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CONTINUOUS MESH ADAPTATION MODELS FOR CFD

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(P): Find the design parameter γ_{opt} for which a functional j is minimum.

γ:aircraft shape parameter
j: performances via CFD approximate model
$\mathcal{M}$:mesh parameterization

$$j(\mathcal{M},\gamma) \quad = \quad J(\mathcal{M},\gamma,U(\mathcal{M},\gamma)) \ ,$$
$$\Psi_{State}(\mathcal{M},\gamma,U(\mathcal{M},\gamma)) = 0 \ .$$

- minimize the functional with respect to shape γ,
- **minimize the error on functional with respect to a mesh parameter,** $\mathcal{M}$.

Adaptation for a functional

- Best ideal mesh density minimizing a continuous model of the error, see Babuska-Strouboulis

- Adjoint based:

. superconvergence for a functional,Giles

. a posteriori error for a functional,Becker-Rannacher

Mesh optimization strategy

Minimize an error functional $\bar{j}$ with respect to mesh parameter, $\mathcal{M}$.

$$\text{with} \quad \Psi_{STATE}(\gamma, U) = 0$$

$$\bar{\mathcal{M}} \quad = \quad \text{ArgMin } \bar{\mathbf{J}}(\mathcal{M}, \gamma, U^{exact} - U)$$
$$\text{with} \quad \Psi_{ERROR}(\mathcal{M}, \gamma, U^{exact} - U) = 0.$$

$$\bar{\mathbf{J}}(U^{exact} - U(\mathcal{M})) \quad = \quad |J_{exacte}(U^{exact})'.(U^{exact} - U(\mathcal{M}))|^2$$

Mesh parameterization: Riemannian metric

We modelize a mesh as a **continuous medium**, with an anisotropic property, the **local metric** (*):

$$\mathcal{M}_{x,y} = \mathcal{R}_{\mathcal{M}}^{-1} \begin{pmatrix} (m_\zeta)^{-2} & 0 \\ 0 & (m_\theta)^{-2} \end{pmatrix} \mathcal{R}_{\mathcal{M}} \ ,$$

(*)(George, Hecht,..., Fortin, Habashi, Vallet,...)

- interpolation error,
- approximation error: elliptic theory, extension to CFD.

P1-Interpolation error

$$\mathcal{E}_{\mathcal{M}} \;=\; \int \left(|\frac{\partial^2 u}{\partial \xi^2}|.m_\xi^2 + |\frac{\partial^2 u}{\partial \eta^2}|.m_\eta^2 \right)^2 \, dxdy$$

where ξ and η are directions of diagonalization of the Hessian of u.

Discontinuous case:

$(u(\xi+\delta,\eta) - 2u(\xi,\eta) + u(\xi-\delta,\eta))/\delta^2$ bounded in $L^{\frac{1}{2}}$.

$$\min_{\mathcal{M}} \;\; \mathcal{E}_{\mathcal{M}}$$

under the constraint $\quad N_{\mathcal{M}} = N.$

Optimal Metric construction.

$$\mathcal{M} = \mathcal{R}\,\tilde{\Lambda}\,\mathcal{R}^{-1}\,,\ \text{where}\ \ \tilde{\Lambda} = diag(\tilde{\lambda}_i)$$

$$\text{and}\ \ \tilde{\lambda}_i = \min\left(\max\left(\frac{c\,|\lambda_i|}{\varepsilon},\frac{1}{h_{max}^2}\right),\frac{1}{h_{min}^2}\right)\ .$$

$$\mathcal{M}_{opt} = \ \frac{C}{N}\,\mathcal{R}^{-1}\ \begin{pmatrix} |\frac{\partial^2 u}{\partial \eta^2}|^{-5/6}|\frac{\partial^2 u}{\partial \xi^2}|^{1/6} & 0 \\ 0 & |\frac{\partial^2 u}{\partial \xi^2}|^{-5/6}|\frac{\partial^2 u}{\partial \eta^2}|^{1/6} \end{pmatrix}\ \mathcal{R}\ .$$

avec:

$$C\ =\ \int\left(|\frac{\partial^2 u}{\partial \xi^2}|.|\frac{\partial^2 u}{\partial \eta^2}|\right)^{\frac{2}{6}}\ dxdy\ .$$

$$\text{Erreur minimale:}\ \ \mathcal{E}_{opt}\ =\ \frac{4C^2}{\mathbf{N^2}}\int\left(|\frac{\partial^2 u}{\partial \xi^2}|^{\mathbf{1/3}}|\frac{\partial^2 u}{\partial \eta^2}|^{\mathbf{1/3}}\right)dxdy$$

Anisotropic mesh reconstruction

Purpose: to generate a mesh such that all edges have a length of (or close to) one in the prescribed metric.

For P a vertex $\mathcal{M}(P)$ the metric at P, an edge PX has a local parametrization $PX = P + t\overrightarrow{PX}$.Its average length is:

