Mesh adaptation for high order finite elements spaces

Olivier Coulaud, Adrien Loseille

INRIA, Paris-Saclay

P_k adaptation

Notations

- $\Omega \subset \mathbb{R}^n$, $n = 2, 3$
- u exact solution
- u_h numerical solution
- Π_k projection operator on the set of polynomial of degree k

Numerical error

$$\|u - \Pi_k u_h\|_{L^p} \leq \underbrace{\|\Pi_k u - \Pi_k u_h\|_{L^p}}_{\text{implicit erreur}} + \underbrace{\|u - \Pi_k u\|_{L^p}}_{\text{interpolation error}}$$

P_k adaptation problem

Find the **best mesh** which minimizes the P_k interpolation error of $\mathbf{u}$

[Tam et al., CMAME 2000], [Pain et al., CMAME 2001], [Picasso, SIAMJSC 2003], [Formaggia et al., ANM 2004], [Bottasso, IJNME 2004], [Li et al., CMAME 2005], [Frey and Alauzet, CMAME 2005], [Gruau and Coupez, CMAME 2005], [Huang, JCP 2005], [Compere et al., 2007], ...

P_1 mesh adaptation : Hessian based methods

[Loseille et al., AIAA 2007], [Alauzet, IJNMF 2008], [Loseille and Alauzet, IMR 2009]

- Optimal control of the interpolation error in L^p norm:

$$\|u - \Pi_1 u\|_{L^p(\Omega)} \leq C\,|\Omega|\,\|H_u\|_{L^p(\Omega)}$$

- From H_u, one deduces a metric space $\mathcal{M}_{L^p}(H_u) \implies P_1$ adapted mesh
- Provides Anisotropic meshes

Outline

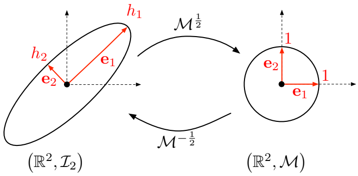
1. **Recalls about P_1 adaptation**
   - **Metric space and unit mesh**
   - **P_1 adaptation: Hessian based methods**
2. Local error estimate: the log-simplex approach
3. Optimal Metric space
4. Numerical applications
   - Smooth analytical functions
   - High order surface mesh generation

Main idea: change geometric quantities

[George, Hecht and Vallet., Adv. Eng. Software 1991]

Fundamental concept: The notion of **metric** and Riemannian metric space

- **Euclidean metric space**
  - $\mathcal{M}$ symmetric definite positive matrix
  - Scalar product $\langle \mathbf{x}\,,\,\mathbf{y}\rangle_{\mathcal{M}} = {}^t\mathbf{x}\mathcal{M}\mathbf{y}$
  - Length $\ell_{\mathcal{M}}(\mathbf{a},\mathbf{b}) = \sqrt{{}^t\mathbf{ab}\ \mathcal{M}\ \mathbf{ab}}$
  - Unit ball $\mathcal{B}_{\mathcal{M}}$
    - ellipse in $\mathbb{R}^2$
    - ellipsoid in $\mathbb{R}^3$

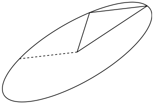


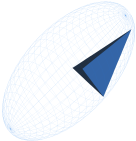
- **Riemannian metric space** $\mathbf{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$ continuous in Ω

Unit Mesh

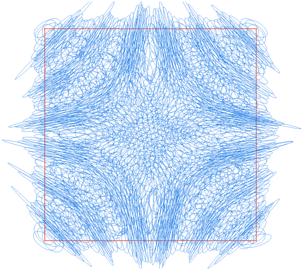
Fundamental concept: Generate a **unit mesh** w.r.t $\mathbf{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$

$$\forall \mathbf{e},\ \ell_{\mathcal{M}}(\mathbf{e}) \approx 1 \text{ and } \forall K,\ |K|_{\mathcal{M}} \approx \begin{cases} \sqrt{3}/4 & \text{in 2D} \\ \sqrt{2}/12 & \text{in 3D} \end{cases}$$

