Old (1 quart) and recent (1 small pint) things in mesh adaption

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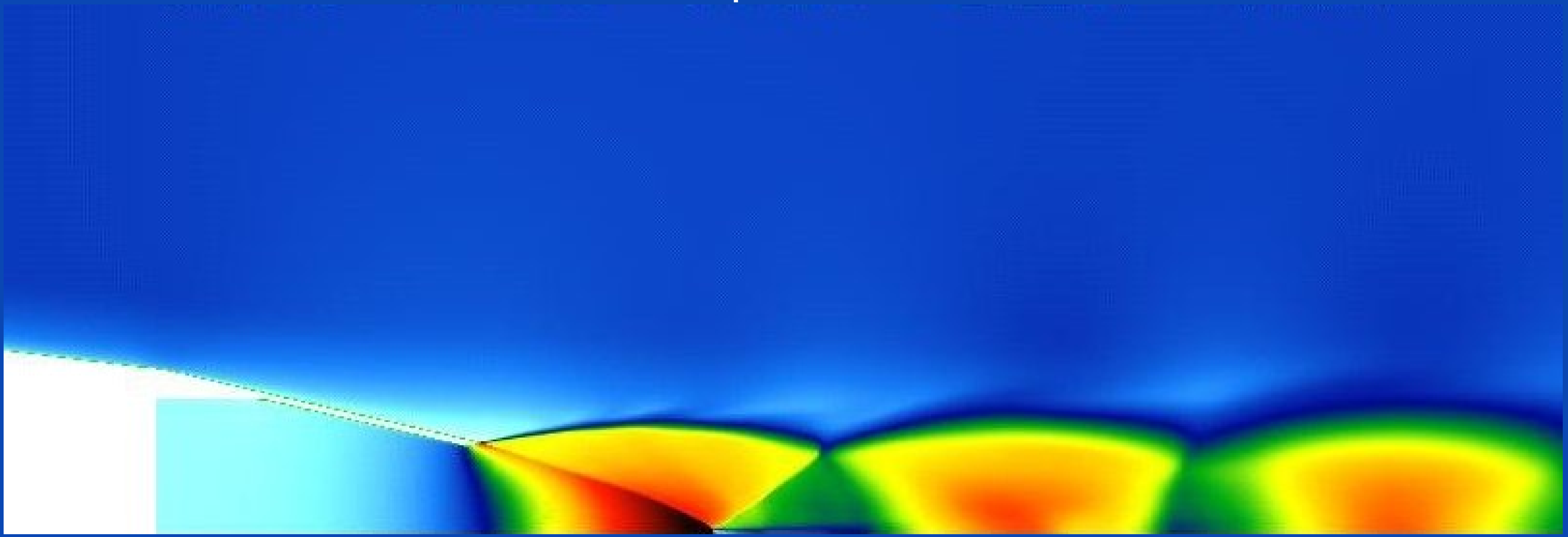
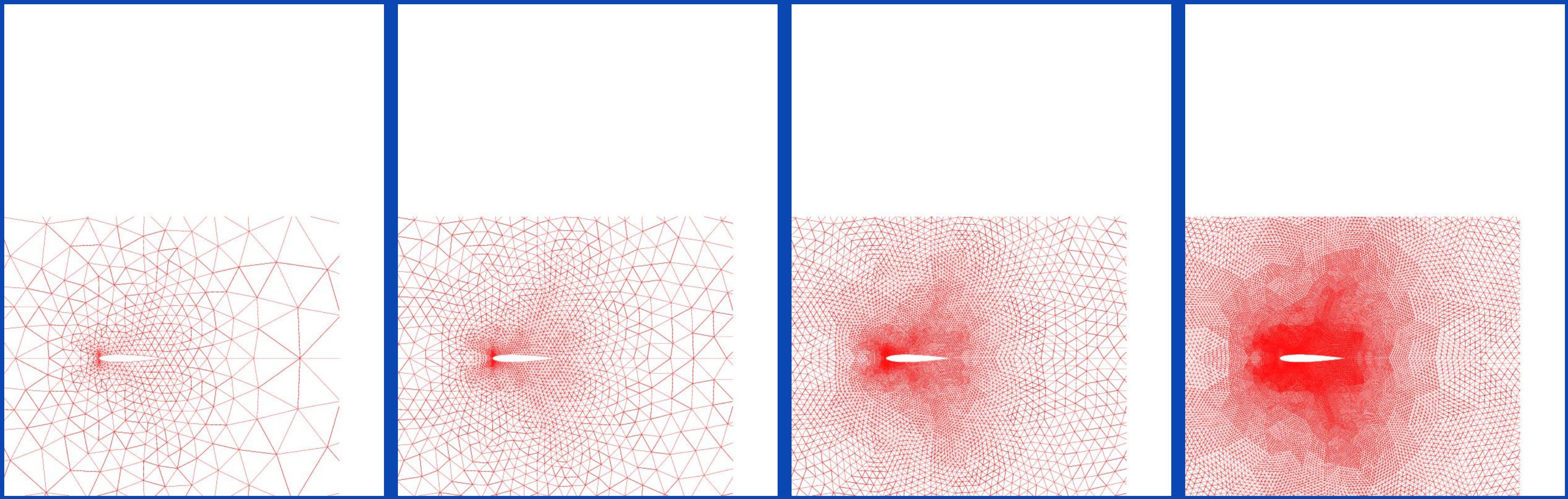
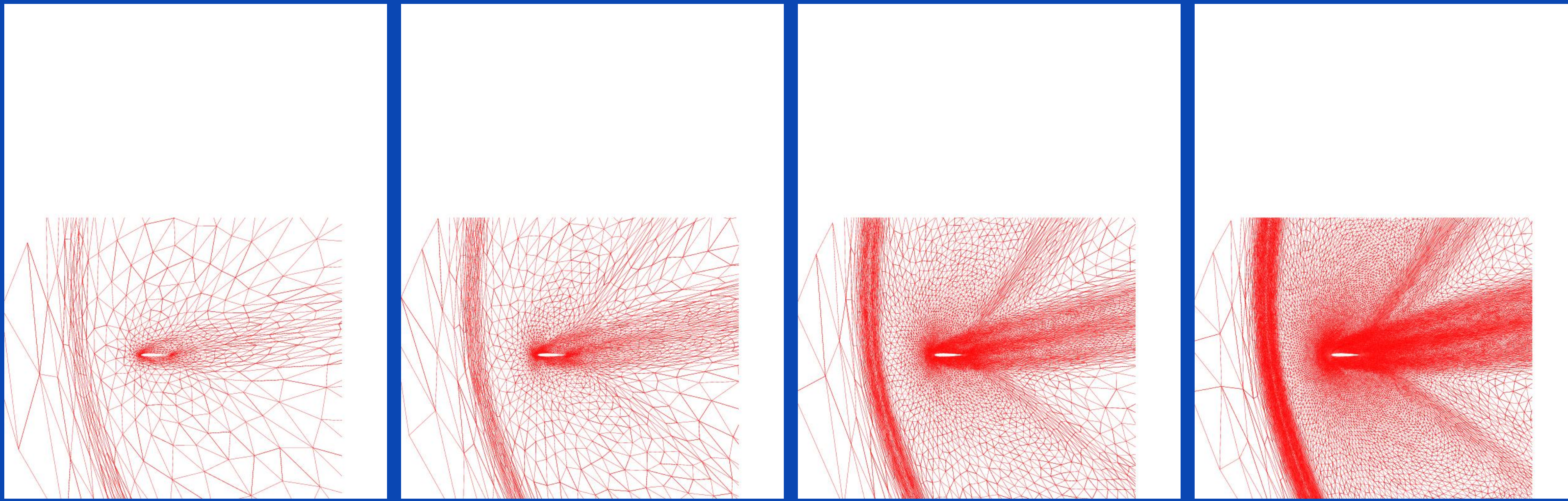


An example



Embedded reft.	mesh 1	mesh 2	mesh 3	mesh 4
# of nodes	800	3114	12284	48792
Numerical order			**0.94**	**1.14**



adaptative reft.	mesh 1	mesh 2	mesh 3	mesh 4
# of nodes	800	3114	11938	40965
numerical order			**1.75**	**1.92**

$$|U_3 - u|_{L^2} \ \leq \ \frac{1}{3}|U_2 - U_3|_{L^2} \ = \ 6.00 \ 10^{-5} \ . \ |U_3 - U_4|_{L^2} \ = \ 5.637 \ 10^{-5} \ .$$

Overview

1. Problem position
2. The optimal metric for interpolation
3. The optimal metric method for an EDP
5. Conclusion

High order adaptation for a discontinuity

u: bounded, piecewise smooth, with a few discontinuities.

