Detection of resonances with Szegö polynomials - Links with the theory of Padé

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AR Systems

AR-Model with characteristic polynomial $L_A(z)$

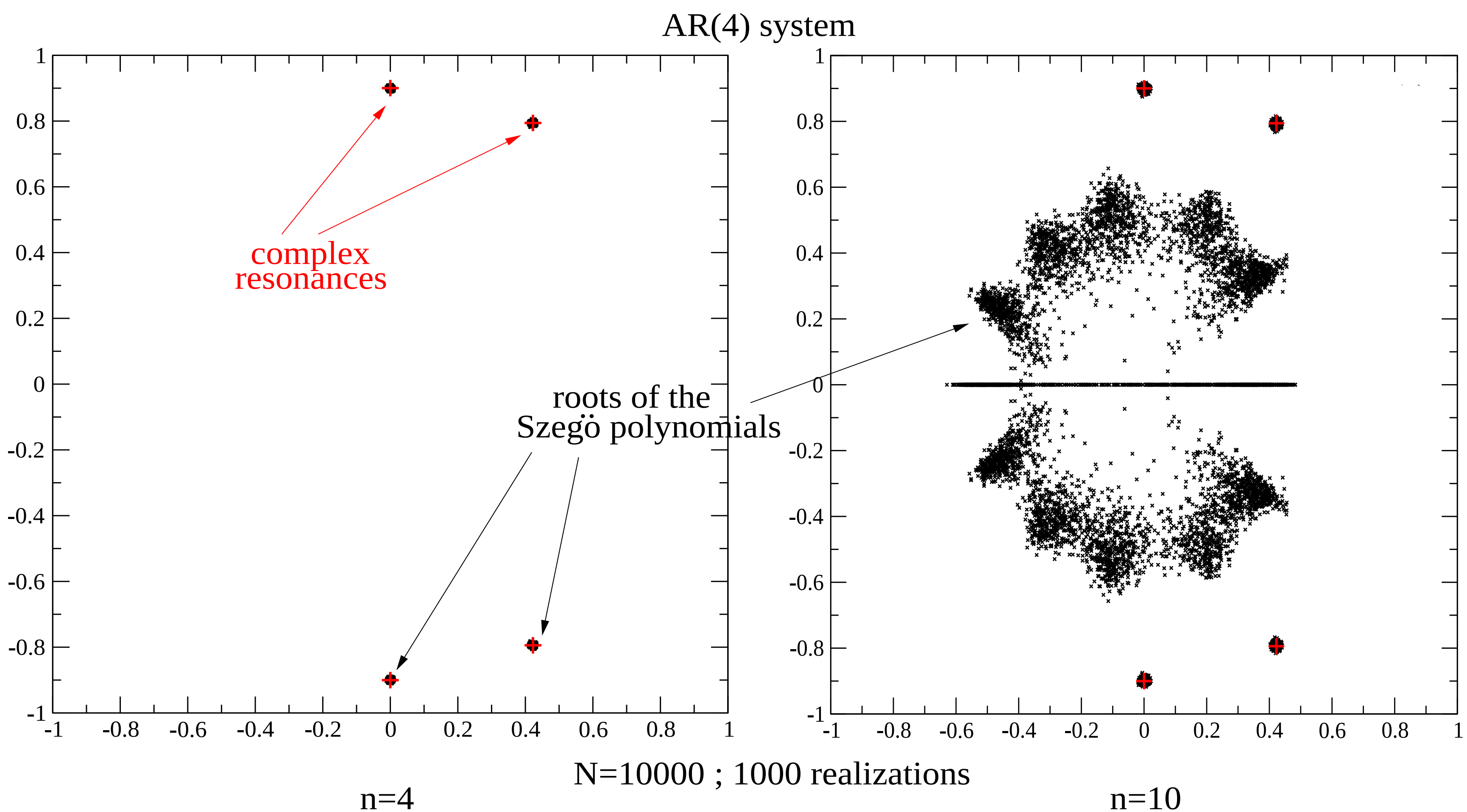
$\hookrightarrow$ Power spectrum

$$\mathscr{E}(z) \;=\; \frac{1}{L^{\dagger}_{A}(z)L^{\dagger}_{A}(1/z)} + \mathcal{O}^{f}(\frac{1}{\sqrt{N}}), \qquad N \gg 1$$