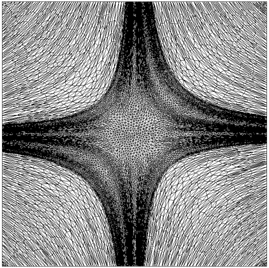




Inputs $(\mathcal{H}_0, \mathcal{M}_i)_{i\in\mathcal{H}}$

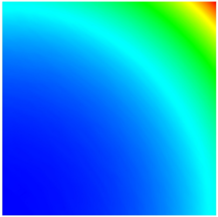


Output $\mathcal{H}$

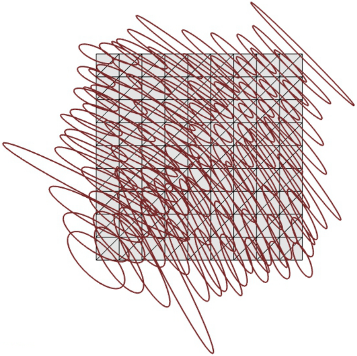


P_1 adaptation: the hessian based methods

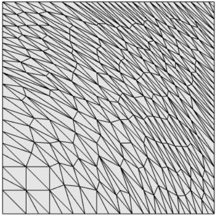
Solution u on Ω



Metric field on a mesh $\mathcal{H}$ of Ω
$\mathcal{M}(x_i) = C(x_i)\,|H_u|\,(x_i)$



Unit Mesh with respect to $\mathbf{M}$



P_k adaptation problem

What is the continuous metric field $\mathbf{M} = (\mathcal{M}(x))_{x\in\Omega}$ minimizing the P_k interpolation error ?

Outline

1. Recalls about P_1 adaptation
   - Metric space and unit mesh
   - P_1 adaptation: Hessian based methods
2. **Local error estimate: the log-simplex approach**
3. Optimal Metric space
4. Numerical applications
   - Smooth analytical functions
   - High order surface mesh generation

Initial problem

Find $\mathcal{H}_{opt}$ having N vertices such that

$$\mathcal{H}_{opt}(u) = \text{Arg}\min_{\mathcal{H}} \|u - \Pi_k u\|_{\mathcal{H},\mathbf{L}^p(\Omega)}$$

Interpolation error estimate

$$|u(x) - \Pi_k u(x)| \leq C \left| d^{(k+1)} u(x_0)(x - x_0) \right| + o\left(\|x - x_0\|_2^{k+1} \right)$$

P_k error model

[Cao, SIAM 2007], [Hecht, J. Comput. Appl. Math 2014], [Mirebeau, Num. Math. 2011]

If we find a continuous metric $\mathbf{Q} = (\mathcal{Q}(\mathbf{x}))_{\mathbf{x}\in\mathbf{\Omega}}$ such that

$$({}^t y \mathcal{Q}(x_0) y)^{\frac{k+1}{2}} \geq d^{(k+1)} u(x_0)(y), \quad \text{for all} \quad x_0 \in \Omega, y \in \mathbb{R}^n$$

then one has

$$|u(x) - \Pi_k u(x)| \leq C \left({}^t(x - x_0) \mathcal{Q}(x_0)(x - x_0) \right)^{\frac{k+1}{2}} + o\left(\|x - x_0\|_2^{k+1} \right)$$

Remark

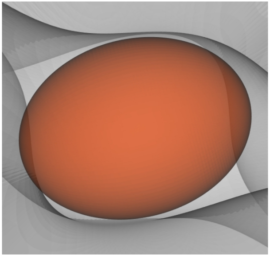
In the P_1 case, $\mathcal{Q} = |H_u|$

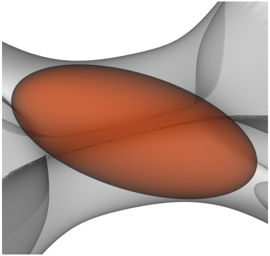
The problem can finally be reduced to the following one

Local problem

For p an homogeneous polynomial of degree $k+1$, find a metric $\mathcal{Q}$ such that

- $x \in \mathcal{B}_{\mathcal{Q}} \Rightarrow |p(x)| < 1$
- $\mathcal{B}_{\mathcal{Q}}$ has the largest volume $\Leftrightarrow \det(\mathcal{Q})$ is minimal



