# MAIDESC

# INRIA-ROCQ+SOPHIA Contribution to MAIDESC

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# An a priori corrector

$$-\Delta u = f \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega$$

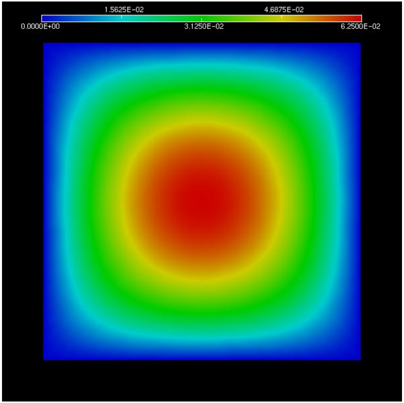
$$\int_\Omega \nabla(\Pi_h u - u_h) \cdot \nabla \phi_h \, \mathrm{d}\Omega = \int_\Omega \nabla \phi_h \nabla(\Pi_h u - u) \, \mathrm{d}\Omega$$
$$= \sum_{\partial T_{ij}} \nabla(\phi_h|_{T_i} - \phi_h|_{T_j}) \cdot \mathbf{n}_{ij} \int_{\partial T_{ij}} (\Pi_h u - u) \, \mathrm{d}\sigma$$

Approximate $\Pi_h u - u \approx H(u) \cdot \delta X \cdot \delta X$ with a superconvergent approximation of the Hessian of $u_h$, obtain $u_h' \approx \Pi_h u - u_h$:

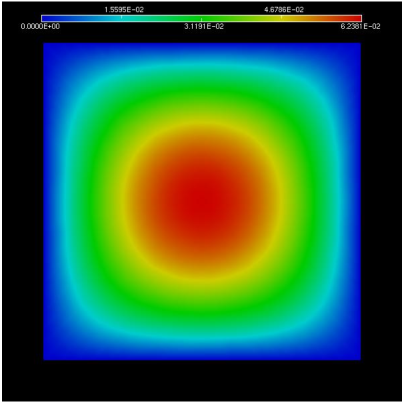
$$\int_\Omega \nabla u_h' \cdot \nabla \phi_h \, \mathrm{d}\Omega = \sum_{\partial T_{ij}} \nabla(\phi_h|_{T_i} - \phi_h|_{T_j}) \cdot \mathbf{n}_{ij} \int_{\partial T_{ij}} H_h(u_h) \cdot \delta X_{\mathscr{M}} \cdot \delta X_{\mathscr{M}} \, \mathrm{d}\sigma.$$

# An a priori corrector

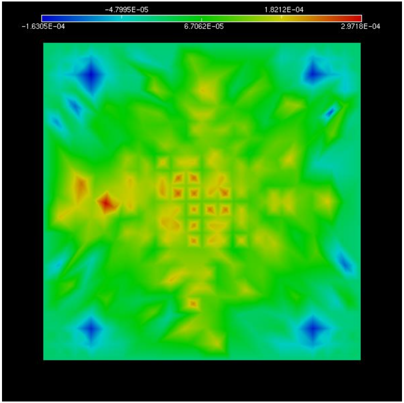
Example: $u(x) = x(1-x)y(1-y)$, 1600 vertices




$\Pi_h u$
values 0 to 6.236 $10^{-2}$

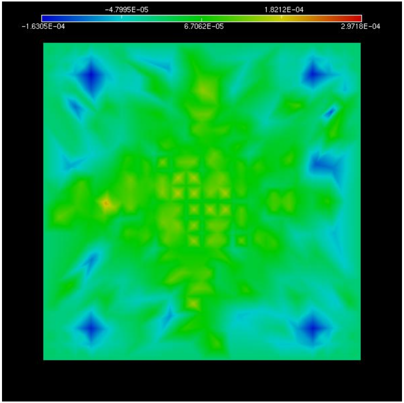



$u_h$
values 0 to 6.236 $10^{-2}$




$\Pi_h u - u_h$

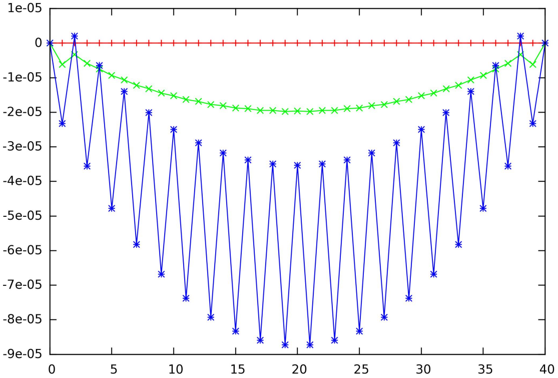
values $-1.63\ 10^{-4}$ to $2.97\ 10^{-4}$




$u'_h$

values $-1.63\ 10^{-4}$ to $2.97\ 10^{-4}$

# An a priori corrector




In red: $\Pi_h u - \Pi_h u = 0$, in blue: $\Pi_h u - u_h$, in green $\Pi_h u - (u_h + u'_h)$

The corrector performs 70% of its work (for a mesh of 1600 vertices with English flag topology).