Prototype : the Heavyside function $+$ a smooth function, on $[0{,}1]$.

Lemma: *For a uniform refinement, the order of accuracy in L^2 of the P1 interpolation is only 1/2. Conversely, there exist adaptative refinements for which the order of accuracy of P1 interpolation is 2.*

Idea of the proof : Divide the interval around discontinuity into eight intervals of same size and divide other intervals into two. Total mesh size is only increased by a factor $2+8/N$ and error is 4 times smaller.

N.B.: For a third-order P_2 interpolation, third-order accuracy is obtained by dividing the singular interval into 16.

The “continuous metrics” approach, 1D, non smooth

In the case of P_1 interpolation, we modelize the error as :

$$\int_0^1 |e_{\mathcal{M}}(x)|^\alpha ds = \int_0^1 (m^2|\delta^{-2}(u(x+\delta) - 2u(x) + u(x-\delta))|)^\alpha ds.$$

where δ is smaller than m.

$\delta^{-2}(u(x+\delta) - 2u(x) + u(x-\delta))$:

- is close to $\frac{\partial^2 u}{\partial x^2}$,

- or to δ^{-2},

- bounded in $L^{1/2}$ independantly of δ.

Continuous metrics adaption for a discontinuity

$$m_{opt}(x) = Cte.|(|\delta^{-2}(u(x+\delta) - 2u(x) + u(x-\delta))|(x))|^{\frac{-2}{5}}.$$

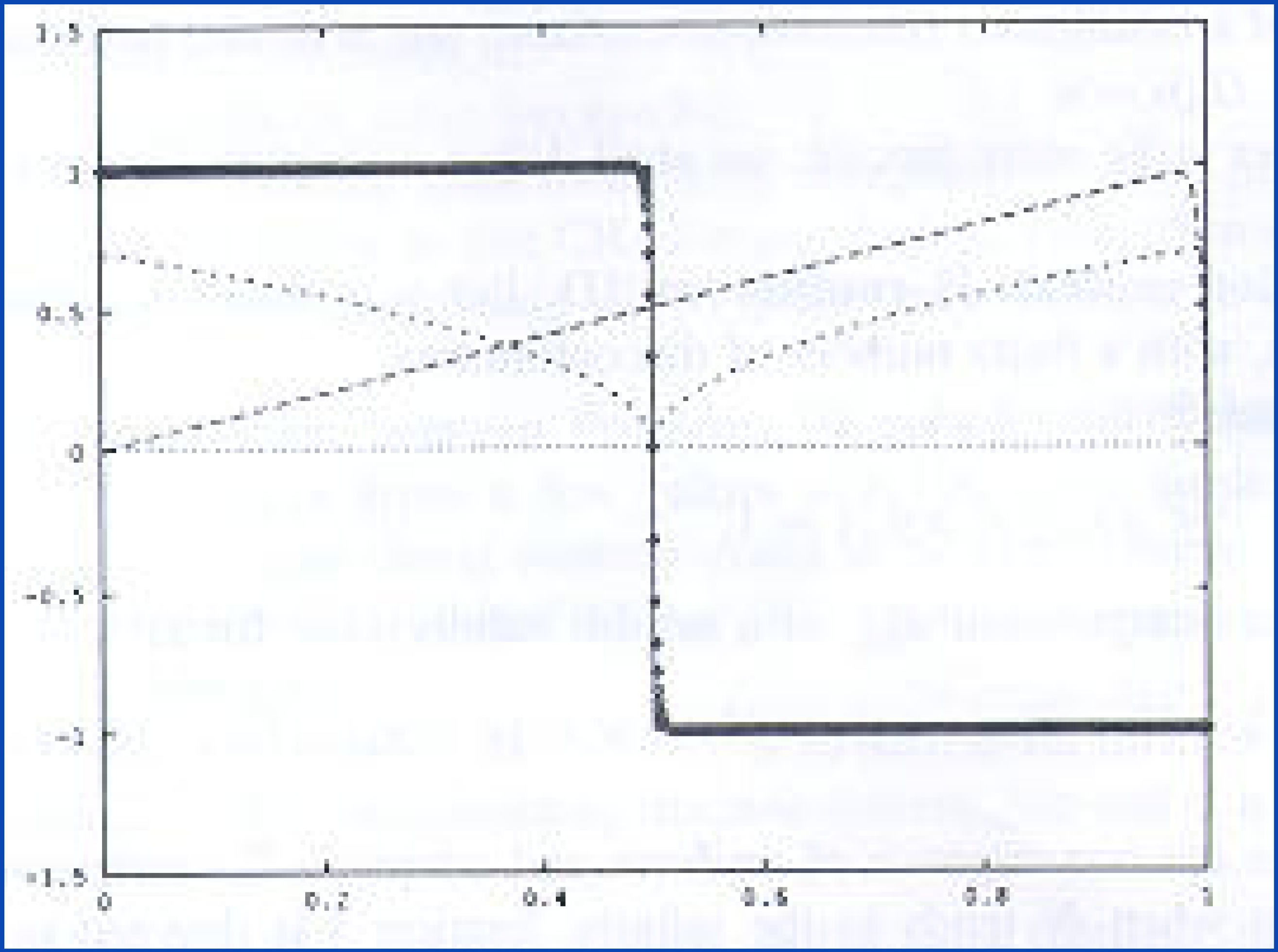
Further the resulting error in L^2 writes:

error $= \frac{2}{N^2} \left(\int |\delta^{-2}(u(x+\delta) - 2u(x) + u(x-\delta))|^{\frac{2}{5}} \right)^{\frac{5}{2}} < \frac{K}{N^2}$

which gives second-order accuracy.

Dicontinuity capturing: Numerical illustration:

Two examples : smooth arctangent, discontinuous Heavyside.



Adaptation on an interval :

Choose a number of nodes N.