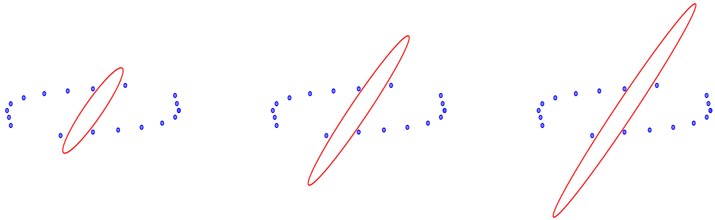


Local discrete problem

For a set $\{x_1, ..., x_m\}$ such that $p(x_i) = 1$, find a metric $\mathcal{Q}$ such that

- ${}^t x_i \mathcal{Q} x_i \geq 1$, for all $x_i \in \{x_1, ..., x_m\}$
- $\det(\mathcal{Q})$ is minimal

This problem is always ill-posed



Log metric

For $\mathcal{Q} = R\Lambda^t R$, let $\mathcal{L} = log(\mathcal{Q}) = R\log(\Lambda)^t R$. One has

$$^t x_i \mathcal{Q} x_i = {}^t x_i \exp(\mathcal{L}) x_i \geq 1.$$

Via convexity inequalities, one has

$$\frac{^t x_i \exp(\mathcal{L}) x_i}{\|x_i\|^2} \geq \exp\left(\frac{^t x_i \mathcal{L} x_i}{\|x_i\|^2}\right), \text{ for all } i \in \{1, ..., m\}.$$

Furthermore $\det(\mathcal{Q}) = \exp(\mathrm{trace}(\mathcal{L}))$.

Log discrete problem

For a set $\{x_1, ..., x_m\}$ such that $p(x_i) = 1$, find a symmetric matrix $\mathcal{L}$ such that

- $^t x_i \mathcal{L} x_i \geq -\|x_i\|^2 \log\left(\|x_i\|^2\right)$, for all $i \in \{1, ..., m\}$
- $\mathrm{trace}(\mathcal{L})$ is minimal

Advantages of the *log*−approximation

- The problem is linear in $\mathcal{L}$
  $\Rightarrow$ solved by a linear simplex algorithm
- It has solutions if the points x_i are well chosen

Proposition

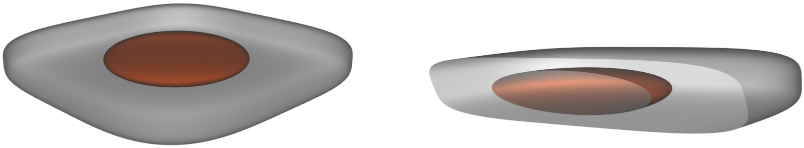
Let $(x_1, ..., x_n)$ be an orthogonal basis of $\mathbb{R}^n$, $n = 2, 3$. If $\mathcal{L}$ is a symmetric matrix such that

$$ {}^t x_i \mathcal{L} x_i \geq -\|x_i\|^2 \log\left(\|x_i\|^2\right), \textit{ for all } i \in \{1, ..., n\}, $$

then there exists $C(x_1, ..., x_n) \in \mathbb{R}$ such that

$$ \textit{trace}(\mathcal{L}) \geq C(x_1, ..., x_n). $$

How to recover the initial constraints ?

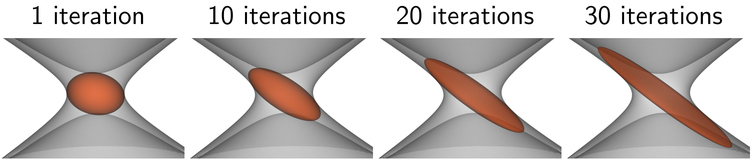


Iterative process

- choose a set $\{x_1, ...x_m\} \in \mathbb{R}^n$ such that $|p(x_i)| = 1$
- compute the optimal metric $\mathcal{Q}$ solution of the *log*−simplex problem
- perform the change of variable $y = \mathcal{Q}^{\frac{1}{2}} x$
- replace p by $p \circ \mathcal{Q}^{-\frac{1}{2}}$

$\Longrightarrow$ If $\mathcal{Q}$ converges, the log contraints and the initial ones are **equivalent**

How to deal with infinite branches ?