# Metric-based mesh optimization

Interpolation error optimization: Hessian-based methods

$$\mathcal{M}_{\mathbf{L}^p} = \mathcal{K}_p(1,u) \;\text{ with }\; \mathcal{K}_p(k,u) = D_{\mathbf{L}^p} \left(\det|kH_u|\right)^{\frac{-1}{2p+2}} |H_u|$$

$$\text{and }\; D_{\mathbf{L}^p} = N^{\frac{2}{2}} \left( \int_\Omega (\det|kH_u|)^{\frac{p}{2p+2}} \right)^{-\frac{2}{2}}, \quad (k=1).$$

Scalar output error optimization: Goal-oriented methods

$$j_{goal}(\mathcal{M}) = (g, \Pi_{\mathcal{M}} u - u_{\mathcal{M}}). \qquad u^*_{g,\mathcal{M}} = \bar{A}^{-*}_{\mathcal{M}} g$$

$$\mathcal{M}_{opt,\mathcal{M}_0} = \mathcal{K}_1(\rho(H(u^*_g), u).$$

Approximation error optimization: Norm-oriented methods

# Norm-oriented mesh optimization

Minimize $j(\mathcal{M}) = |\Pi_{\mathcal{M}} u - u_{\mathcal{M}}|^2$ with respect to the metric $\mathcal{M}$.

$$j'(\mathcal{M}).\delta\mathcal{M} = (\Pi_{\mathcal{M}} u - u_{\mathcal{M}}, \frac{\partial}{\partial \mathcal{M}}(\Pi_{\mathcal{M}} u - u_{\mathcal{M}}).\delta\mathcal{M}).$$

Neglecting high-order terms:

$$\frac{\partial}{\partial \mathcal{M}}(\Pi_{\mathcal{M}} u - u_{\mathcal{M}}).\delta\mathcal{M} \approx \bar{A}_{\mathcal{M}}^{-1} A_{\mathcal{M}} \frac{\partial}{\partial \mathcal{M}}(\Pi_{\mathcal{M}} u - u).\delta\mathcal{M}. \qquad (1)$$

Replacing $\Pi_{\mathcal{M}} u - u_{\mathcal{M}}$ by $u'_{\mathcal{M}} = A_{\mathcal{M}}^{-1} A_{\mathcal{M}} (\Pi_{\mathcal{M}} u - u)$ ,

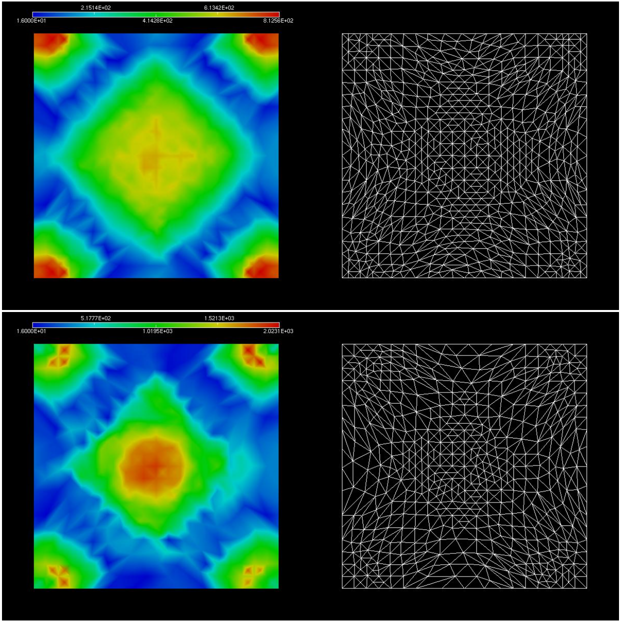
$$\mathcal{M}_{opt,norm} = \mathcal{K}_1(A^*_{\mathcal{M}} \bar{A}_{\mathcal{M}}^{-*} \bar{A}_{\mathcal{M}}^{-1} A_{\mathcal{M}} (\Pi_{\mathcal{M}} u - u), u).$$

# Norm-oriented metric optimization

Minimize $j(\mathcal{M}) = |\Pi_{\mathcal{M}} u - u_{\mathcal{M}}|$ with respect to the metric $\mathcal{M}$.

- Compute corrector $u_{\mathcal{M}}{}'$,
- compute adjoint $u^* = A^{-*} u_{\mathcal{M}}{}'$,
- compute $\bar{H} = H_{\mathcal{M}}(u_{\mathcal{M}})$,
- minimize $|\rho(H(u^*)\bar{H} \cdot \delta X_{\mathcal{M}} \cdot \delta X_{\mathcal{M}}|$ with respect to $\mathcal{M}$,
- reiterate until convergence.

# Hessian-based versus norm-oriented



# CONCLUSIONS

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