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Continuous error analysis for P1-exact schemes

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SUMMARY

We propose a mesh adaptation strategy relying on a continuous error model and its minimisation in a L2 norm. The error model is of a priori type and supposes the solution of a linearised system. The error minimisation takes the form of an optimal control problem with an adjoint state. This is applied to a Dirichlet problem discretised with the usual linear continuous finite-element approximation. A numerical example is given.

KEY WORDS: Finite element; Mesh adaptation; A priori error; Adjoint state.

1 Introduction

Mesh adaption methods are progressing rapidly, but some limitations exist and can be summarized as follows: many heuristics based on *a priori* error estimates lead to specify an ideal mesh, typically from a "metric" field (see [3],[9], [10]), but without sufficient control of the criterion for which this mesh is the best one. Complementarily, with *a posteriori* errors based methods ([1]) and adjoint formulations (see [2],[8],[11]), we can improve the current mesh according to a precise criterion, typically the error in the evaluation of a specified functional depending on the approximate solution. The motivation of our investigation is to find the best mesh for a minimal error on a prescribed output. In contrast to usual *a priori* error+metric strategies which address a local error, we choose which output should be computed with lowest error, and with which norm this error should be the lowest. This is expressed in the cost functional which is the chosen norm of the specified output. In contrast to *a posteriori* error+adjoint methods who evaluate *a posteriori* errors from an existing mesh to be improved, we want to find an ideal "optimal" mesh on the basis of an *a priori*-based evaluation of it. To this end, the global problem is formulated as a complete optimal control problem in which the cost functional is minimized with a gradient iteration.

2 Error analysis

The main strategy adopted consists in defining the different ingredients in a continuous context. The mesh is replaced by a continuous metric, the error analysis is replaced by an expansion of the error with respect to mesh size, in which the first term is a continuous function. This can be applied to interpolation errors. The popular Hessian model is based of the extension:

$$\Pi_h u \;-\; u \;\;=\;\; H(u).\Delta x.\Delta x \;+\; R(h)$$

where $H(u)$ is the Hessian matrix of function u. According to a local error estimate of P1-interpolation, the rest $R(h)$ will be neglected. This model is at the origin of many successful anisotropic mesh adaptations. An interesting property in this estimate is that the Hessian term is a purely continuous (i.e. non-discretized) one as soon as we define Δx as a continuous field ([5]). A continuous optimization problem can be built and analytically solved, yielding the optimal metric, defining optimal meshes, see [1].

In this paper we consider *PDE systems*, a prototype of them is the Dirichlet problem in a domain Ω,

$$(\nabla u, \nabla \phi) \quad = \quad (f, \phi)\ , \quad u \ \in \ H^1_0(\Omega), \quad \forall\ \phi \ \in \ H^1_0(\Omega)$$

and we restrict ourselves to *isotropic metrics*. These metrics prescribe meshes without stretching. They are described by a scalar field, the mesh density. The link with mesh convergence is easily kept by considering the case of an ideal series of meshes with homothetic density: Let m a positive scalar field defined over the computational domain Ω. We consider a set $\mathcal{H}$ of real positive numbers h having zero as accumulation point, such that we can define for any h in $\mathcal{H}$ a mesh density

$$m_h(x,y) \quad = \quad h^2\ m(x,y) \quad \forall (x,y) \in \Omega,$$

and a triangulation $\mathcal{T}_h$ having m_h as density.
We denote by V_h the usual P_1-continuous approximation space for $H^1_0(\Omega)$ on this triangulation. The discretized system writes:

$$(\nabla u_h, \nabla \phi_h) \quad = \quad (f, \phi_h)\ , \quad u_h \ \in \ V_h, \quad \forall\ \phi_h \ \in \ V_h\ .$$

The error expansion can be simplified by expressing the approximation error $u - u_h$ in terms of the interpolation error expansion $\Pi_h u - u$ and a second term,

$$u_h \ - \ u \quad = \quad u_h - \Pi_h u \ + \ \Pi_h u \ - \ u\ ,$$

where $u_h - \Pi_h u$ is called in the sequel the "implicit error". It is the solution of a discrete system which can be written:

Find $u_h - \Pi_h u \ \in \ V_h$, such that $(\nabla(u_h - \Pi_h u), \nabla \phi_h) \ = \ (\nabla(u - \Pi_h u), \nabla \phi_h) \ \ \forall\ \phi_h \ \in \ V_h$.