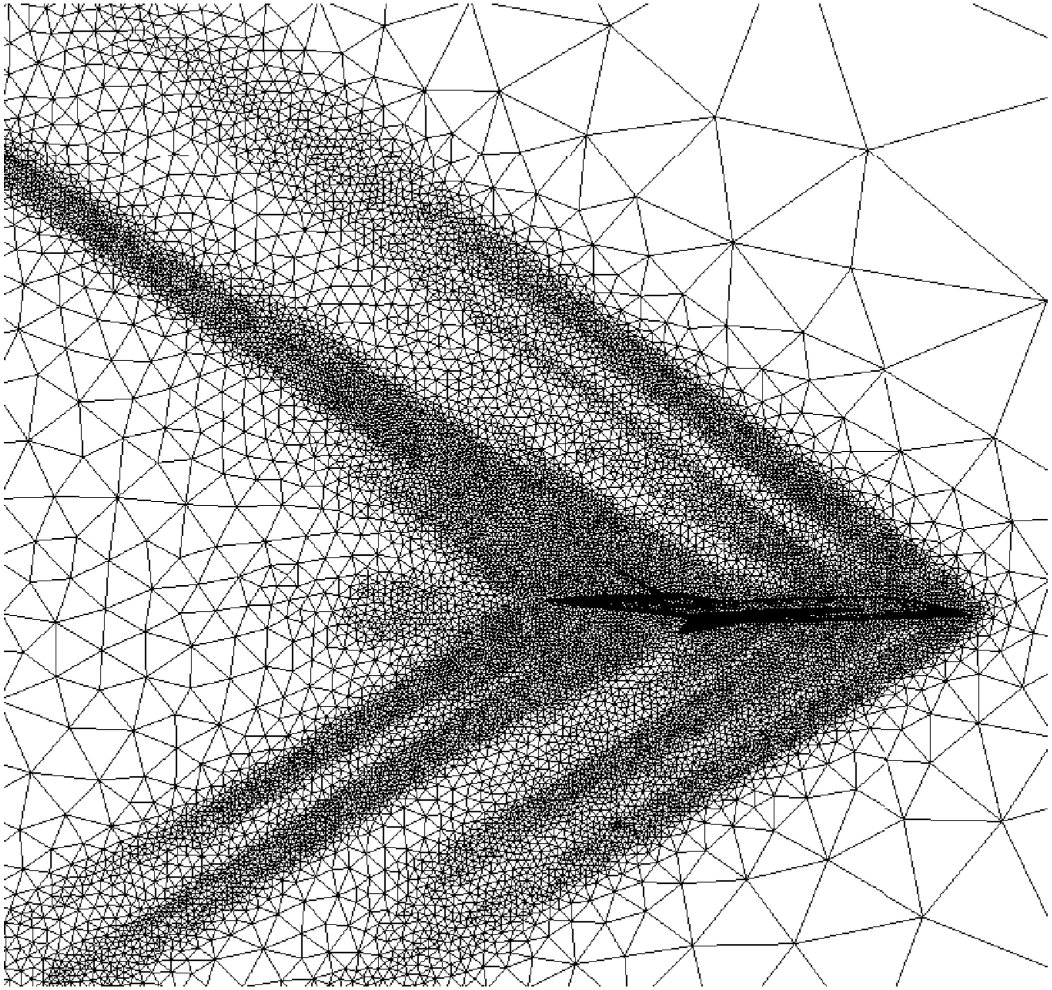
$$l_{\mathcal{M}}(\overrightarrow{PX}) = \int_0^1 \sqrt{{}^t\overrightarrow{PX}\,\mathcal{M}(P + t\overrightarrow{PX}\,)\,\overrightarrow{PX}}\;dt\,. \tag{1}$$

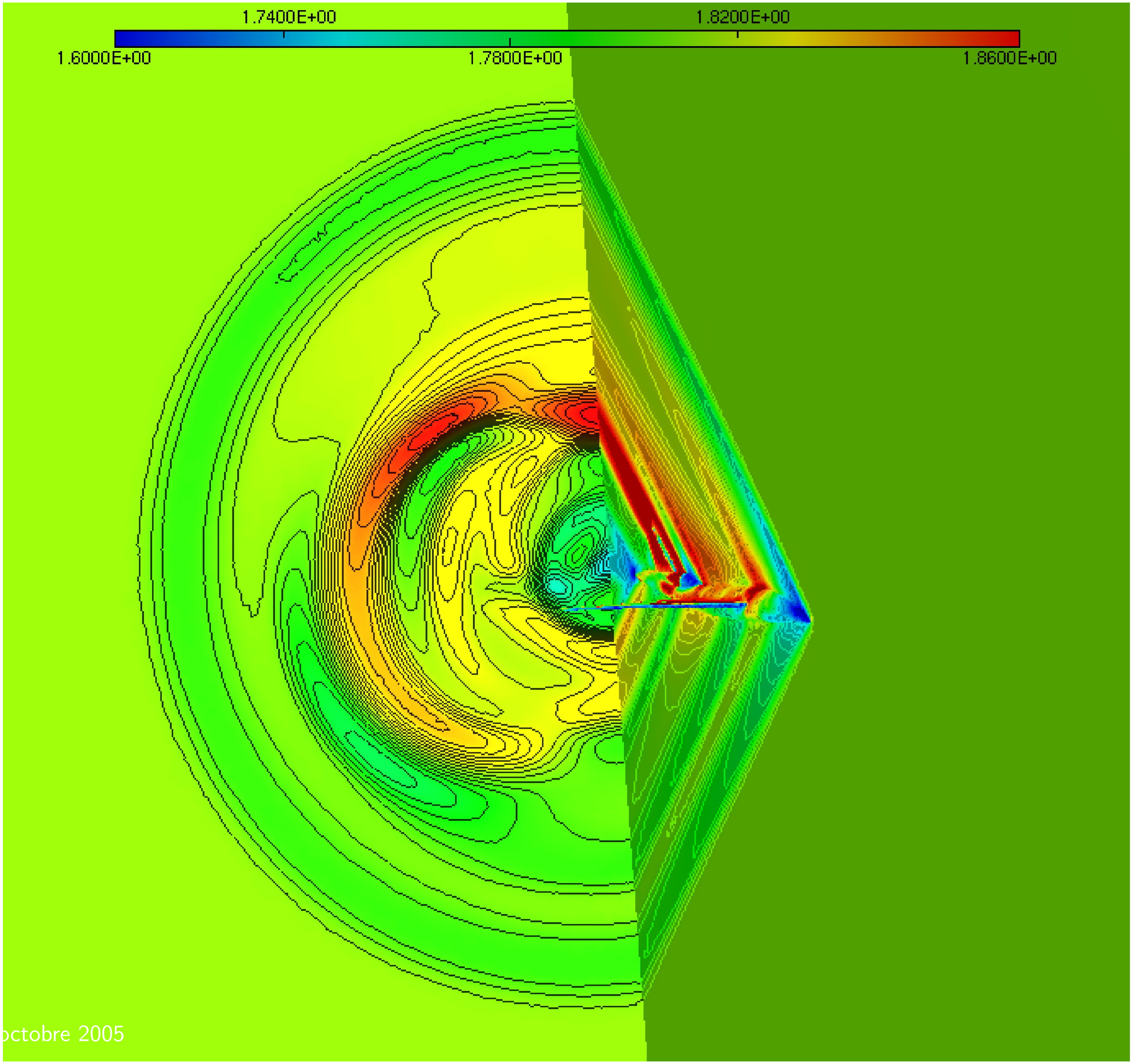
- add nodes in edges longer than 1+...