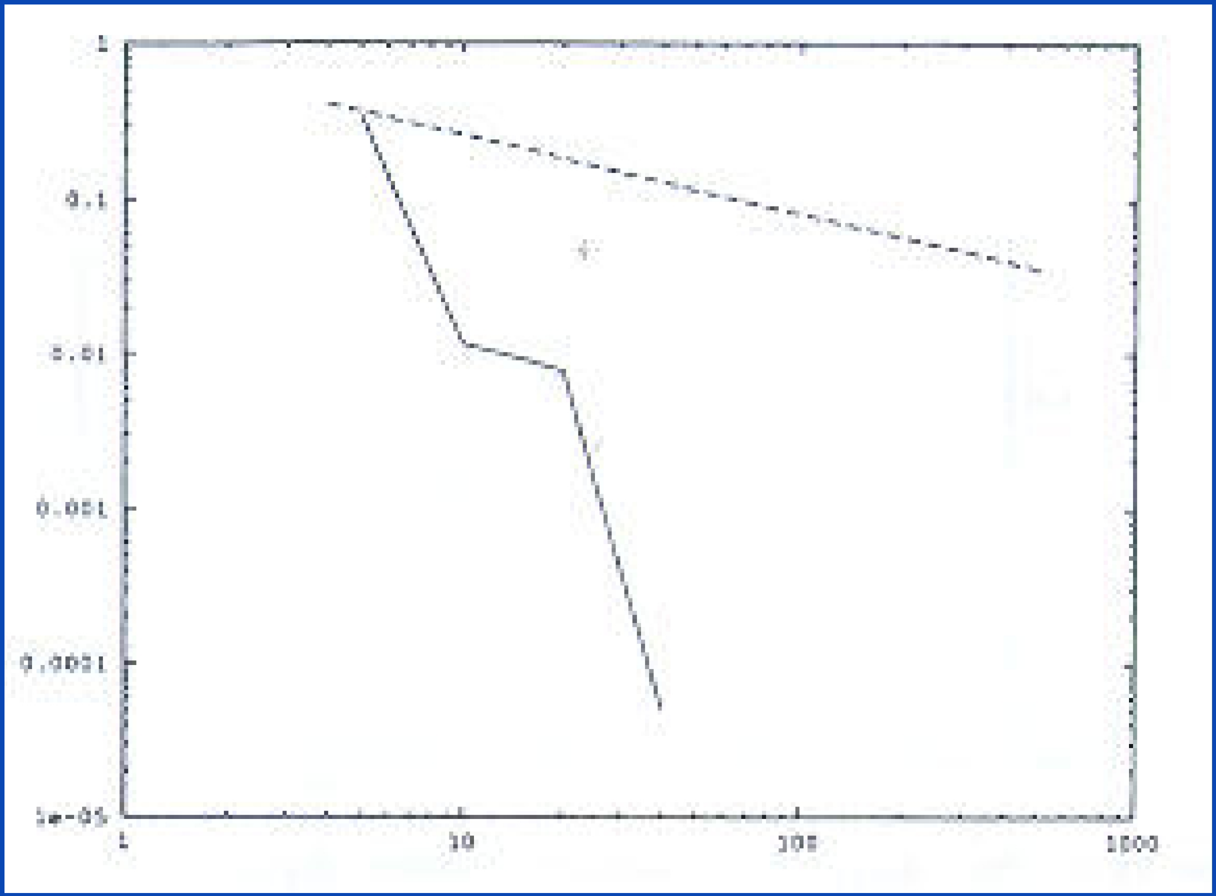
Derive the optimal metrics m.

Define x from:

$$x_0 = 0, \qquad \int_{x_i}^{x_{i+1}} m^{-1} dx = 1 \ ,$$

N.B.: Can also be done by mesh deformation.

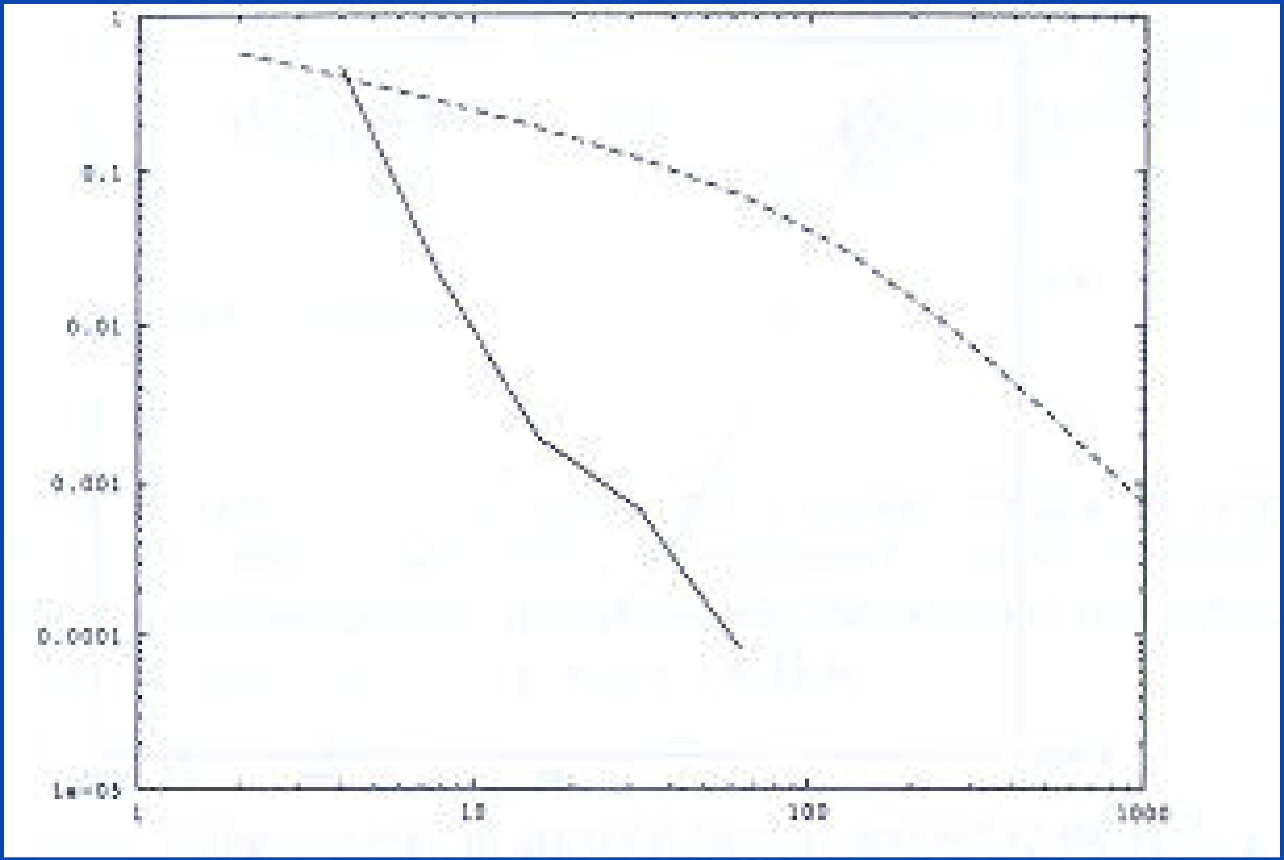
Convergence to the continuous: Heavyside



Convergence towards $y = -sign(x - \frac{1}{2})$

Abscissae : number of nodes; ordinates : interpolation error, Dashes : uniform refinement, line : adaptive refinement.

Convergence to the continuous: Arctangent



Uniform refinement: late capturing

Adaptative refinement : early capturing

Early capturing/late capturing

Uniform refinement needs N_S nodes, where N_s is the inverse of the size of the smallest detail (1D).

A good adaptative refinement needs N_d nodes, where N_d is (1D) the number of details (for example: the function is monotone on N_d intervals).

$$N_d \quad << \quad N_S.$$

Isotropic simplified optimum :

The above calculation can be done with a scalar metrics. It turns like the 1D case.

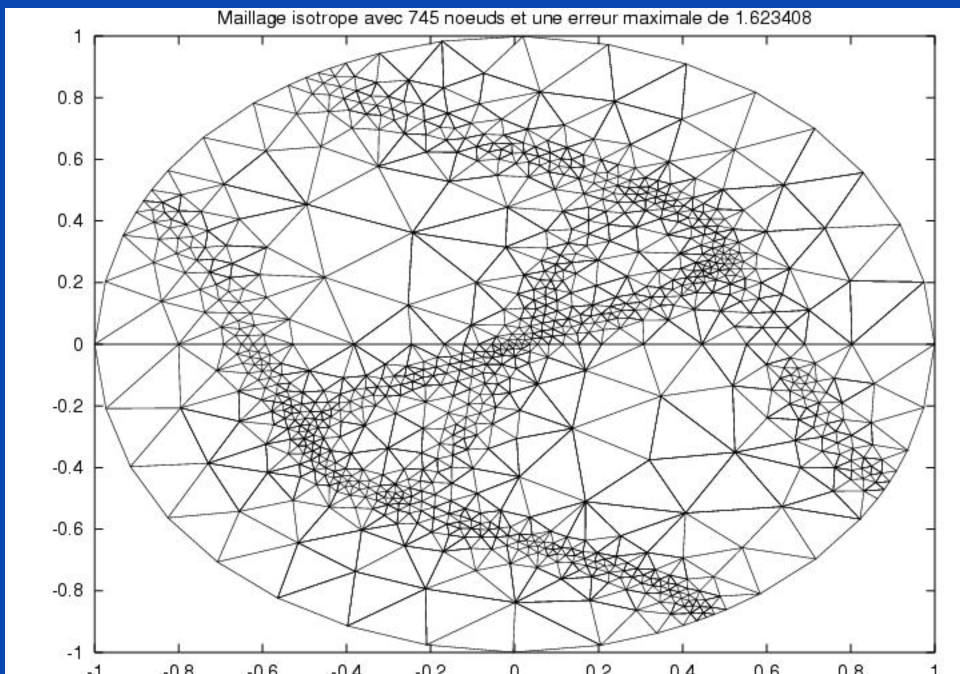
$$e_{\mathcal{M}}(x,y) \;= m^2(x,y) M(x,y)$$

