)$

Given $f \in L^2(I)$, $\psi \in L^2(J)$ and a strictly positive constant M, find $g_0 \in H^2$ such that

$$\|g_0(e^{i\theta}) - \psi(e^{i\theta})\|_{L^2(J)} \leq M \quad \text{and}$$

$$\|f - g_0\|_{L^2(I)} = \min_{\substack{g \in H^2 \\ \|g-\psi\|_{L^2(J)} \leq M}} \|f - g\|_{L^2(I)} . \tag{10}$$

If I is symmetric with respect to $\mathbb{R}$ and f has the conjugate symmetry, then $g_0 \in H^2_{\mathbb{R}}$.

The solution is best expressed upon introducing the Toeplitz operator :

$$\begin{aligned} \phi_{\chi_J}: \; H^2 &\to H^2 \\ g &\mapsto P_{H^2}(\chi_J g) \end{aligned} \tag{11}$$

with symbol χ_J, characteristic function of J.

Theorem If I has positive measure, there is a unique solution g_0 to (10). Moreover, if f is not the restriction to I of a H^2 function whose $L^2(J)$-distance to ψ is less than or equal to M, this unique solution is given by

$$g_0 = \left(1 + \lambda \phi_{\chi_J}\right)^{-1} P_{H^2}(f \vee (1+\lambda)\psi), \tag{12}$$

where $\lambda \in (-1, +\infty)$ is the unique real number such that the right hand side of (12) has $L^2(J)$-norm equal to M.

Note that (12) indeed makes sense because the spectrum of ϕ_{χ_J} is $[0, 1]$.

The theorem is due to J.Partington, J.Leblond, L.B., but germane work was done by M.G.Krein and P.Ya Nudel'man. It is closely related to Carleman-Goluzin-Krylov interpolation, on which many people have worked *e.g.* Patil, Aizenberg, V.P.Havin, Bart. Its principle was abstracted to smooth Banach spaces and applied to the hyperinvariant subspace problem [I.Chalendar, J.Partington, M.Smith].

The theorem provides a constructive means of solving $ABEP\Big(L^2(I), L^2(J)\Big)$ because, although the correct value for λ is not known *a priori*, the $L^2(J)$-norm of the right-hand side in (12) is decreasing with λ so that iterating by dichotomy allows one to converge to the solution.

Because $H^2_{|_I}$ is dense in $L^2(I)$,, the error in (10) can be made very small, but this is at the cost of making M very big unless $f \in H^2_{|_I}$, a circumstance that essentially never happens due to modelling and measurement errors.

In this connection, it is interesting to ask how fast M goes to $+\infty$ as the error $e = \|f - g_0\|_{L^2(I)}$ goes to 0.

Using the constructive diagonalization of Toeplitz operators with multiplicity 1 [Rosenblum-Rovnyak], one can get asymptotic estimates when I is an interval. To state a typical result, put $I = (e^{-ia}, e^{ia})$ with $0 < a < \pi$, and let $\mathcal{W}^{1,1}(I)$ denote the Sobolev space of absolutely continuous functions on I.

$\boxed{\text{Asymptotic estimates}}$

Theorem Let f satisfy :

$$(1 - e^{-i\theta} e^{ia})^{-1/2} (1 - e^{-i\theta} e^{-ia})^{-1/2} f(e^{i\theta}) \in L^1(I)\,, \tag{13}$$

$$(1 - e^{-i\theta} e^{ia})^{1/2} (1 - e^{-i\theta} e^{-ia})^{1/2} f(e^{i\theta}) \in \mathcal{W}^{1,1}(I)\,. \tag{14}$$

If we set $e = \|f - g_0\|^2_{L^2(I)}$, where g_0 is the solution to (10), then to each $K_1 > 0$ there is $K_2 = K_2(f) > 0$ such that

$$M^2 \le K_2\, e^2 \exp\{K_1 e^{-1}\}\,. \tag{15}$$

In the above statement, the factor e^{-1} in the exponent cannot be replaced by $h(e)$ for some function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that $h(x) = o(1/x)$ as $x \searrow 0$.