Proposition: *We assume that the continuous solution u is in $\mathcal{C}^3(\bar{\Omega})$ and that the continuous mesh size m is in $\mathcal{C}^2(\bar{\Omega})$. Then, for any function ϕ of $\mathcal{C}_3(\bar{\Omega})$ with compact support, we have*

$$\begin{aligned} \int_\Omega \frac{\partial (u - \Pi_h u)}{\partial x} \frac{\partial \Pi_h \phi}{\partial x} dM \ + \ \int_\Omega \frac{\partial (u - \Pi_h u)}{\partial y} \frac{\partial \Pi_h \phi}{\partial y} dM & \\ = \ h^2 \int_\Omega g'(m) \phi dM \ + \ O_\phi(h^3) & \qquad (1) \end{aligned}$$

where $h^{-3} O_\phi(h^3)$ is bounded for a fixed ϕ, and where $g'(m) \ = \ g'_1(m) + g'_2(m)$:

$$\begin{aligned} \int_\Omega g'_1(m) \phi dM \quad = \quad & -\frac{3}{48} \int_\Omega \phi \frac{\partial}{\partial y} \left(m^2 \frac{\partial^3 u}{\partial x \partial y^2} \right) dM \\ & + \frac{1}{48} \int_\Omega \phi \frac{\partial}{\partial x} \left(m^2 \frac{\partial^3 u}{\partial x^3} \right) dM \end{aligned}$$

$$\begin{aligned} \int_\Omega g'_2(m) \phi dM \quad = \quad & -\frac{1}{4} \int_\Omega \phi \frac{\partial}{\partial y} \left(\frac{m^2}{6} \frac{\partial^3 u}{\partial x^2 \partial y} \right) + \phi \frac{\partial^2}{\partial y^2} \left(m^2 \frac{\partial^2 u}{\partial x^2} \right) dM \\ & + \frac{3}{24} \int_\Omega \phi \frac{\partial}{\partial y} \left(m^2 \frac{\partial^3 u}{\partial y^3} \right) + \phi \frac{\partial^2}{\partial y^2} \left(m^2 \frac{\partial^2 u}{\partial y^2} \right) dM\ . \end{aligned}$$

Let us define the unique solution of:

$$E(m) \in H^1_0(\Omega), \quad \text{and} \quad (\nabla E(m), \nabla \phi) = \int_\Omega g'(m) \phi dM \quad \forall \phi \in H^1_0(\Omega)\ . \qquad (2)$$

then

$$\int_\Omega \nabla(u_h - \Pi_h u)\nabla\Pi_h\phi dM \quad = \quad + \ h^2 \int_\Omega \nabla E_m \nabla\phi dM \ + \ O_\phi(h^3) \ .\square \tag{3}$$

Detailed proofs of this proposition are given in [7].

With this analysis, it will be possible to investigate which mesh gives the smallest first term of the error.

3 Error model and Optimal Control problem

According to the above analysis, for any smooth enough metric $m(x, y)$, we can define a continuous error model $E(m)$ as follows:

$$E \ = \ E(m) \Leftrightarrow E \ \in \ H^1_0(\Omega) \ , \text{ and } \forall \ \ \phi \ \in \ H^1_0(\Omega) \ , \int_\Omega \nabla E.\nabla\phi \ dM \ = \ \int_\Omega g'(m)\phi \ dM \ . \tag{4}$$

This Dirichlet problem can be interpreted as follows:

$$-\Delta E(m) \quad = \quad g'(m)$$
$$E(m) \quad = \quad 0 \ \text{ on } \ \partial\Omega \ .$$

Now we are able to look for the best mesh through an for minimizing a given error through the formulation of an Optimal Control problem. The control variable is the metric, the state dependant variable is the solution $E(m)$ of the above system and represents the approximation error. Let J be a functional depending on state $E(m)$ and possibly on control m. It represents the precise norm of output error that we want to reduce. Typically, when state error $E(m)$ is zero, so is this functional.

$$J: \ (m, E) \quad \rightarrow \quad J(m, E) \ \ ; \ \ J(m, E) = 0 \ \ if \ \ E \ = \ 0$$

Let us define:

$$j(m) \quad = \quad J\left(m, E(m)\right),$$

the optimisation problem to solve writes:

$$\text{Find } \ m_{opt} \quad = \quad \text{ArgMin } j(m) \ . \tag{5}$$

If it is allowed to take a sequence of metrics m tending towards zero in the convenient norm, then the minimum of j will be also zero. This expresses the fact that with a progressively very fine mesh, the error can be very close to zero. But a very fine mesh is very expensive. Our standpoint is to find among meshes with a given number of nodes the one which will give the smaller error. To attain this end, and since the control variable m si equal to the mesh node density, the above minimum is taken over a set of metrics m with a fixed integral.