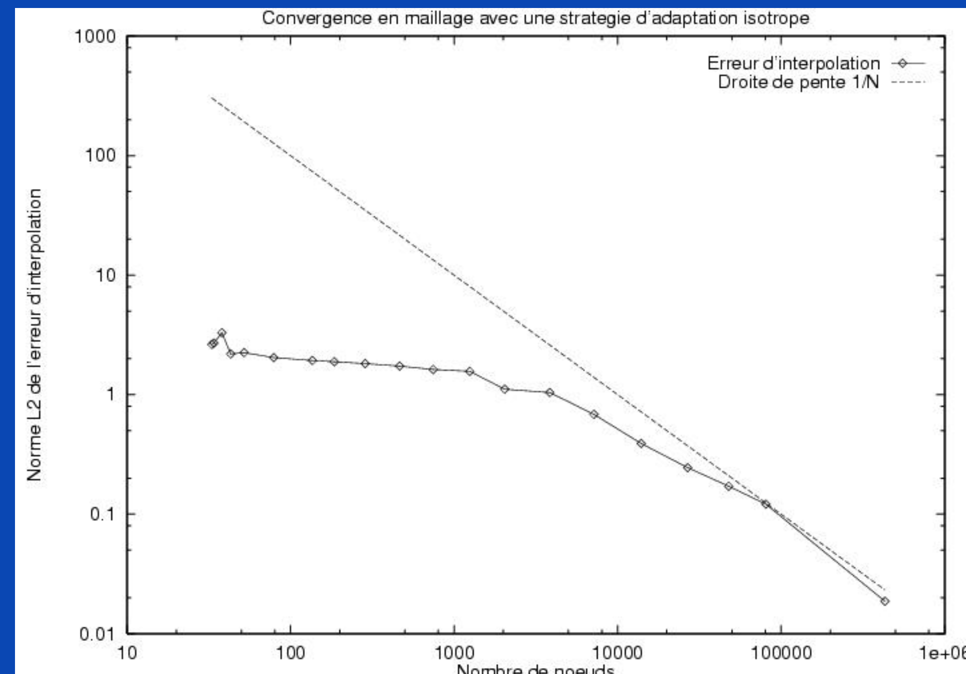
where $M(x,y)$, is $Max(Sp(\mathcal{H}))$, the maximum absolute value of eigenvalues of the local Hessian of u. We obtain the optimum:

$$m_{opt}(x) = \left(\frac{(\int_\Omega M^{\frac{-2}{3}} ds)}{N} \right)^{\frac{1}{2}} \; M(x,y)^{\frac{-1}{3}}.$$

Numerical illustration : 1. Isotropic adaptive refinement

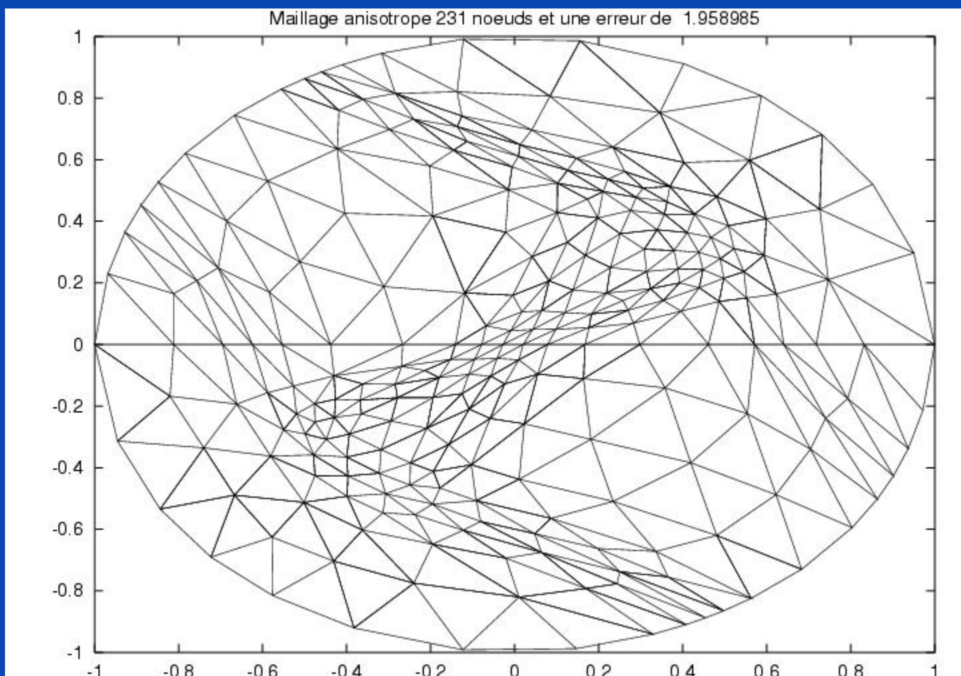
Test case : interpolate a couple of S-shaped arctangent functions
Sensor : scalar field equal to $Max(sp(\mathcal{H}))$.
Controlled Voronoi remeshing. George, Hecht, Saltel, Mohammadi,...

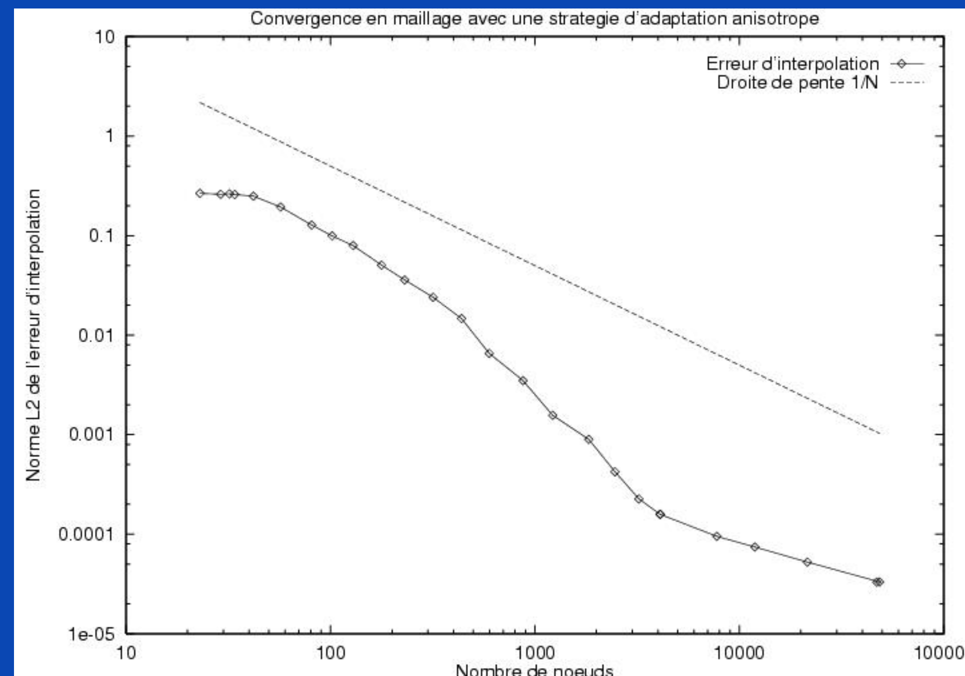




2. Anisotropic adaptive refinement

. Sensor : 2×2 metrics field derived from the Hessian
.Controlled Voronoi remeshing George, Hecht, Saltel, Mohammadi,..