Supersonic business jet

Final mesh (iteration 9) contains
$798{,}756$ vertices,
$38{,}492$ boundary triangles and
$4{,}714{,}162$ tetrahedra.
Total CPU time 44 hours on a 600MHz workstation with 1Gb of memory.
The remeshing time is 2% of total time time.

Supersonic business jet

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Isotropic mesh and asymptotics

How does behave the approximation error?

${m_\zeta}^{\frac{1}{2}} = {m_\theta}^{\frac{1}{2}} = d$: local mesh density

$$\mathcal{M} \;=\; d \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$d_h \;=\; h^{-1}\, d \quad \forall h > 0.$$

A First model: Dirichlet model, Galerkin

Babuska-Rheinbolt: local problems + pollution error

Mesh quality and element location effects

A priori: Projection/Aubin-Nitsche

Our option: extract a global smooth behavior from second-order a priori estimates.

Approximation-error splitting

- $\mathcal{P}_1$-continuous, vertex-centered approximations,

$$\Pi_{\mathcal{M}} : H^k(\Omega) \;\rightarrow\; V_h \;=\; \{v, \text{ continue}, \;\; P_1 \text{ by element } \}$$

$$U^{exact} - U(\mathcal{M}) \;\;=\;\; U^{exact} - \Pi_{\mathcal{M}} U^{exact} \;+\; \Pi_{\mathcal{M}} U^{exact} - U(\mathcal{M})$$

$U^{exact} - \Pi_{\mathcal{M}} U^{exact}$ is interpolation error

$\Pi_{\mathcal{M}} U^{exact} - U(\mathcal{M})$ is the implicit error, solution of a discrete system.

Implicit error : elliptic case

$$< \nabla(u), \nabla\phi > \ = \ \int f\phi \, dx \ .$$

$$< \nabla(u_h), \nabla\Pi_h\phi > \ = \ \int f\phi \, dx \ .$$

$$< \nabla(u_h - \Pi_h u), \nabla\Pi_h\phi > \ = \ < \nabla(u - \Pi_h u), \nabla\Pi_h\phi >$$

Lemma : We assume that the continuous solution u is in $\mathcal{C}^3(\bar{\Omega})$ and that the continuous mesh size m is in $\mathcal{C}^2(\bar{\Omega})$. Then, for any function ϕ of $\mathcal{D}_3(\Omega)$, we have

$$\int_\Omega \frac{\partial(u-\Pi_h u)}{\partial x}\frac{\partial \Pi_h \phi}{\partial x} dM \; + \; \int_\Omega \frac{\partial(u-\Pi_h u)}{\partial y}\frac{\partial \Pi_h \phi}{\partial y} dM = h^2 \int_\Omega g'(m)\phi dM \; + \; O_\phi(h^3) \quad (2)$$

$$g'(m) \;=\; g_1'(m) + g_2'(m):$$

$$(g_1'(m),\phi) \;=\; -\frac{3}{48}\int_\Omega \phi\frac{\partial}{\partial y}\left(m^2\frac{\partial^3 u}{\partial x\partial y^2}\right) dM$$
$$+\frac{1}{48}\int_\Omega \phi\frac{\partial}{\partial x}\left(m^2\frac{\partial^3 u}{\partial x^3}\right) dM$$

$$(g_2'(m),\phi) \;=\; -\frac{1}{4}\int_\Omega \phi\frac{\partial}{\partial y}\left(\frac{m^2}{6}\frac{\partial^3 u}{\partial x^2\partial y}\right) + \phi\frac{\partial^2}{\partial y^2}\left(m^2\frac{\partial^2 u}{\partial x^2}\right) dM$$
$$+\frac{3}{24}\int_\Omega \phi\frac{\partial}{\partial y}\left(m^2\frac{\partial^3 u}{\partial y^3}\right) + \phi\frac{\partial^2}{\partial y^2}\left(m^2\frac{\partial^2 u}{\partial y^2}\right) dM\;.$$

Remark: H^2 pivot space for gradient with respect to m.