$\Rightarrow$ Associated Szegö polynomial

$$S_n(z) \;\stackrel{N\gg 1}{\simeq}\; z^{n-A}\; L^{\dagger}_{A}(z) \quad \forall n \geq A$$

AR -II



AR(4) system
complex
resonances
roots of the
Szegö polynomials
N=10000 ; 1000 realizations
n=4
n=10

ARMA Systems

ARMA model of order (A, B) :

$$X(m) - a_1 X(m-1) - \cdots a_A X(m-A) = \Phi(m) \quad -b_1 \Phi(m-1)$$
$$+\text{CàL} \qquad \cdots \quad -b_B \Phi(m-B)$$

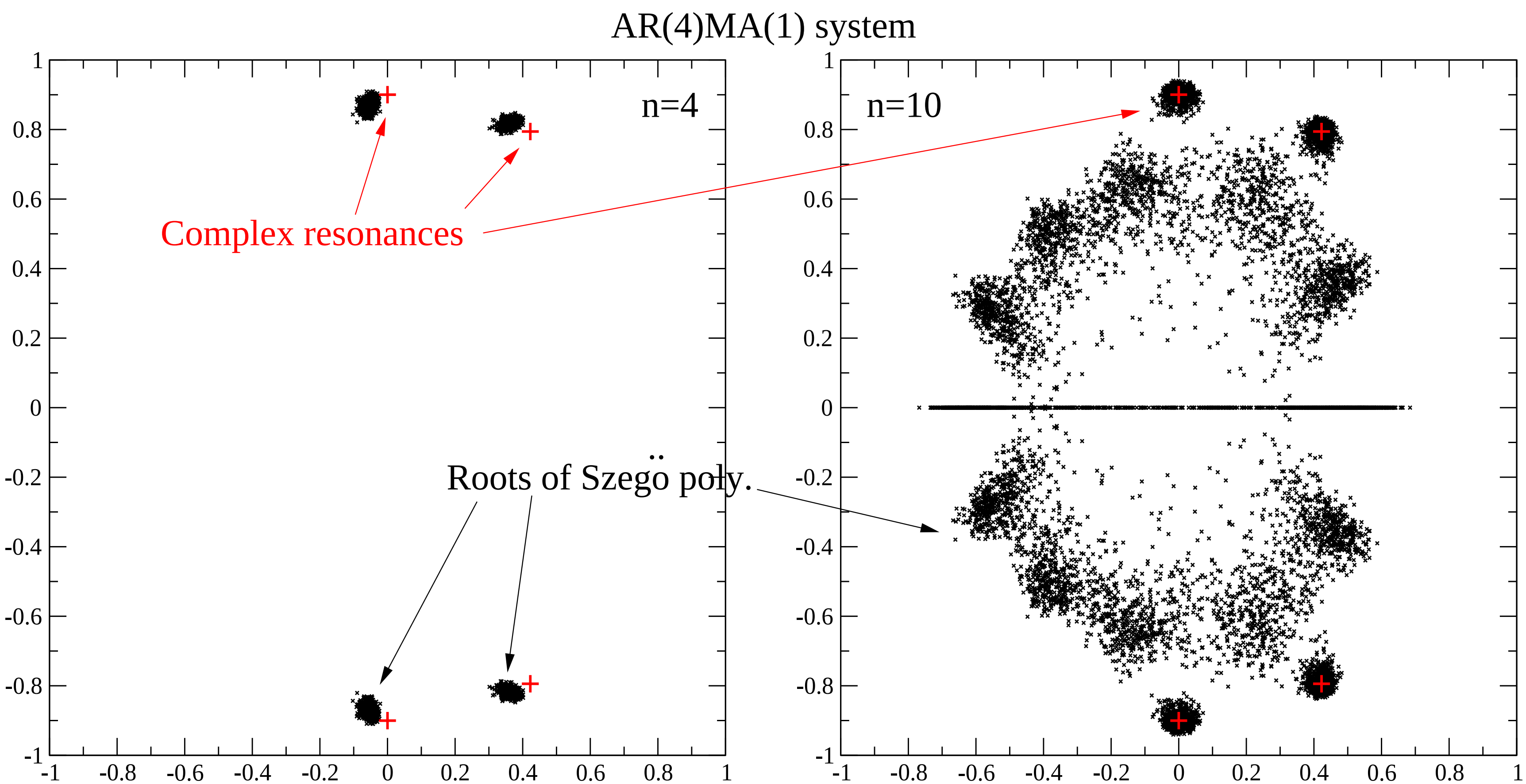
characteristic polynomial of the Moving Average part