Lemma (barriers in L^2):

The convergence order of uniform refinement is at most 1/2,

The convergence order of 2D isotropic adaptative refinement is at most 1.

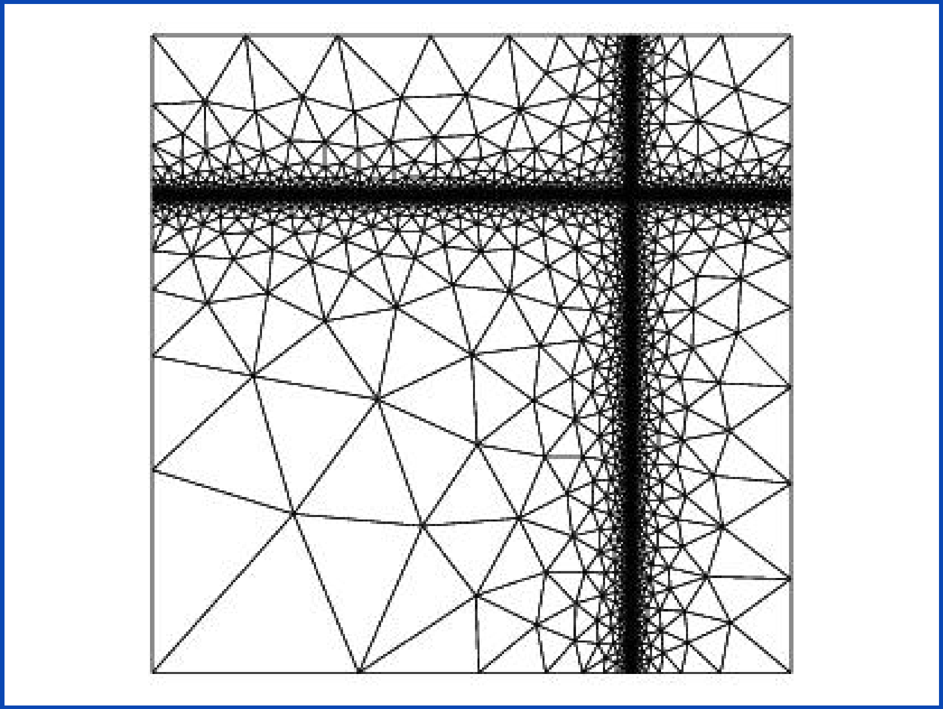
The convergence order of 3D isotropic adaptative uniform refinement is at most 3/4

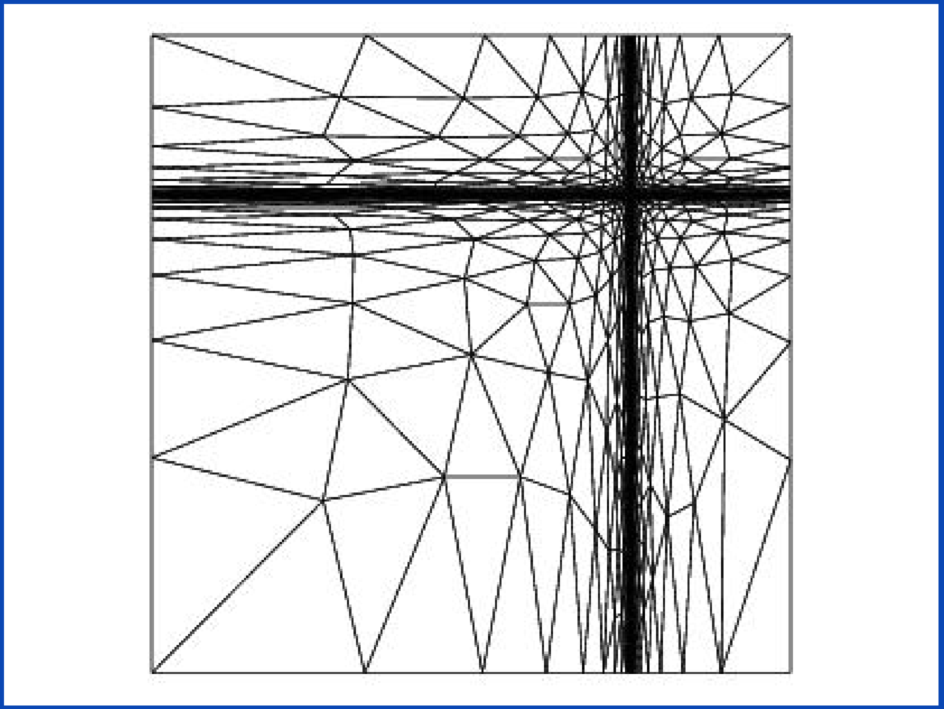
Coudière-Dervieux-Leservoisier-Palmerio, 2001

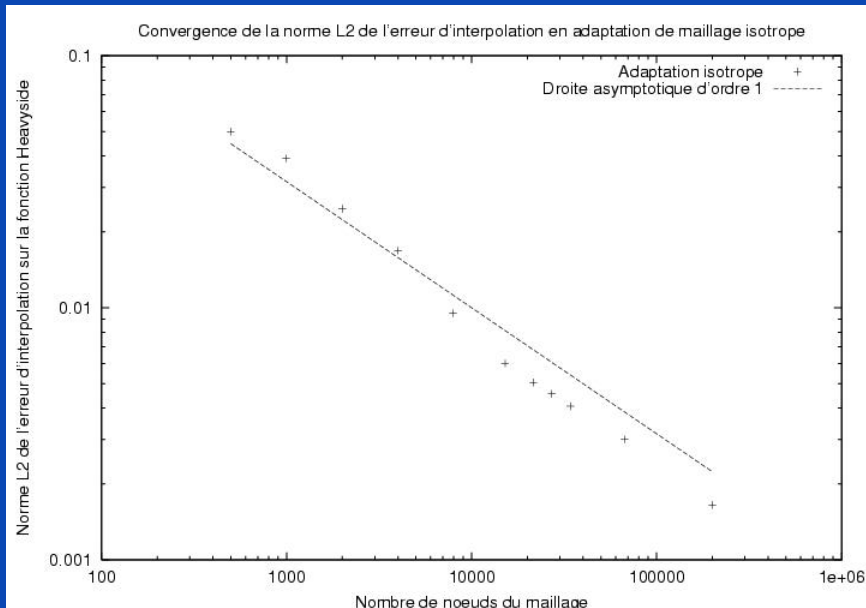
N.B.: This was announced by the continuous metrics model, which produces "the best mesh". Analysis of the resulting error lead to the same barriers.

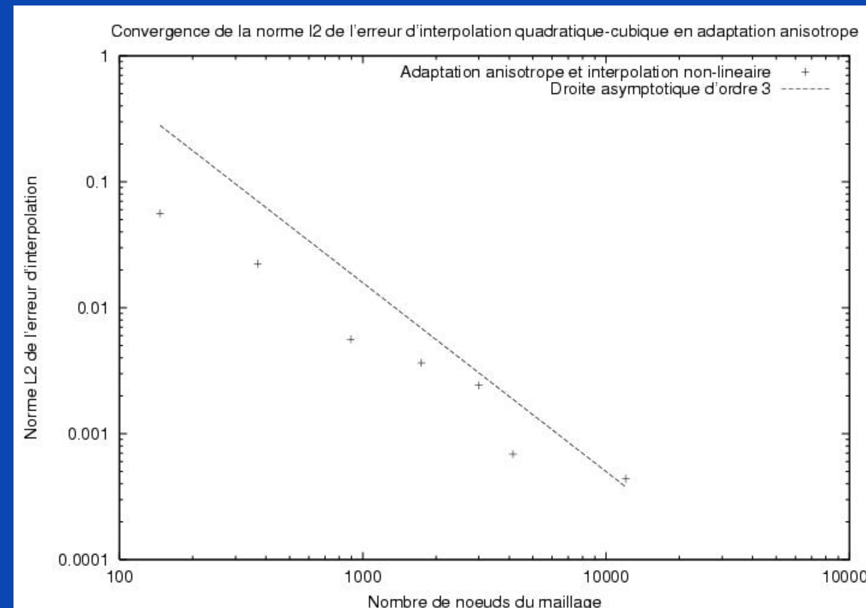
Illustration of the barrier lemma on a couple of two Heavyside functions

A vertical one and an horizontal one.