Resolving this minimisation with respect to a distributed control is generally not so easy since the number of unknowns is high. Sequential Quadratic Programming algorithms can be applied to solve Optimal Control problems with PDE's. In [6],[4] we try to identify some efficient versions.

4 An illustration: Dirichlet problem in a square

The PDE under investigation is set in a two-dimensional unit square Ω. We consider the particular function u:

$$u(x, y) \quad = \quad (x^2 - x)(y^2 - y) \ .$$

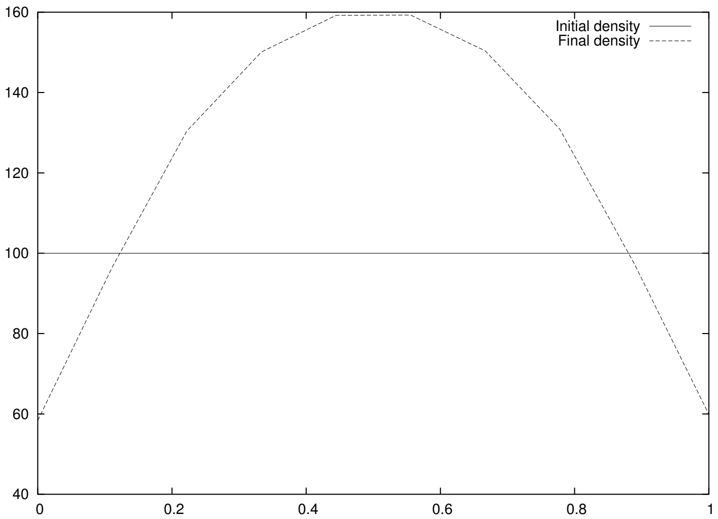

Figure 1: Minimization of L^2 error over the whole domain: central cut of initial (line) and final (dashes) mesh-densities

We consider the minimisation of mesh density in order to have for a given number of nodes an L^2 error as small as possible.

$$\text{Find } \ m_{opt} \ = \ \text{ArgMin} \int_\Omega |E(m)|^2 \ d\Omega \ .$$

The local error terms g' are simplified to the maximum of the absolute values of fourth derivatives ($\eta, \zeta, \kappa, \lambda = x \ or \ y$):

$$\bar{g}'(m)(x,y) \ = \ \max(|u^{(4)}_{\eta\zeta\kappa\lambda}(x,y)| \ .$$

We have to emphasize that, for a Dirichlet problem, choosing an error norm involving boundary values as the previous one is a rather difficult task. Indeed the error is zero at boundary independantly of the mesh density in this region. We refer to [7], [6] for an analysis of this problem. We adopt the solution proposed in these references. We introduce penalty terms in order to master the tendency of the algorithm to give a zero density at boundaries.

A few experiments have been done with a cartesian mesh of 11×11 vertices. The gradient of the discrete functional is obtained by an adjoint method and developed with the help of the TAPENADE Automatic Differentiator([12]). A nonlinear conjugate gradient method allows to obtain a (possibly local) minimum in a hundred iterations. In these preliminary experiments, we are only interested in finding the optimal mesh density. We did not regenerate a new mesh from the optimal density, as should be done when an adapted mesh is really needed.

We start from a uniform node density. It appears that this initial condition is already somewhat quasi optimal. However, the optimisation made the functional further decrease of about 50%. The effect of this mesh optimisation is seen through several vertical cuts at $y = 0.5$:
- the mesh density increases in the center, decreases at boundary, see Fig.1.
- the truncation error displayed in Fig.2, is reduced of 40% at the center of computational domain, and takes larger values at boundary,
- the approximation error model is reduced in any place of the computational domain, see Fig.3.