It is striking to compare this with the analogous result when f is the trace on I of a meromorphic function :

Theorem If f is of the form h/q_N with $h \in H^2$ and q_N a polynomial of degree N whose poles lie at distance $d > 0$ from $\mathbb{T}$. Then

$$M^2 = O\left(N^2 |\log e|\right) , \qquad (16)$$

and the Landau symbol O holds uniformly with respect to $\|h\|_2$ and d, the estimate being sharp in the considered class of functions.

The last two theorems are due to J.Partington, J.Leblond and L.B. They suggest that approximation is much easier if f extends holomorphically in a 2-D neighborhood of I.

Uniform approximation

The $L^\infty(I)/L^\infty(J)$ problem can also be adressed :

$ABEP\Big(L^\infty(I), L^\infty(J)\Big)$

Given $f \in L^\infty(I)$, $\psi \in L^\infty(J)$, and a strictly positive constant M, find $g_0 \in H^\infty$ such that

$$\|g_0(e^{i\theta}) - \psi(e^{i\theta})\|_{L^\infty(J)} \le M \quad \text{and}$$

$$\|f - g_0\|_{L^\infty(I)} = \min_{\substack{g \in H^\infty \\ \|g-\psi\|_{L^\infty(J)} \le M}} \|f - g\|_{L^\infty(I)}. \tag{17}$$

A seemingly more general version is obtained by letting M be a function in $L^\infty(J)$ and the constraint become $|g - \psi| \le M$ a.e. on J. If $\psi/M \in L^\infty(J)$, this version reduces to the present one.

Indeed, either $\log M \notin L^1(J)$ in which case the inequality $\log|g| \leq \log M + \log(1+|\psi/M|)$ shows that $g = 0$ is the only candidate approximant, or else $\log M \in L^1(J)$ and we can form the outer function $w_M \in H^\infty$ having modulus 1 on I and M on J; then, upon replacing f by f/w_M and ψ/w_M and observing that g belongs to H^∞ and satisfies $|g| \leq M$ a.e. on J if, and only if, g/w_M lies in H^∞ and satisfies $g/w_M \leq 1$ a.e. on J (because g/w_M lies by construction in the Nevanlinna class whose intersection with $L^\infty(\mathbb{T})$ is H^∞), we are back to $M = 1$. If $\psi/M \notin L^\infty(J)$ the situation is more complicated.

Hankel Operators.

This time we introduce Hankel rather than Toeplitz operators [N.Nikolskii, J.Partington, V.Peller]. Given $\varphi \in L^\infty$, the *Hankel operator with symbol φ* is

$$\Gamma_\varphi : H^2 \to \bar{H}^2_0$$

given by

$$\Gamma_\varphi g = P_{\bar{H}^2_0}(\varphi g)$$

where $P_{\bar{H}^2_0}$ denotes the orthogonal projection of $L^2(\mathbb{T})$ onto $\bar{H}^2_0$. A Hankel operator is bounded, and it is compact whenever it admits a continuous symbol; note that the operator only characterizes the symbol up to the addition of some $\mathcal{H}^\infty$-function. Thus, whenever $\varphi \in H^\infty + C(\mathbb{T})$ (the latter is in fact an algebra), the operator Γ_φ is compact and therefore it has a maximizing vector $v_0 \in H^2$, namely a function of unit norm such that $\|\Gamma_\varphi(v_0)\|_2 = |||\Gamma_\varphi|||$, the norm of Γ_φ. Let us mention Kronecker's theorem that Hankel operators of finite rank are those admitting a rational symbol.