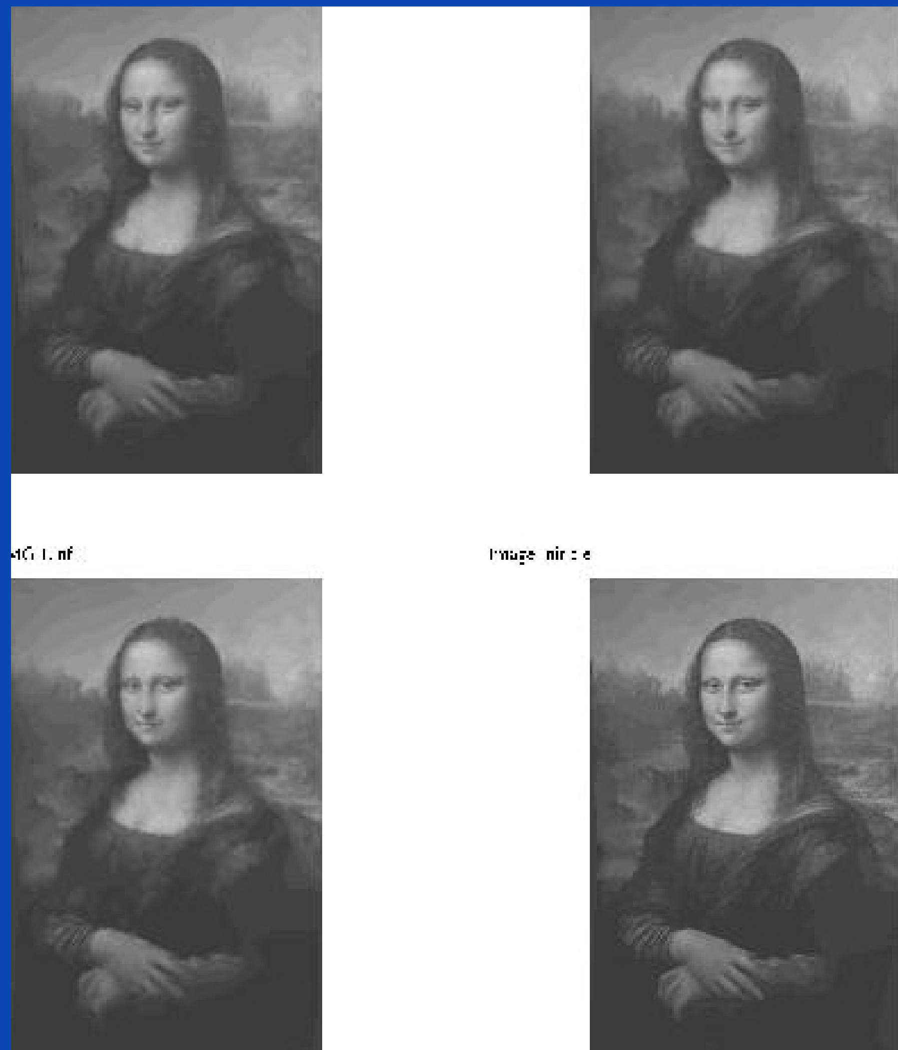








Isotropic : 1st order, anisotropic : 2nd order accuracy

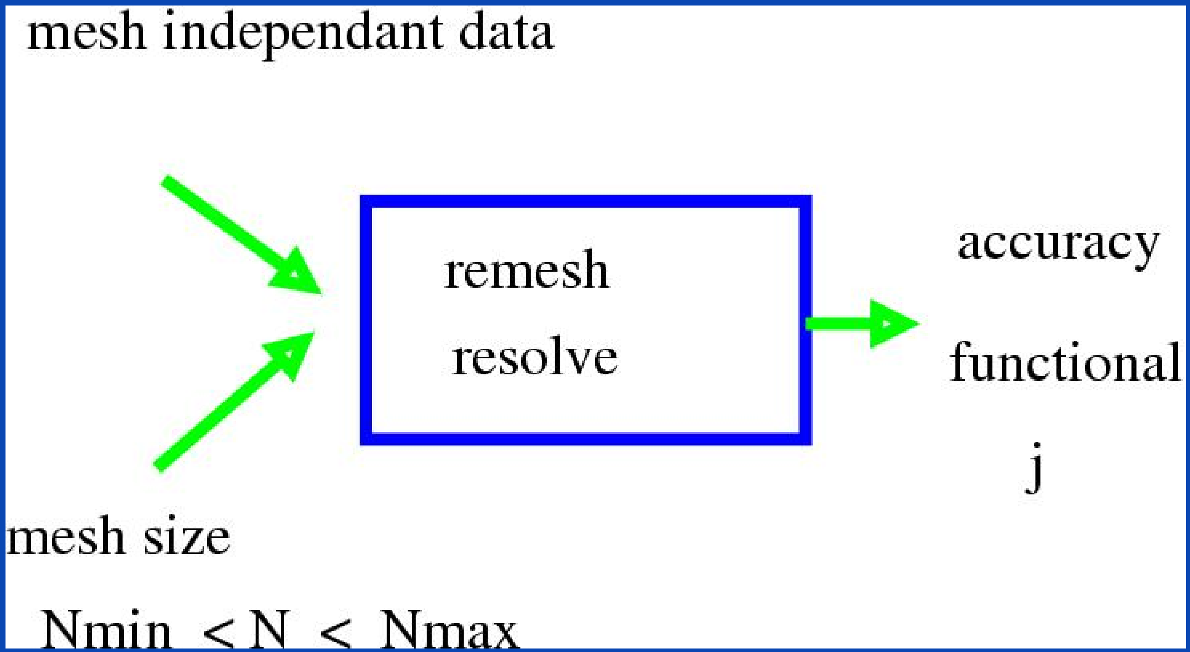


Anisotropic image compression

1. EDP: PROBLEM POSITION

We consider the research of an ideal "best mesh":

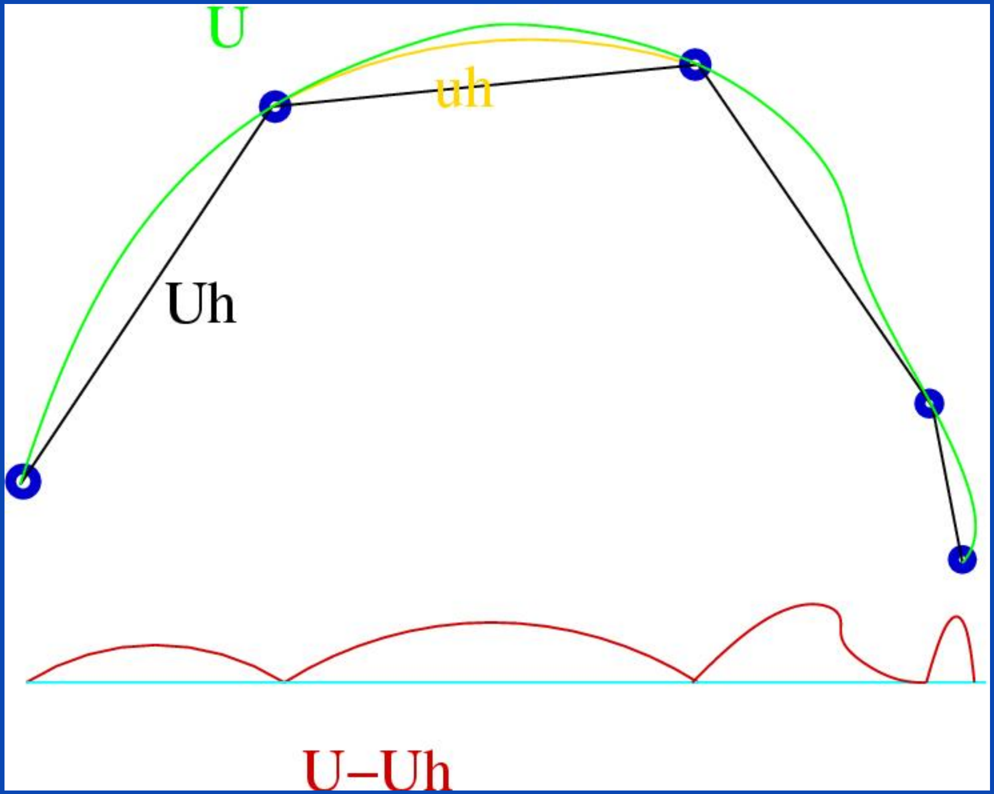
- **best**: for a specified criterion/cost function