A particular interest of the implicit or adjoint based error formulation with respect to local truncation error is that we can minimise the approximation error measured on a specified subset of the computational domain. Let us for example restrict the cost function to an integral over the

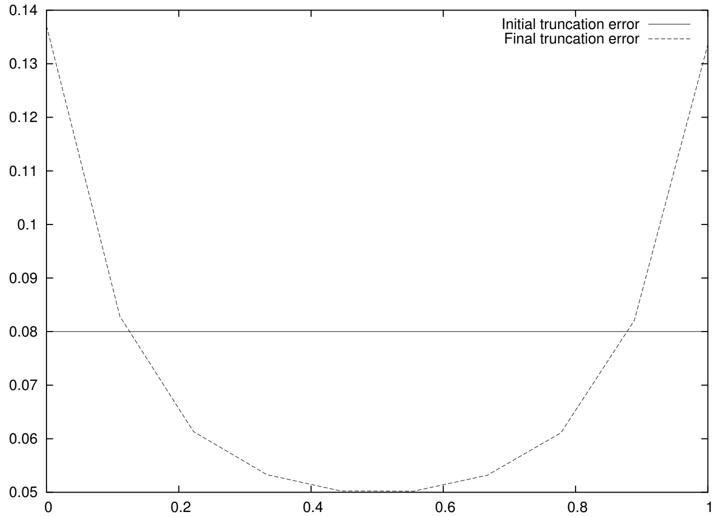

Figure 2: Minimization of L^2 error over the whole domain: central cut of initial (line) and final (dashes) truncation errors

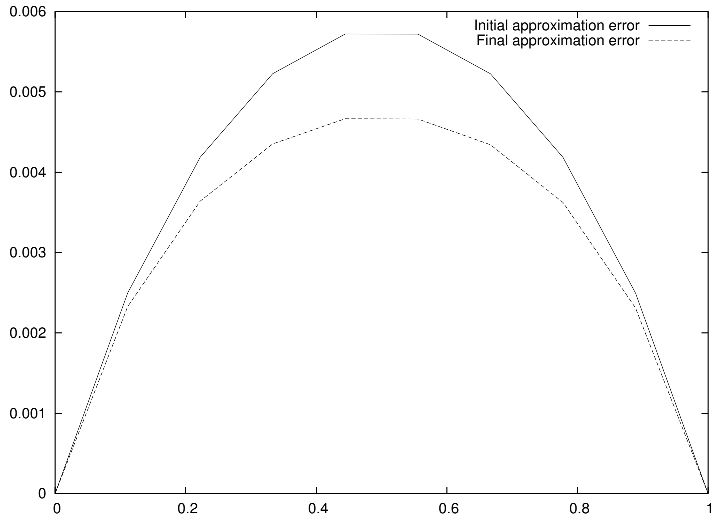

Figure 3: Minimization of L^2 error over the whole domain: central cut of initial (line) and final (dashes) approximation errors

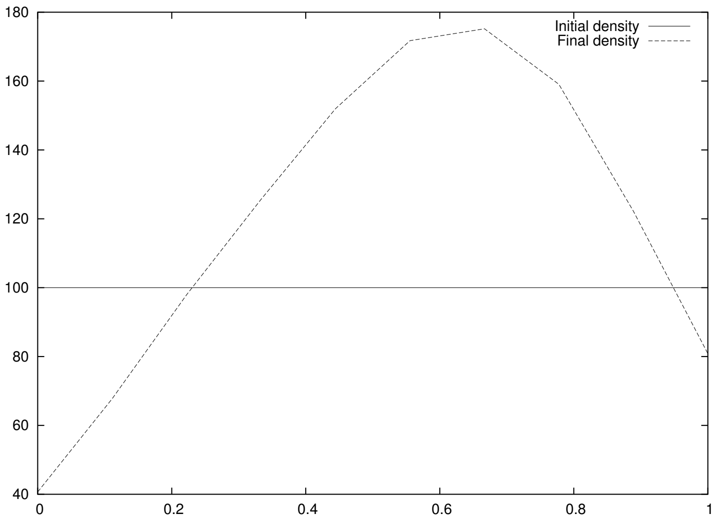

Figure 4: Minimization of L^2 error over right half-domain: central cut of initial (line) and final (dashes) mesh-densities

half right part Ω_{right} of the domain.