Theorem Assume that I has positive measure and that ψ extends continuously to $\overline{J}$. Then, there is a solution g_0 to (17). Moreover, if f is not the restriction to I of a H^∞ function whose $L^\infty(J)$-distance to ψ is less than M, so that the value β of the problem is strictly positive, and if moreover $f \vee \psi \in H^\infty + C(\mathbb{T})$, this solution is unique and given by :

$$g_0 = w_{M/\beta}^{-1} \frac{P_{H^2}\left((f \vee \psi) w_{M/\beta} v_0\right)}{v_0}, \qquad (18)$$

where $w_{M/\beta}$ is the outer function wih modulus M/β on I and modulus 1 on J, and v_0 is a maximazing vector of the Hankel operator $\Gamma_{(f \vee \psi) w_{M/\beta}}$.

Here again, the solution has conjugate symmetry if the data do.

The theorem is due to J. Leblond, J.Partington and L.B., but makes essential use of Nehari's theorem to be reviewed shortly. Although the value β of the problem is not known *a priori*, it is the unique positive real number such that the right hand side of (18) has modulus M a.e. on J, and so the theorem allows for us to constructively solve $ABEP\Big(L^\infty(I), L^\infty(J)\Big)$ if a maximizing vector of $\Gamma_{(f\vee\psi)w_{M/\beta}}$ can be computed for given β. Generically convergent algorithms to this effect have been given in the case where I is an interval and $f \vee \psi$ is C^1-smooth.

Bounded extremal problems can be posed in various contexts. Versions with constraint on real and imaginary part with application to inverse problems of the 2-D Laplacean were developed by A. Ben Abda, J. Leblond,J. Partington, and others. The mixed L^2/L^∞ problem also can be solved, and gives rise to a spectral equation for an unbounded Toeplitz operator, whose solution was developed through a polynomial approximation scheme by F. Seyfert. Abstract versions in Hilbert and Banach spaces are due to Chalendar, Leblond, Partington, Smith.

But we leave such problems for now, and we turn briefly to rational and meromorphic approximation.

AAK Theory

We saw that rational approximation is an important issue in the modelling of linear dynamical systems. Let us review the Adamjan-Arov-Krein theory (in short : AAK) which deals with a related problem, namely *meromorphic approximation* in the uniform norm.

For $k = 0, 1, 2, \ldots$, recall that the *singular values* of Γ_φ are defined by the formula:

$$s_k(\Gamma_\varphi) := \inf \left\{ |||\Gamma_\varphi - A|||; \quad A \text{ an operator of rank } \leq k \text{ on } H^2 \right\}.$$

When $\varphi \in H^\infty + C(\mathbb{T})$, the singular values are, by compactness, the square roots of the eigenvalues of $\Gamma_\varphi^* \Gamma_\varphi$ arranged in non-increasing order; a k-th *singular vector* is an eigenvector of unit norm associated to $s_k(\Gamma_\varphi)$.

A celebrated connexion between the spectral theory of Hankel operators and best meromorphic approximation on the unit circle is given by AAK theory as follows.

$$\inf_{g \in H_n^\infty} \|\varphi - g\|_\infty = s_n(\Gamma_\varphi) \qquad (19)$$

where the *infimum* is attained; moreover, the *unique* minimizer is given by the formula

$$g_n = \varphi - \frac{\Gamma_\varphi v_n}{v_n} = \frac{P_{H^2}(\varphi v_n)}{v_n}, \qquad (20)$$

where v_n is *any* n-th singular vector of Γ_φ. Formula (20) entails in particular that the inner factor of v_n is a Blaschke product of degree at most n. The error function $\varphi - g_n$ has further remarkable properties; for instance it has constant modulus $s_n(\Gamma_\varphi)$ a.e. on $\mathbb{T}$.