Polynomial reduction

[Borouchaki, George]

There exist symmetric matrices $\{H_{ijk}\}_{i+j+k=d-2}$ such that

$$p(x,y,z) = \sum_{i+j+k=d-2} x^i y^j z^k \left({}^t X H_{ijk} X\right), \quad X = (x,y,z)$$

We replace p by $q(x,y,z) = \sum_{i+j+k=d-2} \left|x^i y^j z^k\right| \left({}^t X \left|H_{ijk}\right| X\right)$.

Input

- Mesh $\mathcal{H}$ of Ω
- Solution u at the nodes of $\mathcal{H}$

Output

- Metric field $\mathbf{Q}$ at the vertices of $\mathcal{H}$

For each vertex $x_0 \in \mathcal{H}$

1. Compute $p = d^{k+1}u(x_0)$
   - On each tetrahedron, build u_k the interpolation of u of order k
   - Compute $d^k u_k$ on each tetrahedron
   - Differentiate $d^k u_k$ on the whole mesh by L^2-projection
2. Factorize p in order to get rid of the infinite branches
3. Perform the iterative log-simplex method with constraint points on the level set of $p \circ \mathcal{Q}^{-\frac{1}{2}}$ at each step

Outline

1. Recalls about P_1 adaptation
   - Metric space and unit mesh
   - P_1 adaptation: Hessian based methods
2. Local error estimate: the log-simplex approach
3. **Optimal Metric space**
4. Numerical applications
   - Smooth analytical functions
   - High order surface mesh generation

Continuous problem

- Given $\mathbf{Q} = (\mathcal{Q}(\mathbf{x}))_{\mathbf{x}\in\mathbf{\Omega}}$ such that

$$\left({}^t x Q(x_0) x\right)^{\frac{k+1}{2}} \geq d^{k+1} u(x_0)(x), \qquad \text{for all } x_0 \in \Omega,\ x \in \mathbb{R}^n,$$

- Find $\mathbf{M}_{\mathbf{L}^p} = (\mathcal{M}_{\mathbf{L}^p}(\mathbf{x}))_{\mathbf{x}\in\Omega}$ with complexity N which minimizes

$$E_{\mathbf{L}^p}(\mathbf{M}) = \left(\int_\Omega \left| \operatorname{trace}\left(\mathcal{M}^{-\frac{1}{2}}(x) \mathcal{Q}(x) \mathcal{M}^{-\frac{1}{2}}(x) \right) \right|^{\frac{p(k+1)}{2}} dx \right)^{\frac{1}{p}}.$$

$\Longrightarrow$ Solved by a calculus of variations

Remark

$$E_{\mathbf{L}^p}(\mathbf{M}) \simeq \| u - \Pi_k u \|_{\mathcal{H},\mathbf{L}^p(\Omega)},$$

for $\mathcal{H}$ a unit mesh with respect to $\mathbf{M}$.

Optimal metric for P_k adaptation

$$\mathcal{M}_{\mathbf{L^p}}^{k} = N^{\frac{2}{n}} \underbrace{\left(\int_\Omega (\det \mathcal{Q})^{\frac{p(k+1)}{2p(k+1)+2n}}\right)^{-\frac{2}{n}}}_{1} \underbrace{(\det \mathcal{Q})^{\frac{-1}{p(k+1)+n}}}_{2} \underbrace{\mathcal{Q}}_{3}$$

1 Global normalization: to reach the constraint complexity N

2 Local normalization: sensitivity to small solution variations, depends on $\mathbf{L}^p$ norm

3 Optimal directions and optimal lengths

Properties

- $E_{\mathbf{L^p}}(\mathbf{M}_{\mathbf{L^p}}^k) = 3^{\frac{k+1}{p(k+1)+2n}} N^{-\frac{k+1}{n}} \left(\int_\Omega \mathcal{Q}^{\frac{p(k+1)}{2p(k+1)+2n}}\right)^{\frac{p(k+1)+n}{3p}}$
- Asymptotic convergence $\quad E_{\mathbf{L^p}}(\mathbf{M}_{\mathbf{L^p}}^k) \leq \dfrac{Cst}{N^{\frac{k+1}{n}}}, \quad$ when $N \gg 1$