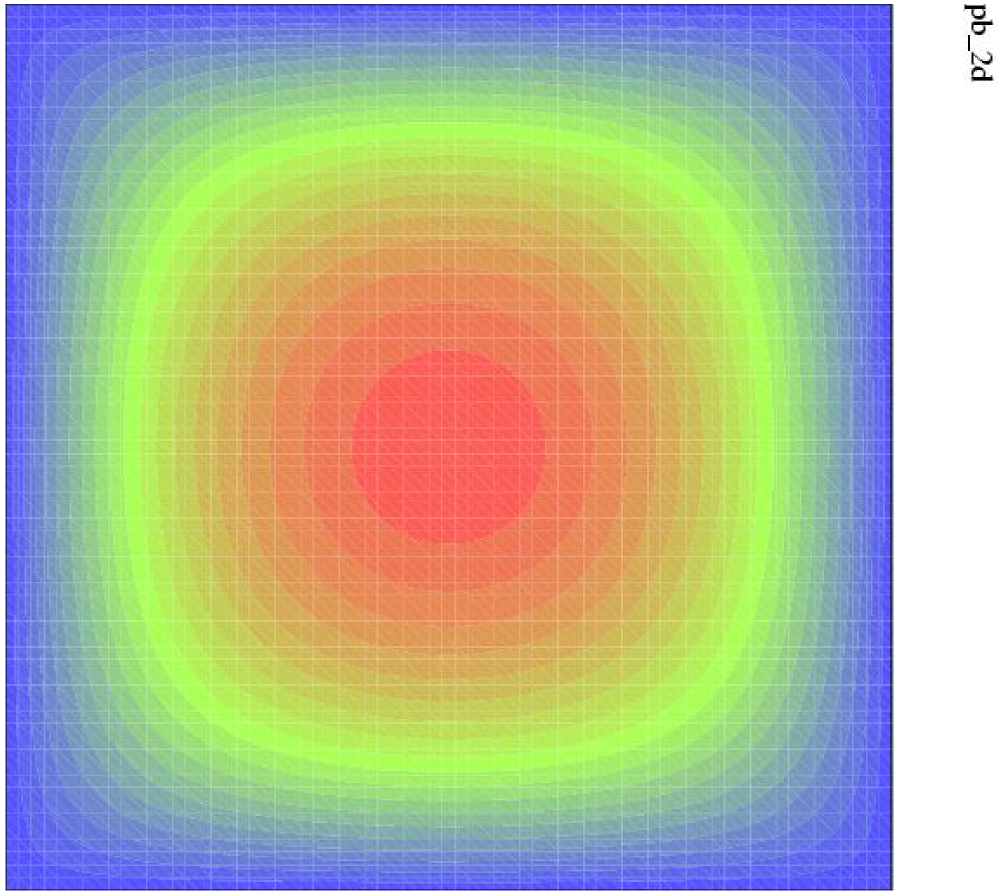
Numerical implementation

Given u and its derivatives, the continuous optimization problem is discretized on a background mesh.

Its adjoint and gradient are obtained by reverse-mode Automated Differentiation with TAPENADE.

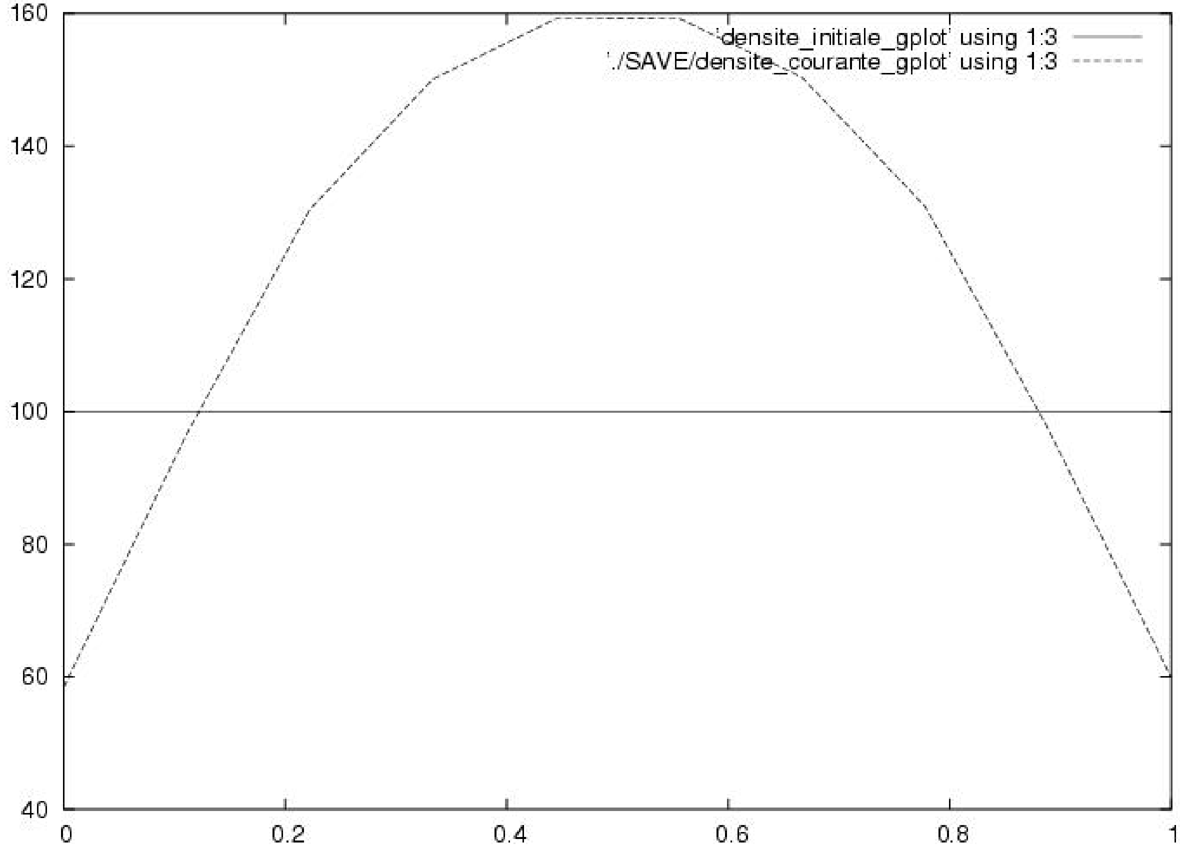
Optimization : One-Shot SQP and multi-level preconditioner.

Example: Dirichlet problem, Minimization of L^2 error on whole domain, on left half domain.