Glover's bound

From the point of view of constructive approximation, it is remarkable that the infimum in (19) can be computed, and the problem as to whether one can pass from the optimal meromorphic approximant in to a nearly optimal *rational* approximant has attracted much attention. Most notably, it was shown by [Glover] that $P_{\bar{H}_0^2}(g_n)$, which is rational in $\mathcal{R}_{n-1,n}$, produces an L^∞ error within

$$2 \sum_{j=n+1}^{\infty} s_j(\Gamma_\varphi) \tag{21}$$

of the optimal one out of $\mathcal{R}_{n-1,n}$. To estimate how good this bound requires a link between the decay of the singular values of Γ_φ and the smoothness of φ.

Decay and smoothness.

The summability of the singular values is equivalent to the belonging of $\overline{P_{\bar{H}_0^2}(\varphi)}$ to the Besov class B_1^1 of the disk as shown by V.Peller, but this does not tell how fast the series converges.

When φ is analytic outside some compact $K \subset \mathbb{D}$, it is an estimate of Walsh that, if e_n is the optimal error in uniform approximation to φ from $\mathcal{R}_{n,n}$ on $\overline{\mathbb{C}} \setminus \mathbb{D}$, then

$$\limsup e_n^{1/n} \leq e^{-1/(C)} \tag{22}$$

where C is the capacity of the condenser $(K, \mathbb{T})$.

$\boxed{\text{Green Capacity.}}$

To make for a definition of capacity here, let us merely mention if K is non polar that there is a unique μ among all probability measures on K that minimizes the Green energy:

$$I(\mu) = \int \int \log \left| \frac{1 - \bar{z}t}{t - z} \right| d\mu(z) d\mu(t).$$

This μ is, by definition, the Green equilibrium distribution on the plate K of the condenser $(K, \mathbb{T})$, whose capacity is $1/I(\mu)$ [Saff and Totik]
.

Geometric rates

Walsh's estimate shows that the decay of the singular values is geometric when φ is analytic outside $\mathbb{D}$ and across $\mathbb{T}$, and allows for an appraisal of the tail in Glover's estimates, although this appraisal is pessimistic in that, as was conjectured by Gonchar and proved by Parfenov and Prokhorov :

$$\liminf e_m^{1/m} \leq e^{-1/(2C)}.$$

However, no algorithmic process is known to construct an optimal subsequence. But for functions defined by Cauchy integrals over so-called symmetric arcs, the lim inf is a true limit as shown by Stahl, Gonchar and Rachmanov. For Markov functions (Cauchy transforms of positive measures), sharp estimates are available [Prokhorov, Saff, L.B.]

Rational approximation.