Non trivial numerator

$$\Rightarrow \mathscr{E}(z) \overset{N \gg 1}{\simeq} z^{A-B} \frac{L_B(z) L_B^\dagger(z)}{L_A(z) L_A^\dagger(z)}$$

$\rightarrow$ no such general formula for the Szegö polynomials.

ARMA -II

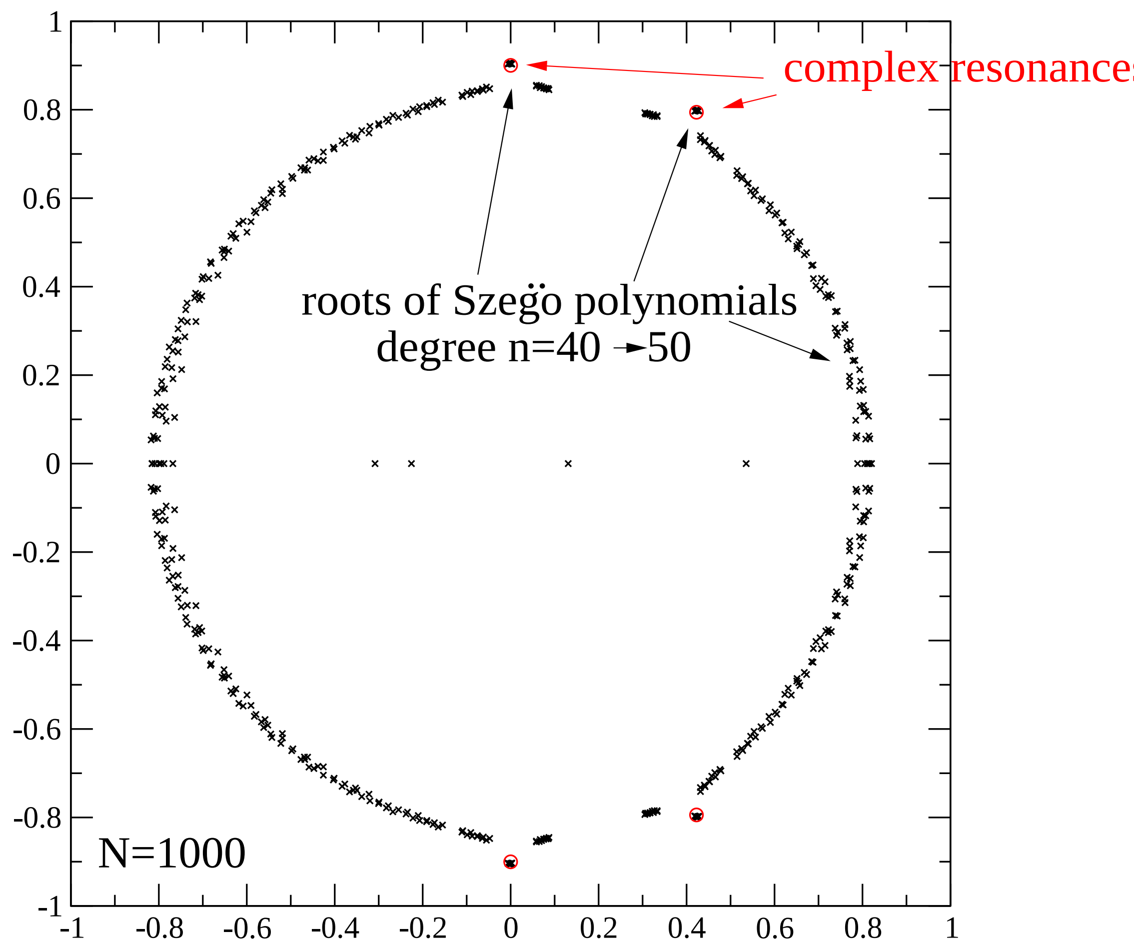


N=1000, 1000 real.

Deterministic signal

sum of M attenuated sinusoids
$\Rightarrow 2M$ complex resonances
in the unit disc.

Same phenomenon than for ARMA with
$B \leftarrow M - 1$



Szegö Polynomial & Padé Approximant

$\rightsquigarrow$ Links with the theory of Rational Approximation ?

Theorem : *(Jones, Njåstad, Saff, 1989.)*

$S_n^\dagger(z)$ is the numerator of the P.A $[n/N-1]$

$$S_n^\dagger(z) = P_n^{[n/N-1]}(z)_{|F_N(z)}$$

for the $2(N-1)$ poles rational fraction

$$F_N(z) \equiv \frac{1}{z^{N-1}\mathscr{E}(z)}$$

determinantal formulæ

$$\begin{vmatrix} C_0 & C_1 & \cdots & C_n \\ C_{-1} & C_0 & \cdots & C_{n-1} \\ \vdots & & & \vdots \\ C_{-n+1} & & \cdots & C_1 \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix} = S_n^\dagger(z) = z^n S_n(\frac{1}{z})$$

C_k = autocorrelation of the signal