Remark

If $k = 1$, then $\mathbf{Q} = |H_u|$, with H_u the hessian matrix of u

Input

- Mesh $\mathcal{H}_{in}$ of Ω
- Solution u at the nodes of $\mathcal{H}_{in}$
- Complexity N

Output

- Adapted mesh $\mathcal{H}_{out}$

Do

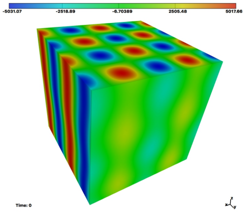
1. For each $x \in \mathcal{H}_{in}$, compute $d^{k+1}u(x)$
2. For each $x \in \mathcal{H}_{in}$, compute $\mathcal{Q}(x)$ which approximates $d^{k+1}u(x)$
3. Normalize $\mathbf{Q} = (\mathcal{Q}(\mathbf{x}))_{\mathbf{x}\in\mathcal{H}_{\mathbf{in}}}$ and get $\mathbf{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\mathcal{H}_{\mathbf{in}}}$ with complexity N
4. Remesh $\mathcal{H}_{in}$ and get $\mathcal{H}_{out}$, which is unit with respect to $\mathbf{M}$

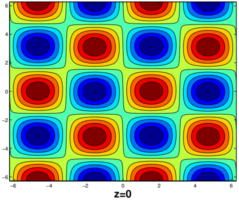
Outline

1. Recalls about P_1 adaptation
   - Metric space and unit mesh
   - P_1 adaptation: Hessian based methods
2. Local error estimate: the log-simplex approach
3. Optimal Metric space
4. **Numerical applications**
   - **Smooth analytical functions**
   - **High order surface mesh generation**

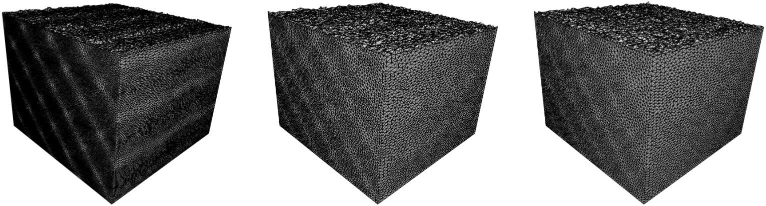
Numerical simulations: gyroide

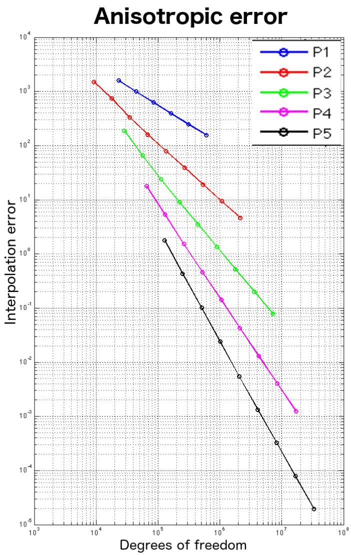
- $\Omega = [-2\pi, 2\pi]^3$
- $u = 5000\cos(x)\sin(y) + 500\cos(y)\sin(z) + 100\cos(z)\sin(x)$
  $-2\cos(2x)\cos(2y) + \cos(2y)\cos(2z) + 2\cos(2x)\cos(2z) - 10$

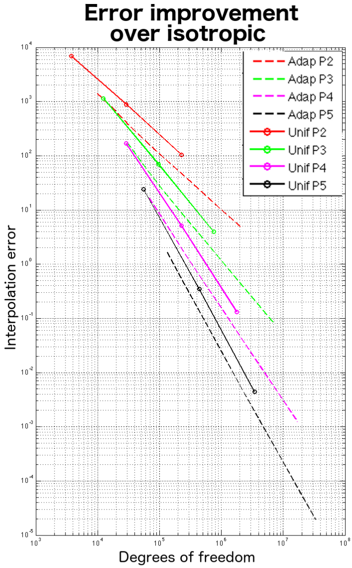




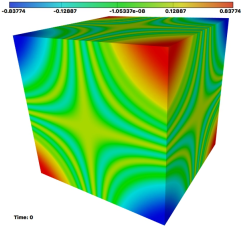
P_1, P_3 and P_5 adapted meshes

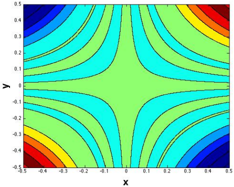



Anisotropic error
P1
P2
P3
P4
P5
Interpolation error
Degrees of freedom


Error improvement over isotropic
Adap P2
Adap P3
Adap P4
Adap P5
Unif P2
Unif P3
Unif P4
Unif P5
Interpolation error
Degrees of freedom