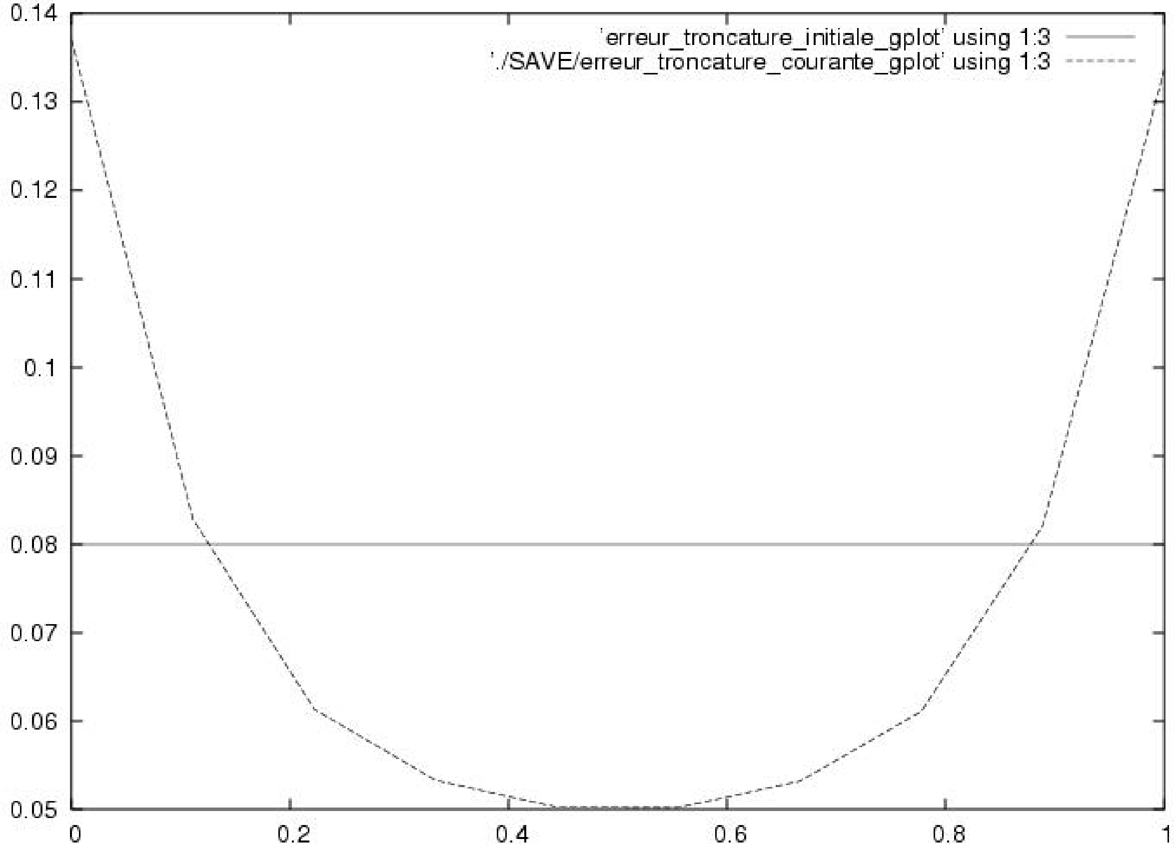


Optimization of mesh for a Dirichlet problem

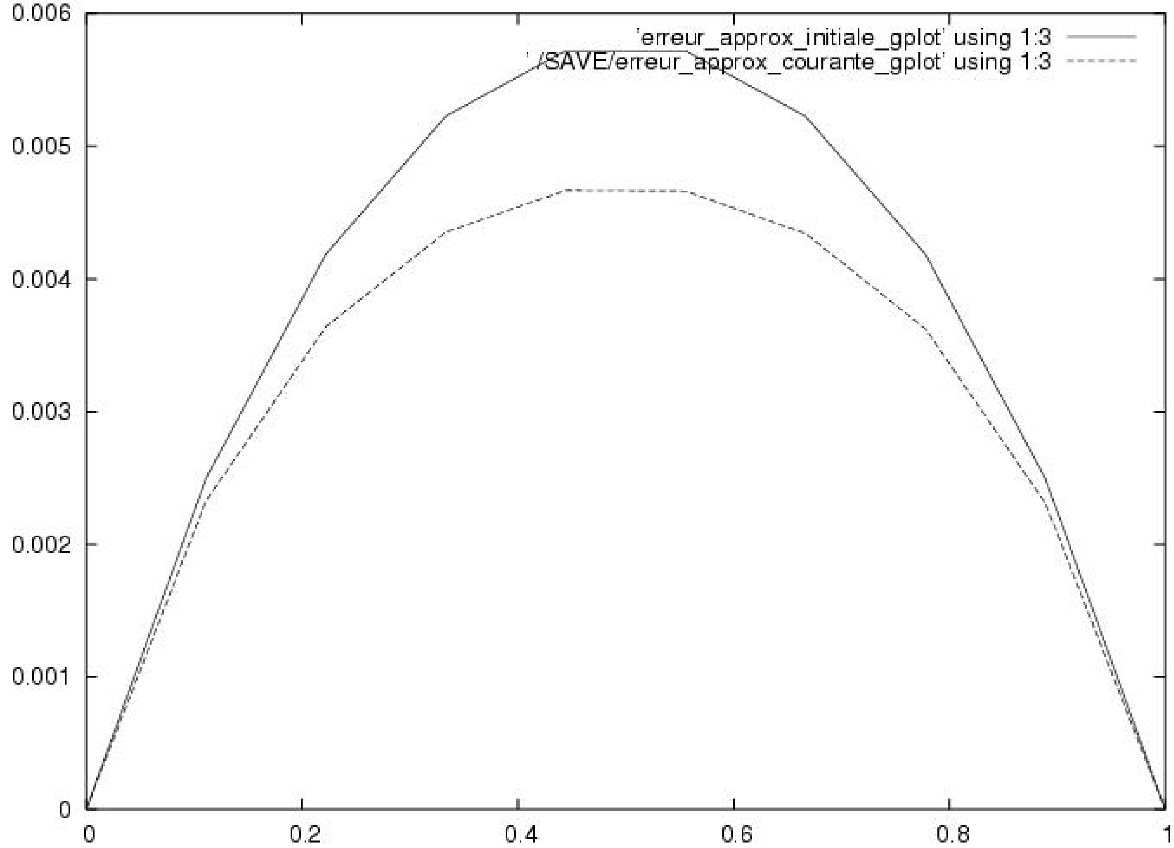
case 1: Min $\int_0^1 \int_0^1 |u - u_h|^2 \; dxdy$, case 2: Min $\int_0^{\frac{1}{2}} (\int_0^1 |u - u_h|^2 \; dy) dx$,