$$Q_n^{[m/n]}(z)_{|f} = \begin{vmatrix} f_m & f_{m+1} & \cdots & f_{m+n} \\ f_{m-1} & & \cdots & f_{m+n-1} \\ \vdots & & & \vdots \\ f_{m-n+1} & & \cdots & f_{m+1} \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix}$$

Frobœnius formula
for the denominator of the Padé

f_k = coeff. of the Taylor expansion of f

appropriate function

$$D_d(z) \equiv C_{-d} + C_{-d+1} z + \cdots + C_0\ z^d + \sum_{k=1}^{N-1} C_k\ z^{d+k}$$

$$\rightsquigarrow \forall d \in [n..\ N-1] \qquad S_n^{\dagger}(z) = Q_n^{[d/n]}(z)_{|D_d}$$

N.B : $d = N-1 \quad \Rightarrow \quad D_{N-1}(z) = z^{N-1}\mathscr{E}(z)$

- the P.A $[m/n]$ of a rational fraction (p/q) is equal to the fraction itself as soon as $m \geq p$ and $n \geq q$. -

AR & ARMA cases

What happens in the case of the AR system ?
D_d = first terms of the Taylor expansion of a rational fraction

$$\frac{\text{poly. of } \partial^0 \ d}{L_A(z)}$$

$$\rightsquigarrow Q_n^{[d/n]}(z)_{|D_d} = S_n^\dagger(z) = L_A(z) \quad \Rightarrow S_n(z) = z^{n-A} L_A^\dagger(z)$$

ARMA or deterministic case : $D_d \rightsquigarrow$ fraction

$$\frac{\text{poly. of } \partial^0 \ d + B}{L_A(z)}$$

convergence theorems $\Rightarrow$ asymptotic detection of the resonances