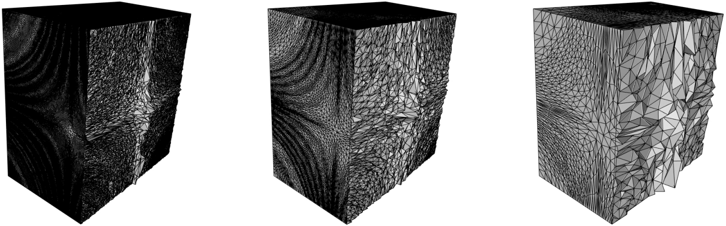
- $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^3$
- $u(x,y,z) = 8\,\|xyz\|_2^2 \sin\left(5\pi\,\|xyz\|_2^2\right)^4 + \frac{1}{10}\left(1 - sin(5\pi\,\|xyz\|_2^8\right)^8 cos\left(100\pi\,\|xyz\|_2^2\right)$



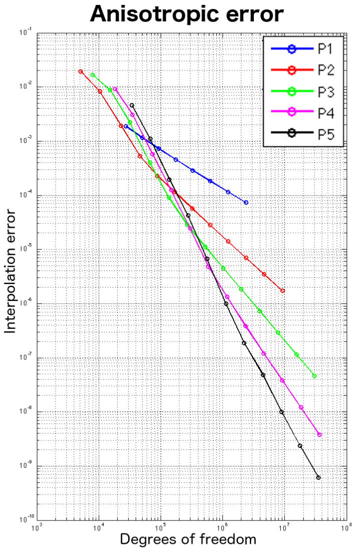


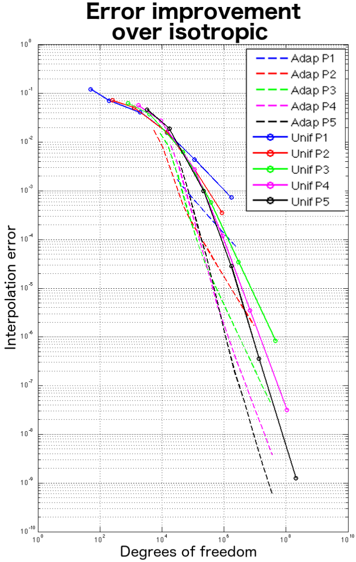
$x = 0.5$

P_1, P_2 and P_4 adapted meshes



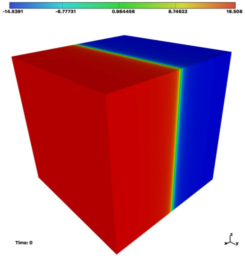
P_k interpolation error: various frequencies

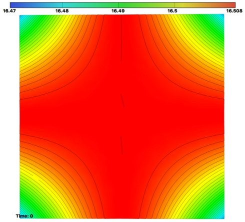




Numerical simulations: smooth shock

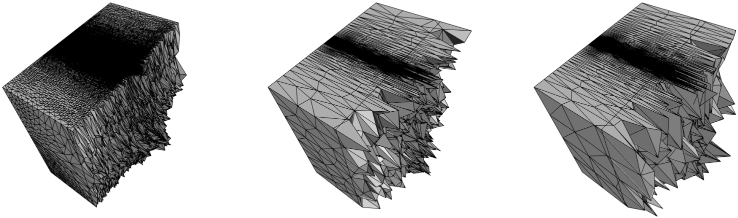
- $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right]^3$
- $u(x, y, z) = 10\,\mathrm{arctan}(100x) + \cos(yz)$



