GALAAD
Geometry, Algebra and Algorithms

B. Mourrain
GALAAD, INRIA Méditerranée, Sophia Antipolis
GALAAD: joint team between INRIA and UNSA

Permanent staff
▶ L. Busé (CR1 INRIA)
▶ A. Galligo (Prof. UNSA)
▶ E. Hubert (CR1 INRIA)
▶ B. Mourrain (DR2 INRIA)

Current Ph. D. Student
▶ Marta Abril Bucero (ANR, 2011-2014)
▶ Matthieu Collowald (MESR, 2011-2014)
▶ Abdallah Lachaal (EU Terrific, 2011-2014)

Current post-doc.:
▶ A. Bernardi (IEF Marie Curie, 2010-2012)
▶ N. Botbol (ITN Marie Curie SAGA, 2011-2012)

Temp. Engineer:
▶ M. Perrinel (INRIA, ADT IJD)

Collaborator:
▶ I.Z. Emiris (Prof. Univ. of Athens).

Assitant:
▶ S. Honnorat (INRIA, IA)

Among former Ph.D. students:
▶ A. Mantzaflaris (Ph.D. SAGA 2008-2011)
▶ T. Lu Baa (Ph.D. 2008-2011)
Post. Doct. in NKUA.
Our approach

▶ centered on algebraic representations of the geometry;
▶ in order to provide rich, compact, high quality models;
▶ and efficient tools for exploiting these models;

Topics of investigation:

1. Algebraic algorithms for geometric computing
2. Symbolic-numeric analysis
3. Algebraic representations for geometric modeling
I Algebraic algorithms for Geometric Computing

- points, curves, surfaces, volumes, ...
- represented by parametric, implicit models
- require specific algebraic methods.

Problems:

- Change from parametric to implicit representations,
- **Intersection** points or curves, **autointersection**, **singular** points,
- **Closest point** or distance computation, Collision.
- **Pipes**, **canal surfaces**, **offset**, **fillet**, **blending**, ...
- **Medial axis** or skeleton structure.
- ...

Study

- systems polynomial equations, algorithms for their solution in geometric applications.
Resultant and syzygy methods:

- Adapted resultant matrices for the self-intersection problem (generic case);
- Matrix representations of rational hypersurfaces and of rational curves;
- Intersection problems with rational curves; reduction of non-square to square one-dimensional pencils of matrices;
- Implicit equation of rational ruled surfaces and of canal surfaces;
- Singularity of rational curves as invariant factors of a one-dimensional pencil of matrices;
- Link between adjoint curves to a rational plane curve and some equations of a Rees algebra;
Algebraic solvers using Border basis:

- Effective rep. of $A = \mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_s)$;
- Generalisation of Gröbner basis; better for approximate computation;
- Further analysis of algebraic properties;
- Intrisinc characterisation; Hilbert Scheme equations;
- Extension to real algebraic geometry and optimisation problems.

Subdivision methods:

- Univariate subdivision solvers; Bernstein or monomial basis; Continued Fraction approximation; one of the most efficient implementations;
- Multivariate Bernstein basis; preconditionning and reduction; sleeves; practical efficiency;
- Multivariate continued fraction;
- Complexity analysis; new (quasi-optimal) bounds for separation; condition number of systems.
II Symbolic-numeric analysis

- Input data may be known with uncertainty;
- Computation may be performed approximately;
- Non-linearity leads to unstability;

Problems:

- Analyze family of input data of the same “shape”;
- Certification of the result even if approximate computation is used;
- Develop robust (stable) algebraic algorithms for geometric analysis;
- Analyze the conditionement of the problems, of the methods;
Topology and arrangements of curves:

- New approach for topology;
- Isolate first the extremal and singular points;
- Subdivision based on regularity test from cell boundary;
- Use topological degree to analyse the real branches at a singular point;

Topology of algebraic surfaces:

- Exploit properties of subresultants;
- Compute an explicit Whitney stratification;
- Improved complexity bounds compared to Cylindrical Algebraic Decomposition;
- Analysis of iterated resultants and discriminants;

Algebraic decomposition and absolute factorisation:

- Use monodromy or Wood theorem to split fiber points;
- Construct a smallest algebraic extension field to represent the coefficients;
- Practical efficiency: absolute factorisation of pol. of degree 400.
III Algebraic representations for geometric modeling

- Implicit or parametric representation;
- Piecewise algebraic descriptions (splines) associated to subdivisions;

Problems:

- **Interpolation** of points or curves by surfaces;
- Curves or surfaces **fitting**;
- Surfaces **filling** holes with boundary constraints;
- Approximation of data/observations by **compact algebraic models**;
- Representation/analysis of **functions** on a given geometry;
Tensor decomposition:

- Decompose a tensor as a minimal sum of tensors of rank 1 (product or power of linear forms);
- Geometry of secants of Veronese and Segre varieties;
- New algebraic method extending Sylvester approach;
- New links with truncated moment problems;

Tubular and convolution surfaces:

- Envelop of spheres; implicit equations; curves in the space of spheres;
- Skeleton based Geometric Modeling for Computer Graphics;
- General formulae for convolution surfaces based on sets of line segments and circular arcs;
- Efficient formulae for convolution surfaces based on planar polygons.
**Isogeometry:**

- Use the same function basis for the geometry and the simulation; exact description of the geometry;
- High order numerical scheme;
- Optimisation of the parameterisation for a given problem;
- Local refinement of function spaces;
- Interaction with shape optimisation;
Software developments

**Mathemagix**

- Free computer algebra system with an interpreter, a compiler, a front-end mathematical editor **TeXmacs**, a geometric modeler **Axel**.
- Algebraic and numeric computation; most efficient known algorithms.
- Efficient dedicated autonomous C++ packages as plugins in the interpreter; Connections with **GMP**, **MPFR**, **LAPACK**, **CDD**, ...
- Collaborative project with J. van der Hoeven and G. Lecerf (CNRS, Palaiseau); Ph. Trébuchet (LIP6); ...
- GPL Licence; 400 000 lines of code; Automatic tools for configuration, documentation; nightly tests;

**Axel**

- Algebraic Geometric modeler: polynomial parametrisation, B-Spline, implicit curves and surfaces, ...; intersection, self-intersection, implicitization, topology and arrangement of implicit (singular) curves or surfaces, ...
- Isogeometric analysis tools; link with gotools (SINTEF), isogeometric toolbox (EXCITING).
- Plugins for optimization and PDE solvers (R. Duvigneau, OPALE), algebraic knot invariants (M. Hodorog, RICAM, Linz, Austria), ray GPU tracing (J. Seland, SINTEF, Oslo, Norway).
- GPL Licence; 200 000 lines of code;
National collaborations

- ANR: GEOLMI (Geometry and Algebra of Linear Matrix Inequalities with Systems Control Applications).

International collaborations

- European projects: SAGA (ShApe Geometry and Algebra) ITN Marie-Curie; Exciting (Isogeometry); TERRIFIC, STREP, “Factory of the Future”; DECONSTRUCT, IEF Marie-Curie.
- Recent bilateral collab.: Barcelona, Buenos Aires, Hong Kong, Seoul, ...