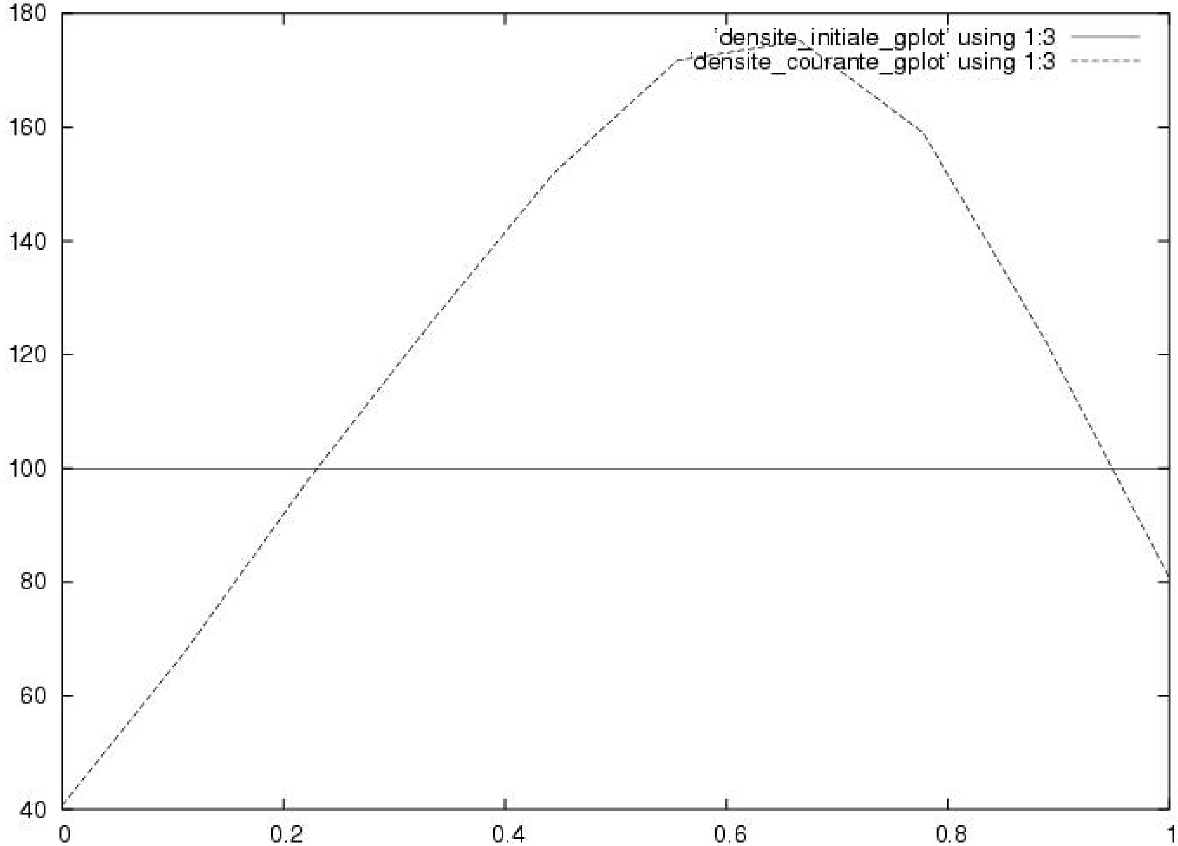
Case 1: Node-density initial(line)/final(dashes)



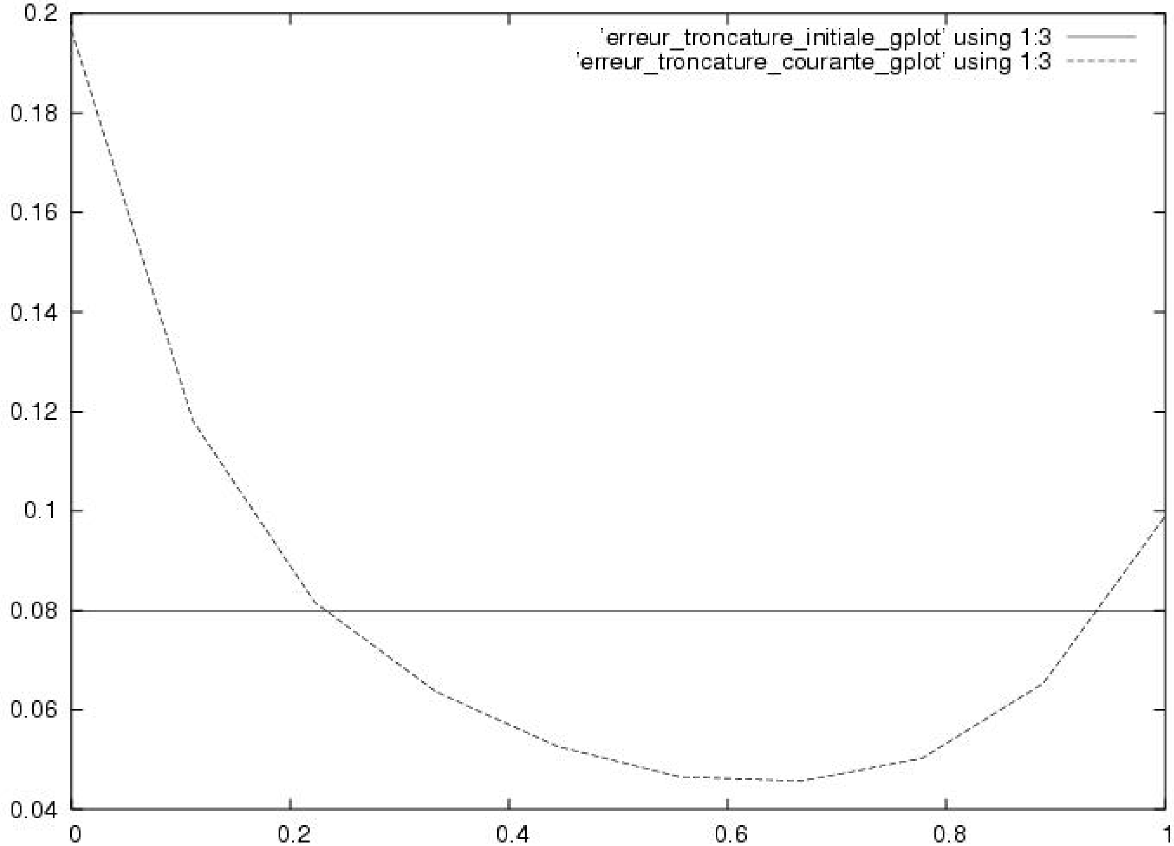
Case 1: Truncation error initial(line)/final(dashes)