- **ideal**: we are not interested by the mesh we start with. We want specify the best mesh. This mesh is of perfect quality. We build it rather independantly of the initial mesh.

2. THE OPTIMAL METRIC FOR INTERPOLATION

2.1 Continuous piecewise-P1 interpolation



Local error/Pollution error Babuska.

2.2 Ideal mesh

We modelize a mesh as a **continuous medium**, with an anisotropic property, the **local metric** (*):

$$\mathcal{M}_{x,y} = \mathcal{R}_{\mathcal{M}}^{-1} \begin{pmatrix} (m_\xi)^{-2} & 0 \\ 0 & (m_\eta)^{-2} \end{pmatrix} \mathcal{R}_{\mathcal{M}} \ ,$$

(*)(George, Hecht,..., Fortin, Habashi,...)

and a **number of nodes**:

$$N_{\mathcal{M}} \quad = \quad \int m_\xi^{-1} \, m_\eta^{-1} \, dxdy.$$

Continuous metrics method for P1 interpolation(2D)

For any $\mathcal{M}$, any function u: **local P1-interpolation error**:

$$\mathcal{E}_{\mathcal{M}} \quad = \quad \int \left(|\frac{\partial^2 u}{\partial \xi^2}|.m_\xi^2 + |\frac{\partial^2 u}{\partial \eta^2}|.m_\eta^2 \right)^2 \; dxdy$$

where ξ and η are directions of diagonalization of the Hessian of u.

Discontinuous case:

use $(u(\xi+\delta,\eta) - 2u(\xi,\eta) + u(\xi-\delta,\eta))/\delta^2$, bounded in $L^{\frac{1}{2}}$.

Optimal metric problem:

$\min_{\mathcal{M}} \;\; \mathcal{E}_{\mathcal{M}}$ under the constraint $N_{\mathcal{M}} = N$.

Solving the optimal problem

Step 1: Pointwise optimization

Given at a point a mesh density $d = (m_\xi m_\eta)^{-1}$, the optimal direction and strength of stretching give:
- optimal stretching direction: $\mathcal{R}_\mathcal{M} = \mathcal{R}_u$,
- optimal stretching strength: $e = m_\xi / m_\eta = (|u_{\eta\eta}|/|u_{\xi\xi}|)^{1/2}$.

Step 2: constrained global optimization

$$\min_d \ \mathcal{E}_d = \int \left(d^{-1} e |\frac{\partial^2 u}{\partial \xi^2}| + d^{-1} e^{-1} |\frac{\partial^2 u}{\partial \eta^2}| \right)^2 dxdy$$

under the constraint $\int d \, dxdy = N.$

2.3 Optimality

$$\mathcal{M}_{opt} = \frac{C}{N}\,\mathcal{R}^{-1} \begin{pmatrix} |\frac{\partial^2 u}{\partial \eta^2}|^{-5/6} |\frac{\partial^2 u}{\partial \xi^2}|^{1/6} & 0 \\ 0 & |\frac{\partial^2 u}{\partial \xi^2}|^{-5/6} |\frac{\partial^2 u}{\partial \eta^2}|^{1/6} \end{pmatrix} \mathcal{R}\ .$$

with:

$$C \;=\; \int \left(|\frac{\partial^2 u}{\partial \xi^2}| . |\frac{\partial^2 u}{\partial \eta^2}| \right)^{\frac{2}{6}} dxdy\ .$$

Minimal error: $\mathcal{E}_{opt} \;=\; \frac{4C^2}{\mathbf{N^2}} \int \left(|\frac{\partial^2 u}{\partial \xi^2}|^{\mathbf{1/3}} |\frac{\partial^2 u}{\partial \eta^2}|^{\mathbf{1/3}} \right) dxdy$

2.4 Convergence properties

Conv. order	Isotropic	Anisotropic
Theory	$\leq$ 1 (*)	≤ 2
Optimal Metric Theory	1	2
Optimal Metric Num. exp. Heavyside 2D	1	2

(*) Coudière-Dervieux-Leservoisier-Palmerio, 2001

3. EDP's

2.1 A short review, type of errors

a priori,.. Babuska pollution ... a posteriori...functionals..

2.1 P1 interpolation and exactness

2.2 Error system

2.3 A minimum problem

2.4 A paradox...

2.5 Truncation error model

Orientation:

Which kind of accuracy is exactly expected?

- skin quantities,

- accuracy in automated processes: optimal design.

Choice of a functional, introduction of two adjoints, w,w_h.

Giles, Pierce, Suli, Becker, Rannacher,Venditi, Darmofal,...

2.3. PARTIAL DIFFERENTIAL EQUATIONS

$$j(\mathcal{M},\gamma) \quad = \quad J(\mathcal{M},\gamma,W(\mathcal{M},\gamma))\ ,\ \ \Psi_{State}(\mathcal{M},\gamma,W(\mathcal{M},\gamma)) = 0\ .$$

Typical example : $\mathcal{M}$: **mesh** , γ: **aircraft shape**, “State” system is Navier-Stokes.

- Minimize the functional with respect to shape γ,

- Minimize **the error on functional** with respect to metric $\mathcal{M}$.

Mesh optimization strategy

$$\text{with} \quad \Psi_{Euler}(\gamma, W) = 0$$

$$\bar{\mathcal{M}} \quad = \quad \text{ArgMin } \bar{\mathbf{J}}(\mathcal{M}, \gamma, W^{exact} - W)$$

$$\text{with} \quad \Psi_{ERROR}(\mathcal{M}, \gamma, W^{exact} - W) = 0.$$

$$\bar{\mathbf{J}}(W^{exact} - W(\mathcal{M})) \quad = \quad |J_{exacte}(W^{exact})'.(W^{exact} - W(\mathcal{M}))|^2$$

Approximation error splitting

- $\mathcal{P}_1$-continuous, vertex-centered approximations,

$$\Pi_{\mathcal{M}} : H^k(\Omega) \;\rightarrow\; V_h \;=\; \{v,\text{ continue, } \; P_1 \text{ by element } \}$$

$$W^{exact} - W(\mathcal{M}) \;\;=\;\; W^{exact} - \Pi_{\mathcal{M}} W^{exact} + \Pi_{\mathcal{M}} W^{exact} - W(\mathcal{M})$$

$W^{exact} - \Pi_{\mathcal{M}} W^{exact}$ is interpolation error

$\Pi_{\mathcal{M}} W^{exact} - W(\mathcal{M})$ is the implicit error, solution of a discrete system.