$$\Omega_{right} = \Omega \cap \{x > \frac{1}{2}\} \quad . \text{ Find } \; m_{opt} \quad = \quad \text{ArgMin} \int_{\Omega_{right}} |E(m)|^2 \; d\Omega \; .$$

If we worked only with a local truncation model, then we would obtain an optimal mesh without nodes on the left part of the domain. This is good for the interpolation error over Ω_{right}, but not so good for the approximation error in Ω_{right}. Indeed, for that latter error, values over the mesh are strongly correlated. The optimal mesh-density given by our algorithm shows only a slight concentration of nodes in the right part of computational domain, see Figs.4, 5,6.

5 Conclusions

We have explored a continuous method for mesh optimisation. The method relies on a complete optimal control theory, with a state system, a functional to minimize, an adjoint state for computing a functional gradient.

Some important potential advantages of the proposed methodology are:
- The accuracy can be specified by user under the form of an error norm to minimize.
- Second, by applying a linearisation, the proposed method can apply to minimizing the error on a prescribed functional. This opens the door to a combination of adaptation loop with an optimal design loop.
- As in [5], the continuous model can be extended to the case where the solution presents some singularities. In that case error order is modelled and can be predicted as second-order for optimal meshes.

Before this, several difficulties have to be solved:
- A sufficiently accurate approximation for the third or fourth derivatives of the unknown have to be introduced in order to treat a generic mesh adaptation problem.
- further analysis is needed in order to have a more complete analysis of the relation between the model and discrete solutions.

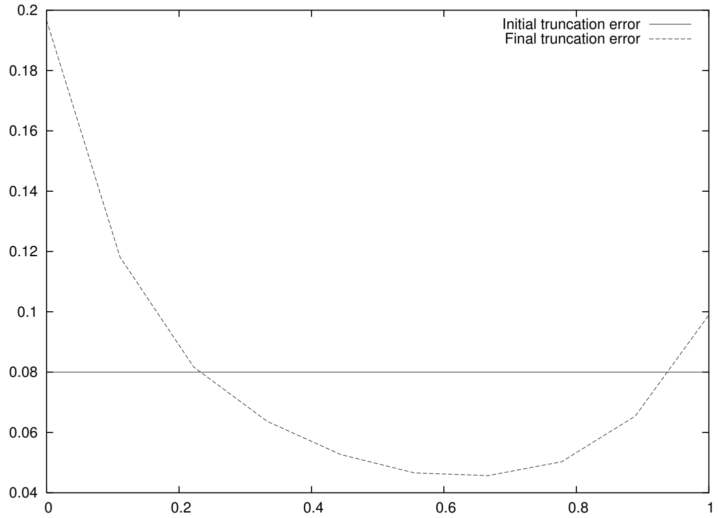

Figure 5: Minimization of L^2 error over right half-domain: central cut of initial (line) and final (dashes) truncation errors

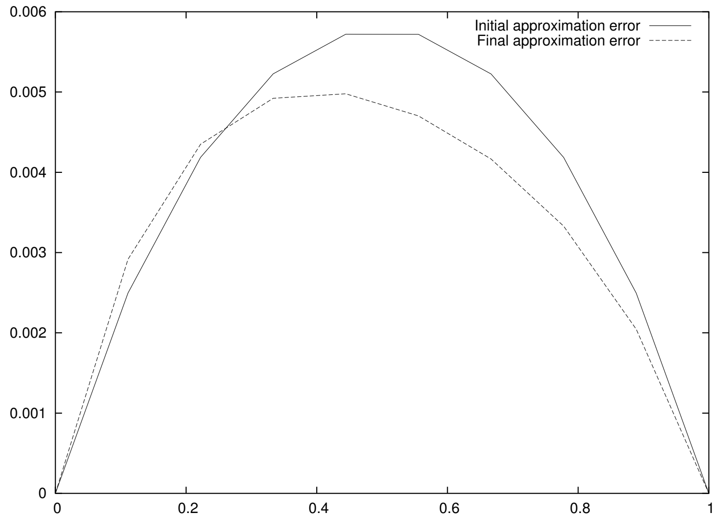

Figure 6: Minimization of L^2 error over right half-domain: central cut of initial (line) and final (dashes) approximation errors

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