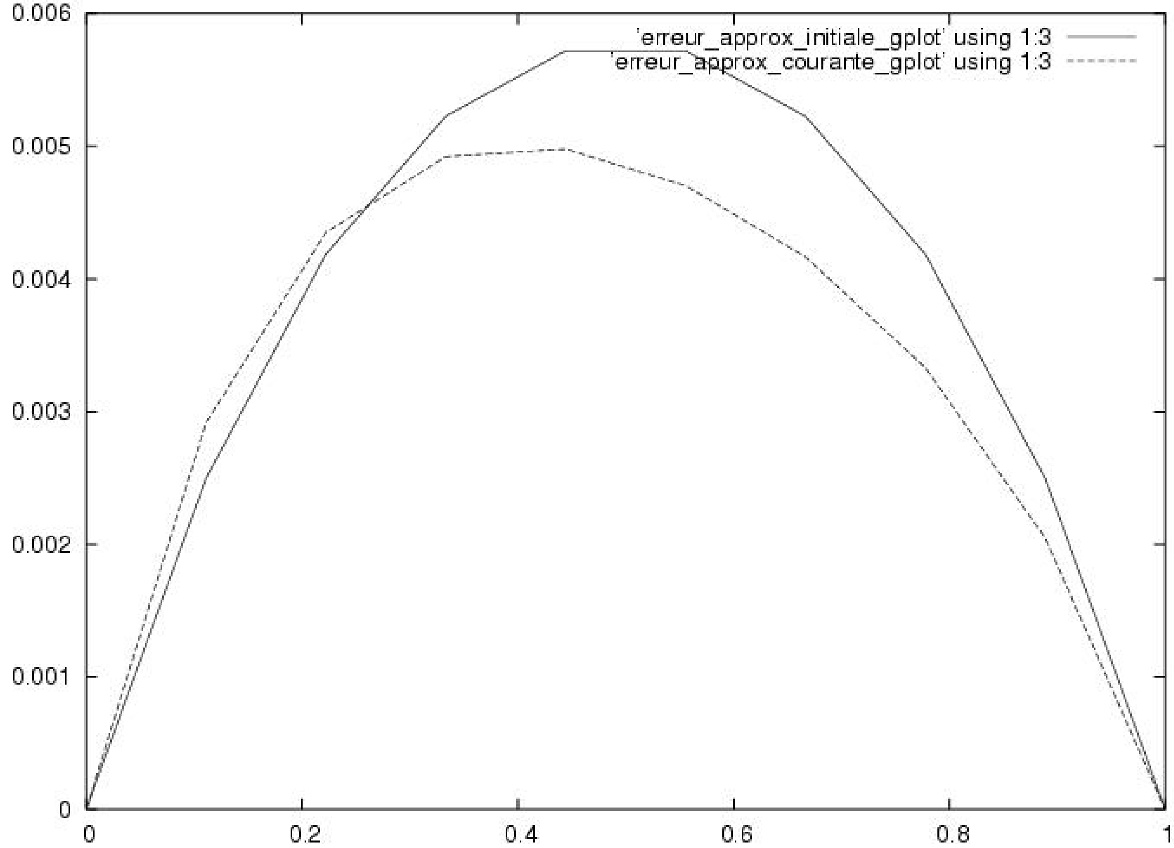
Case 1: Approximation error initial(line)/final(dashes)



Case 2: Node-density initial(line)/final(dashes)



Case 2: truncation error initial(line)/final(dashes)



Case 2: approximation error initial(line)/final(dashes)

Towards the anisotropic case:

Step 1: Pointwise optimization

Given at a point a mesh density $d = (m_\zeta m_\theta)^{-1}$, find **numerically** at each node the:
- optimal stretching direction,
- optimal stretching strength.

Build the reduced error model based on these outputs.

Step 2: constrained global optimization

Minimize the reduced error model with respect to mesh density.

A VARIATIONAL MODEL FOR EULER EQUATIONS

A simplifying assumption: pure Galerkin formulation W and $\phi \in \left(H^1(\Omega)\right)^5$,$V_h$ is included in V.

$$\int_\Omega \phi \nabla.\mathcal{F}(W) d\Omega - \int_{\partial\Omega} \phi \bar{\mathcal{F}}(W).n d\partial\Omega \ = \ 0.$$

$$\int_\Omega \phi_h \nabla.\mathcal{F}_h(W_h) d\Omega - \int_{\partial\Omega} \phi_h \bar{\mathcal{F}}_h(W_h).n d\partial\Omega \ = \ 0$$

where $\mathcal{F}_h(W)$ is the interpolate of $\mathcal{F}$, i.e. $\mathcal{F}_h(W_h) = \Pi_h \mathcal{F}(W_h)$, and same for $\bar{\mathcal{F}}_h(W_h)$.

Replace ϕ by $\phi_h = \Pi_h \phi$:

$$\int_\Omega \phi_h \nabla . \mathcal{F}_h(W_h) d\Omega - \int_{\partial\Omega} \phi_h \bar{\mathcal{F}}_h(W_h).n d\partial\Omega \ = \int_\Omega \phi_h \nabla . \mathcal{F}(W) d\Omega - \int_{\partial\Omega} \phi_h \bar{\mathcal{F}}(W).n d\partial\Omega$$

Boundary data are involved inside $\bar{\mathcal{F}}(W)$ and $\bar{\mathcal{F}}_h(W)$ Assumption: these terms can be split in W-dependant terms, denoted respectively by $\bar{\mathcal{F}}^{out}(W)$ and ${\bar{\mathcal{F}}_h}^{out}(W_h)$, and constant terms, denoted $\bar{\mathcal{F}}^{in}$ and ${\bar{\mathcal{F}}_h}^{in}$.