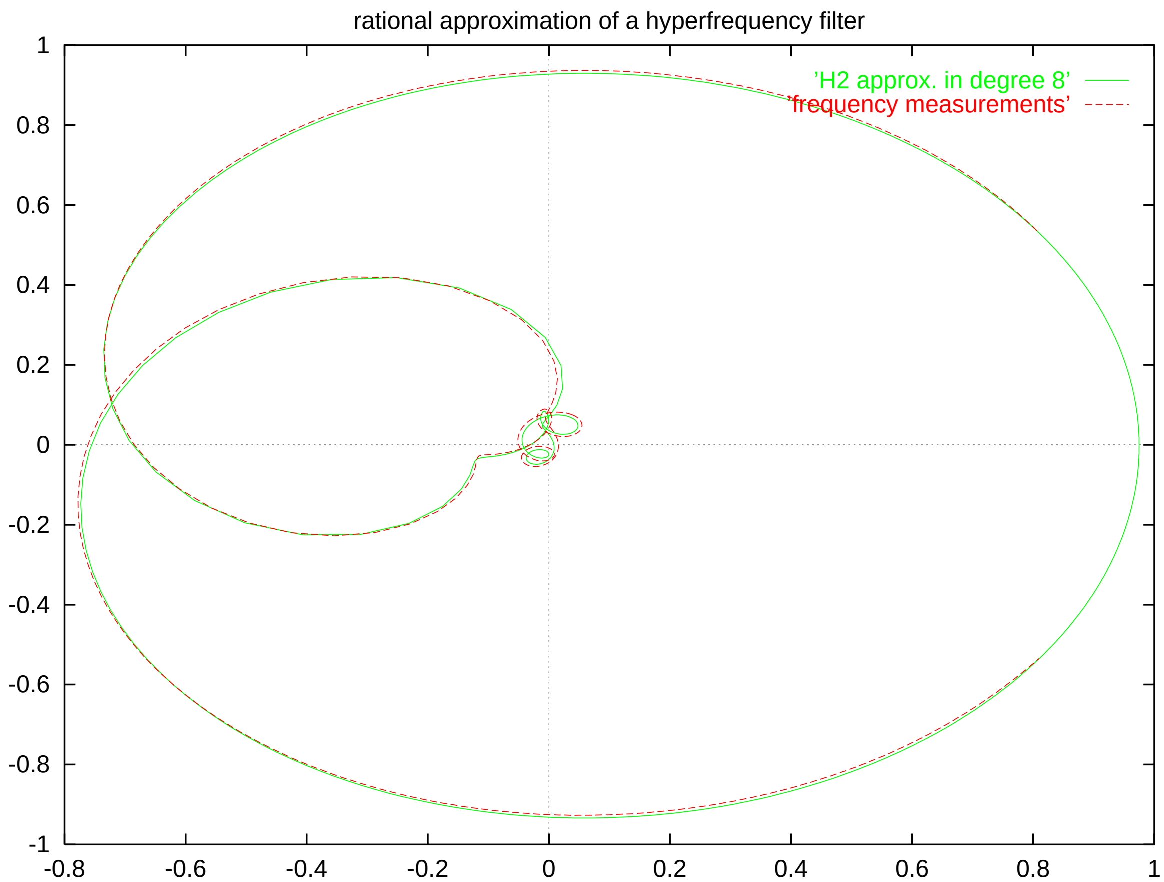
Let us conclude with a few words concerning $\bar{H}^2$ rational approximation of type (n,n). For a comparison with AAK theory, let us point out that

$$\inf_{g\in\mathcal{R}_{n-1,n}\cap\bar{H}_0^2}\|\varphi-g\|_2=s_n(\Gamma'_\varphi) \tag{23}$$

where Γ'_φ is again the Hankel operator but this time $H^\infty \to \bar{H}_0^2$, and the error formula is still valid while the maximizing vector is necessarily a Blaschke product of degree n. The nonlinear character of the set of Blaschke products of given degree makes for a much more difficult problem, and practical algorithms have to rely on numerical searches with the usual burden of local minima.

In the special case of Markov functions, also correspond to the transfer functions of so-called relaxation systems a lot is again known including sharp error rates [Prokhorov, Saff, Stahl, Wielonsky, L.B.], asymptotic uniqueness of a critical point for Szegö-smooth measures [Stahl, Wielonsky] and uniqueness for all orders and small support [Wielonsky, L.B.]. For certain entire functions like the exponential, sharp error rates and asymptotic uniqueness of a critical point have also been derived [Saff, Wielonsky, L.B.], but for most classes of functions the situation is still open. Finally we point out that, despite the lack of a general theory, rather efficient algorithms are available to generate local minima [Caedelli, Fulcheri, Grimm, Marmorat, Olivi, L.B.].

[htbp]



The dotted line in this diagram is the Nyquist plot (*i.e.* the image of the bandwidth on the imaginary axis) of the transfer function of the reflexion of a hyperfrequency filter measured by the French CNES (Toulouse). The data were first completed by solving an $\bar{H}^2$

bounded extremal problem and then approximated by a rational function of degree 8 whose Nyquist plot has been superimposed on the figure. The locus is not conjugate-symmetric because a low-pass transformation sending the central frequency to the origin was performed on the data. This illustrates that approximation with complex Fourier coefficients can be useful in system identification, even though the physical system is real.