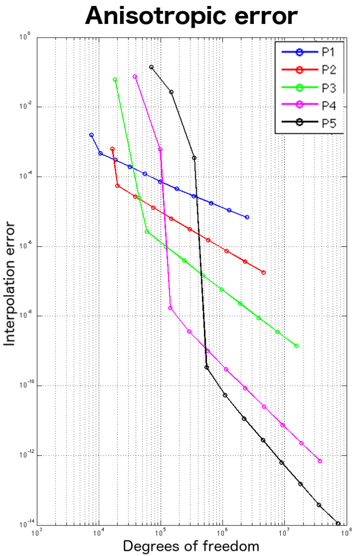


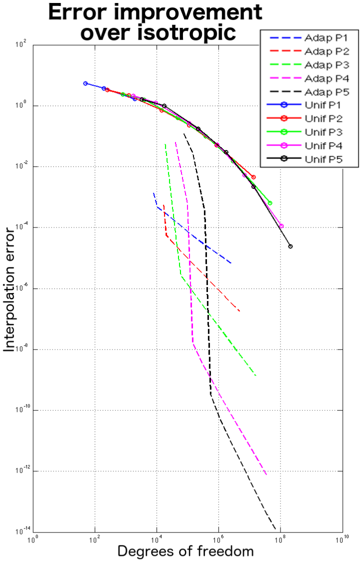
$x = 0.5$

P_1, P_3 and P_4 adapted meshes



P_k interpolation error: smooth shock





CPU times

The gyroide function

	P1	P2	P3	P4	P5
degrees of freedom	606 834	535 177	444 435	523 941	510 630
interpolation error	155.271	19.274	3.513	0.458	0.101
total CPU time (s)	264	82	31	26	24
derivatives (s)	1	14	8	12	12
metric field (s)	6	48	14	9	9
remeshing (s)	247	15	7	4	2

The function with various frequencies

	P1	P2	P3	P4	P5
degrees of freedom	2 374 794	2 376 164	1 989 277	2 329 110	2 220 443
interpolation error	7.2×10^{-5}	6.9×10^{-6}	1.8×10^{-6}	3.9×10^{-7}	1.8×10^{-7}
total CPU time (s)	365	604	153	120	115
derivatives (s)	2	102	37	40	45
metric field (s)	26	409	100	70	64
remeshing (s)	330	63	11	6	4

The smooth shock function

	P1	P2	P3	P4	P5
degrees of freedom	2 463 091	2 299 983	1 926 453	2 299 810	2 219 674
interpolation error	6.8×10^{-6}	3.7×10^{-7}	2.3×10^{-8}	8.5×10^{-11}	1.2×10^{-11}
total CPU time (s)	547	517	382	138	144
derivatives (s)	3	96	99	61	70
metric field (s)	19	327	228	61	66
remeshing (s)	501	62	31	11	7

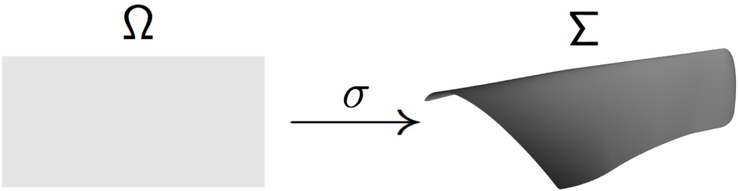
Outline

1. Recalls about P_1 adaptation
   - Metric space and unit mesh
   - P_1 adaptation: Hessian based methods
2. Local error estimate: the log-simplex approach
3. Optimal Metric space
4. **Numerical applications**
   - **Smooth analytical functions**
   - **High order surface mesh generation**

Notations

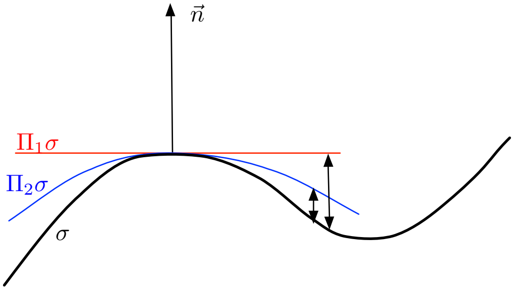
- $\Sigma \subset \mathbb{R}^3$ surface of $\mathbb{R}^3$
- $\Omega \subset \mathbb{R}^2$ parametrization space
- Surface parametrization $\sigma : \Omega \to \Sigma$
- The unit normal vector $n(u_0, v_0) = \dfrac{\partial_u \sigma \times \partial_v \sigma}{\|\partial_u \sigma \times \partial_v \sigma\|_2}(u_0, v_0)$