Therefore:

$$\int_{\Omega} \phi_h \nabla . \mathcal{F}_h(W_h) d\Omega - \int_{\partial\Omega} \phi_h \bar{\mathcal{F}_h}^{out}(W_h).n d\partial\Omega \ = \int_{\Omega} \phi_h \nabla . \mathcal{F}(W) d\Omega - \int_{\partial\Omega} \phi_h \bar{\mathcal{F}}^{out}(W).n d\partial\Omega$$

Error estimate

Assumption: both W and ϕ are several time continuously differentiable.

$$\int_{\Omega} \phi_h \nabla.(\mathcal{F}_h(W_h) - \Pi_h \mathcal{F}(W)) d\Omega -$$

$$\int_{\partial\Omega} \phi_h (\bar{\mathcal{F}_h}^{out}(W_h) - \Pi_h \bar{\mathcal{F}}^{out}(W)).n d\partial\Omega \; =$$

$$\int_{\Omega} \phi_h \nabla.(\mathcal{F}(W) - \Pi_h \mathcal{F}(W)) d\Omega -$$

$$\int_{\partial\Omega} \phi_h (\bar{\mathcal{F}}^{out}(W) - \Pi_h \bar{\mathcal{F}}^{out}(W)).n d\partial\Omega.$$

The left-hand side will be inverted and the right-hand side will be expanded to get the error estimate.

Interpolation errors

We recall that $\Pi_h \mathcal{F}(W) = \mathcal{F}_h(W)$. The left-hand side writes:

$$LHS = \int_\Omega \phi_h \nabla.(\mathcal{F}_h(W_h) - \mathcal{F}_h(W)) d\Omega - \int_{\partial\Omega} \phi_h (\bar{\mathcal{F}_h}^{out}(W_h) - \bar{\mathcal{F}}_h^{out}(W)).n d\partial\Omega.$$

We linearize it as follows:

$$LHS \;=\; \int_{\Omega} \phi_h \nabla.(\Pi_h \frac{\partial \mathcal{F}}{\partial W}(W_h - W))d\Omega - \int_{\partial\Omega} \phi_h (\Pi_h \frac{\partial \bar{\mathcal{F}_h}^{out}}{\partial W}(W_h - W)).n d\partial\Omega.$$

Where the derivatives $\frac{\partial \mathcal{F}}{\partial W}$ and $\frac{\partial \bar{\mathcal{F}_h}^{out}}{\partial W}$ are evaluated from vertex values of W.

$$LHS \;=\; \mathcal{A}_h(W)(W_h - \Pi_h W)$$

$$LHS \;=\; \mathcal{A}_h(W)(W_h - \Pi_h W)$$

Assumption:the Jacobian $\mathcal{A}_h(W)$ of the discretized Euler system is invertible. The **implicit error** $W_h - \Pi_h W$ is the unique solution of:

$$W_h - \Pi_h W \;=\; (\mathcal{A}_h(W))^{-1} \;\; RHS \;.$$

In the right-hand side:

$$RHS = \int_{\Omega} \phi_h \nabla.(\mathcal{F}(W) - \Pi_h \mathcal{F}(W)) d\Omega - \int_{\partial\Omega} \phi_h (\bar{\mathcal{F}}^{out}(W) - \Pi_h \bar{\mathcal{F}}^{out}(W)).n d\partial\Omega$$

we recall that $\phi_h = \Pi_h \phi$ and we add and substract a ϕ term:

$$RHS = RHS_1 + RHS_2$$

with:

$$RHS_1 = \int_{\Omega} (\Pi_h \phi - \phi) \nabla.(\mathcal{F}(W) - \Pi_h \mathcal{F}(W)) d\Omega - \int_{\partial\Omega} (\Pi_h \phi - \phi)(\bar{\mathcal{F}}^{out}(W) - \Pi_h \bar{\mathcal{F}}^{out}(W)).n d\partial\Omega$$

Assuming smoothness of ϕ and $\mathcal{F}(W)$, we deduce that on Ω, interpolation errors are of order two and their gradients are of order one, same on boundary, and RHS_1 is thus of order three.

$$RHS_1 \;\leq\; \textsf{const.}h^3$$

The second term writes:

$$RHS_2 \;=\; \int_\Omega \phi\nabla.(\mathcal{F}(W) - \Pi_h\mathcal{F}(W))d\Omega - \int_{\partial\Omega} \phi(\bar{\mathcal{F}}^{out}(W) - \Pi_h\bar{\mathcal{F}}^{out}(W)).nd\partial\Omega$$

and we transform it as follows:

$$\begin{aligned} RHS_2 \; = \; & -\int_{\Omega}(\nabla\phi).(\mathcal{F}(W)-\Pi_h\mathcal{F}(W))d\Omega \\ & +\int_{\partial\Omega}\phi(\mathcal{F}(W)-\Pi_h\mathcal{F}(W)).nd\partial\Omega - \\ & \int_{\partial\Omega}\phi(\bar{\mathcal{F}}^{out}(W)-\Pi_h\bar{\mathcal{F}}^{out}(W)).nd\partial\Omega \end{aligned}$$

Same as for elliptic (easier):

$$RHS_2 \; \leq \; \text{const.}h^2.$$

$$RHS_2 \; = \; h^2 \; (G(W,d),\phi) \; + \; R \quad , \quad R \; = \; o(h^2).$$

Provisional conclusion

The above study shows that the implicit error $W_h - \Pi_h W$ is a linear function of the interpolation error $W - \Pi_h W$.

A first option consists in reducing the interpolation error. This option is studied in Sections 4 and 5.

In Section 6 we shall define how to minimize with the implicit error by introducing an adjoint.

Implicit error reduction, general case

$$\frac{\partial \Psi}{\partial W}(W,d)Y = G(W,d),$$

W is the unknown, d mesh density, Y implicit error.

$$\begin{aligned}
\frac{\partial \Psi}{\partial W}(W,d)Y - G(W,d) &= 0 \\
\left(\frac{\partial \Psi}{\partial W}(W,d)\right)^{*}\Pi - \frac{\partial \bar{J}}{\partial Y}(Y,d) &= 0 \\
\bar{j}' = -\left(\frac{\partial(\frac{\partial \Psi}{\partial W}Y - G)}{\partial d}(W,d)\right)^{*}\Pi + \frac{\partial \bar{J}}{\partial d}(Y,d) &= 0
\end{aligned}$$

which can be solved as for elliptic case.

CONCLUSION

This talk proposes rather sophisticated algorithms involving:
flow solver,
local error evaluation,
sensitivity analysis by Automated Differentiation,
adjoint and gradient based optimization,
controlled generation of next mesh,
interpolation of solution,
till a convergence of a global loop is attained.

These sophisticated algorithms will not bring only robustness, but also efficiency.