Implicit error : elliptic case

$$< \nabla(u), \nabla\phi > \;=\; \int f\phi \, dx \; .$$

$$< \nabla(u_h), \nabla\Pi_h\phi > \;=\; \int f\phi \, dx \; .$$

$$< \nabla(u_h - \Pi_h u), \nabla\Pi_h\phi > \;=\; < \nabla(u - \Pi_h u), \nabla\Pi_h\phi >$$

Lemma : We assume that the continuous solution u is in $\mathcal{C}^3(\bar{\Omega})$ and that the continuous mesh size m is in $\mathcal{C}^2(\bar{\Omega})$. Then, for any function ϕ of $\mathcal{D}_3(\Omega)$, we have

$$\int_\Omega \frac{\partial(u - \Pi_h u)}{\partial x}\frac{\partial \Pi_h \phi}{\partial x} dM \; + \; \int_\Omega \frac{\partial(u - \Pi_h u)}{\partial y}\frac{\partial \Pi_h \phi}{\partial y} dM$$
$$= \; h^2 \int_\Omega g'(m)\phi dM \; + \; O_\phi(h^3) \tag{1}$$

$g'(m) \ = \ g_1'(m) + g_2'(m)$:

$$
\begin{aligned}
(g_1'(m),\phi) \ &= \ -\frac{3}{48}\int_\Omega \phi\frac{\partial}{\partial y}\left(m^2\frac{\partial^3 u}{\partial x\partial y^2}\right) dM \\
&\quad +\frac{1}{48}\int_\Omega \phi\frac{\partial}{\partial x}\left(m^2\frac{\partial^3 u}{\partial x^3}\right) dM
\end{aligned}
$$

$$
\begin{aligned}
(g_2'(m),\phi) \ &= \ -\frac{1}{4}\int_\Omega \phi\frac{\partial}{\partial y}\left(\frac{m^2}{6}\frac{\partial^3 u}{\partial x^2\partial y}\right) + \phi\frac{\partial^2}{\partial y^2}\left(m^2\frac{\partial^2 u}{\partial x^2}\right) dM \\
&\quad +\frac{3}{24}\int_\Omega \phi\frac{\partial}{\partial y}\left(m^2\frac{\partial^3 u}{\partial y^3}\right) + \phi\frac{\partial^2}{\partial y^2}\left(m^2\frac{\partial^2 u}{\partial y^2}\right) dM \ .
\end{aligned}
$$

Remark: H^2 pivot space for gradient with respect to m.

Second formulation:

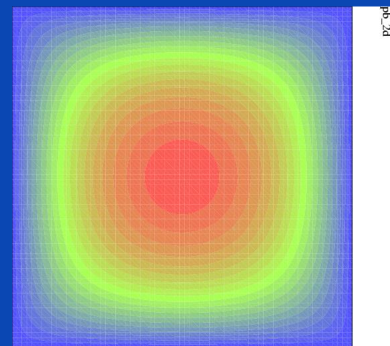
$$\tilde{E}_m \in H_0^1(\Omega), \text{ and } \left(\nabla \tilde{E}_m, \nabla \phi\right) = (g'(m), \phi) \quad \forall \phi \in H_0^1(\Omega) \ .$$

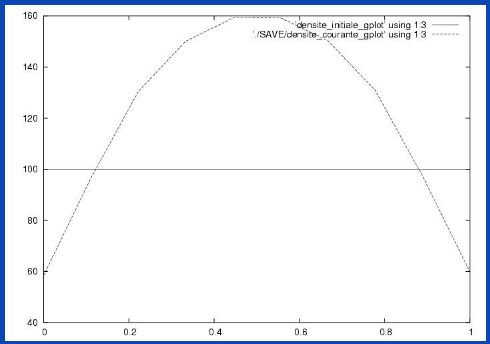
Since $g(m)$ is smooth. So is $\tilde{E}_{\bar{m}}$.

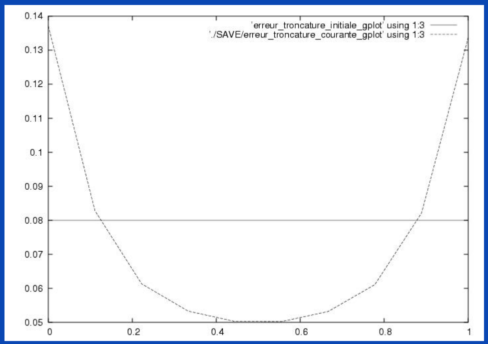
$$\begin{aligned} \int_\Omega \nabla u_h \nabla \Pi_h \phi dM \quad = & \\ & \int_\Omega \nabla u \nabla \phi dM \\ & + \ h^2 \ \int_\Omega \nabla \tilde{E}_m \nabla \phi dM \\ & + \ h^2 \ (g''(m,u), \phi) \ + \ O_\phi(h^3) \ . \end{aligned}$$

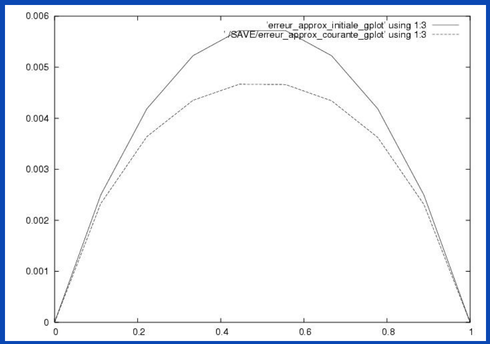
where g'' a non-discrete smooth term involving $\tilde{E}_m$ implicit terms and local terms.

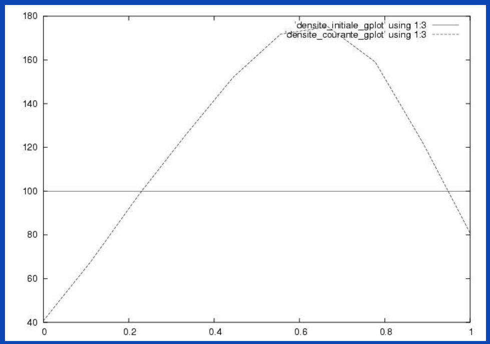
An illustration: Dirichlet problem in a square

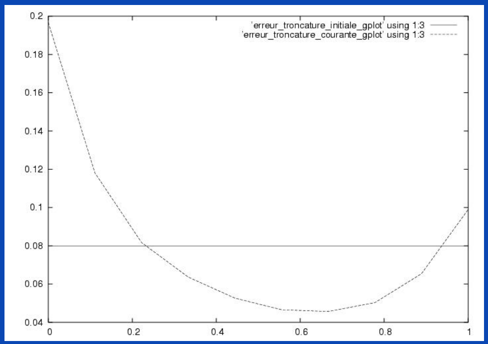


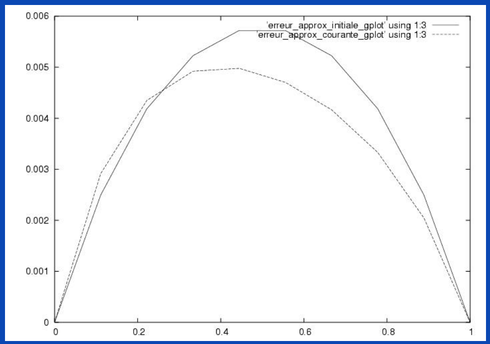












Towards the anisotropic case:

$$\Delta x = m_\xi cos(\theta) + m_\eta sin(\theta) \qquad \Delta y = -m_\xi sin(\theta) + m_\eta cos(\theta)$$

where m_ξ,m_η,θ are functions of (x,y). We have to admit that the space derivatives of these can be neglected during the pointwise optimization:

Step 1: Pointwise optimization

Given at a point a mesh density $d = (m_\xi m_\eta)^{-1}$, find **numerically** at each node the:
- optimal stretching direction,
- optimal stretching strength.

Build the modified error model based on these outputs.

Step 2: constrained global optimization

Minimize the modified error model with respect to mesh density.

CONCLUSIONS

We have specified a mesh adaptation issue as an optimal control problem, where we can minimize a specified error term.

For this we have analysed and then remodelised in a continuous context the approximation error of the numerical model.

Post-evaluation of the method can rely on numerical convergence degree.

This approach has a good potential to propose a rather natural formulation of isotropic and anisotropic mesh adaption.