Surface parametrization



Error model

$$\left|(\sigma(u,v)-\Pi_k\sigma(u,v)).n(u_0,v_0)\right| \leq \left|d^{k+1}\sigma(u_0,v_0)(u-u_0,v-v_0).n(u_0,v_0)\right| + o\left(\|(u-u_0,v-v_0)\|_2^{k+1}\right)$$



P_k adaptation in Ω

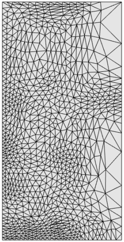
- For all $x_i \in \mathcal{H}$, obtain $\mathcal{M}(x_i)$ via the log-simplex algorithm performed with

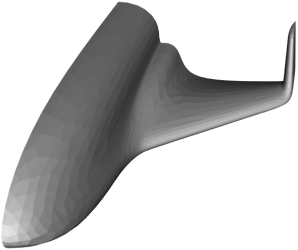
$$p(u,v) = d^{k+1}\sigma(x_i)(u,v).n(x_i)$$

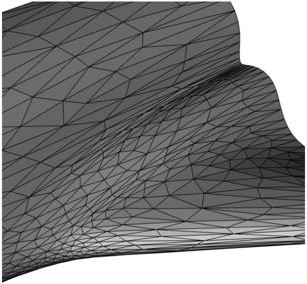
- Build a unit mesh $\mathcal{H}_{opt}$ with respect to $\mathbf{M} = (\mathcal{M}(\mathbf{x_i}))_{\mathbf{x_i}\in\mathcal{H}}$
- Map the vertices of $\mathcal{H}_{opt}$ and the interpolation nodes on Σ

High order surface mesh generation: space shuttle

P_1 **adaptation**: 967 vertices, 1804 triangles

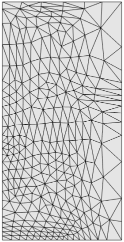


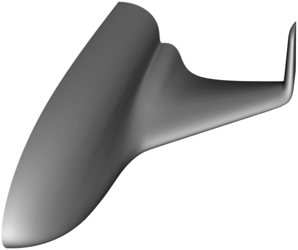


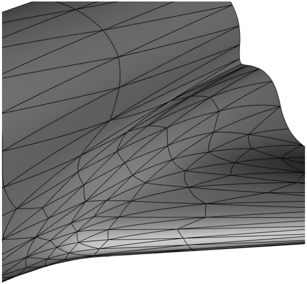


P_2 **adaptation**: 1130 vertices and nodes, 533 P2 triangles

Time: 0

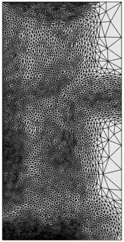


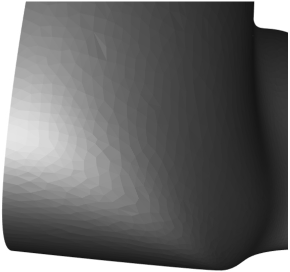




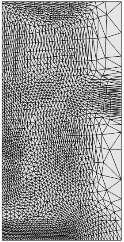
High order surface mesh generation: space shuttle

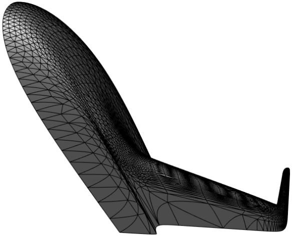
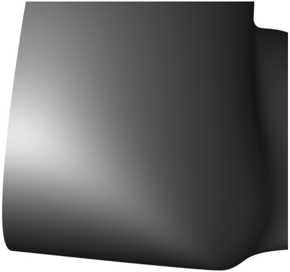
P_1 **adaptation**: 7389 vertices, 14415 triangles





P_2 **adaptation**: 8959 vertices and nodes, 4392 P2 triangles



Conclusions and Perspectives

Achievements

1. Theoretical extension of the Hessian methods for P_k adaptation
2. Numerical implementation for smooth analytical functions
3. Application to the generation of curved surface meshes

Perspectives

1. Treat the discontinuities of the interpolated function
2. Extend the Sylvester reduction to the 3 dimensional case
3. Consider curved meshes
4. Application to high-order resolution of PDE