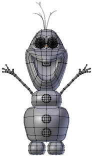


MAIDESC

Contribution INRIA-Sophia à Maidesc, M30

Éléonore Gauci[*+], Frédéric Alauzet[+], Alain Dervieux[*]

(*) INRIA , Sophia-Antipolis, France
(+) INRIA , Saclay, France

13 avril 2016

Overview

Proposal

T3-D3 (M18) : Space-time error for a third-order CENO scheme for Euler model, Theoretical report.
T3-D4 (M18) : Space-time error for a norm-oriented formulation, Theoretical report.
T5-D3 : Norm-oriented adaptive FMG for a 2D model. M12 Theoretical report, M42 Theory, implementation and numerical results (ATC1, ATC4).
Test case : ATC1, falling water column.

Task status

T3-D3-4, delivered.
T5-D3, delivered + extension with Lemma.
The ALE thematic of Montpellier was transfered to Sophia.

Theses

Proposal

Thesis 1 : co-advised by Rocquencourt and Sophia : High-order curved mesh adaptation for Computational Fluids Dynamics : error estimate, flow solver and mesh adaptation.
Thesis 3 : co-advised by Montpellier and Sophia : Estimates for ALE, optimal mesh for third order, parallel multi- rate time advancing.
Thesis 4 : Lemma-Sophia : Norm-oriented error estimates and FMG adaptation (Gautier Brethes, started October 2012, supported by Lemma and region PACA).
Thesis 5 : Rocquencourt Sophia : Goal-oriented third-order mesh adaption (Alexandre Carabias, started in ANR- ECINADS, end supported by INRIA proper resources).

Theses status

Thesis 1 : Tranformed to ALE analysis (Éléonore Gauci).
Thesis 3 : Parallel multirate time advancing (Emmanuelle Itam).
Thesis 4 : Norm-oriented, dec 2015 (Gautier Brèthes).
Thesis 5 : Third order, dec. 2014 (Alexandre Carabias).

Analysis of goal-oriented criterion for ALE

Plan

Formulation ALE
Local Transient Fixed-Point
Continuous ALE state equations
Time-discrete ALE state equations
Discrete ALE state equations
Error estimate.

Formulation ALE

- $\phi : \Omega_0 \times [0,T] \to \mathbb{R}^d, (\xi, t) \mapsto \phi(\xi, t)$
- $\Omega_{t_n+1} = \{x_{n+1} = \phi_n(x_n, t_n), x_n \in \Omega_{t_n}\}$

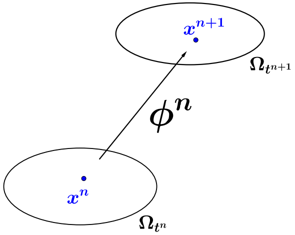


FIGURE: *Transformation* ϕ

$$Q_T = \bigcup_{t=0}^{t=T} \phi(\Omega_0, t) \times \{t\}.$$

Local (=Hessian-based) Transient Fixed-Point for deforming domain

For successively the k_{max} time sub-intervals $[t_k, t_{k+1}]$:

Do until convergence :

Step 1 : generate the $\mathscr{H}_k$ from the metric $\mathscr{M}_k$ and compute the discrete state $W_{\mathscr{M}}$ from t_k to t_{k+1} on the deforming meshes defined by deforming the $\mathscr{H}_k$ with the ALE mapping,

Step 2 : compute sensor $s_{\mathscr{M}} = s(u_{\mathscr{M}})$ and the n optimal metrics

$\mathscr{M}_{inter}^{opt,k} = \bigcap_{t=t^k}^{t^{k+1}} \mathscr{K}_2\Big(H_u(\phi_k(\mathbf{x}_k,t)\big[Jac(\phi_k)(\mathbf{x}_k,t)\big]^{\frac{1}{2}}\Big)$

or

$\mathscr{M}_{integr}^{opt,k} = \mathscr{K}_2\Big(\int_{t^k}^{t^{k+1}} H_u(\phi_k(\mathbf{x}_k,t)\big[Jac(\phi_k)(\mathbf{x}_k,t)\big]^{\frac{1}{2}} \quad dt\Big)$

Step 3 : $\mathscr{M}_k = \mathscr{M}_{inter/integr}^{opt,k}$

Go to step 1. Stop when fixed point is converged.

Next k :

Numerical example for Local Transient Fixed-Point for deforming domain

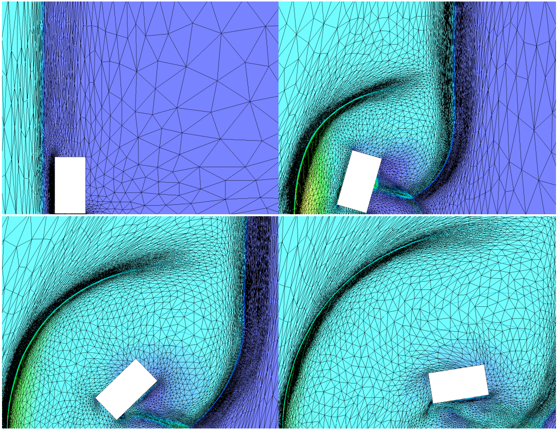


FIGURE: Application of the Fixed-Point algorithm to a blast wave inpinging an obstacle, Géraldine Olivier, Frédéric Alauzet

Equation-based criterion

We want to replace the Hessian criterion by a goal-oriented criterion.

The rest of the talk is devoted to the theoretical evaluation of a goal-oriented criterion for an Arbitrary-Lagrangian-Eulerian formulation of the unsteady Euler system.

Théorie

- Hessian $\Rightarrow$ cost function g.
- Minimize $\delta j_h = |j - j_h| = |(g, W) - (g, W_h)|$
- We introduce the adjoint state W^*
$$\left(\frac{\partial \Psi_{(h)}}{\partial W_{(h)}} \varphi_{(h)}, W^*_{(h)}\right) = (g, \varphi_{(h)})$$
- $\delta j_h \;\approx\; (W^*, \Psi_h(W) - \Psi(W))$
- To minimize $\delta j_h \Rightarrow$ *a priori* error estimate

ALE formulations of Euler equations

$$\Omega_t = \{x = \phi(\xi,t), \xi \in \Omega_{t_0}\}$$

$$\frac{1}{\mathscr{J}} \frac{\partial \mathscr{J} W}{\partial t} + \nabla_x \mathscr{F}(W) = 0 \qquad \text{in} \quad \Omega_t$$

où

$$\mathscr{J}(\xi,t) = det \frac{\partial \phi}{\partial \xi}(\xi,t)$$

et

$$\mathscr{F}(W) = \begin{pmatrix} \rho(\mathbf{u} - \dot{\phi}) \\ \rho(\mathbf{u} - \dot{\phi})u + p\mathbf{e}_x \\ \rho(\mathbf{u} - \dot{\phi})v + p\mathbf{e}_y \\ \rho(\mathbf{u} - \dot{\phi})E + \mathbf{u}p \end{pmatrix}.$$

Continuous state equations

ALE continuous formulation of Euler equations

$$\frac{1}{\mathscr{J}(\xi,t)}\frac{\partial\,\mathscr{J}\,\hat{W}}{\partial t}|_{\xi}(\xi,t)+\widehat{\nabla_{\mathbf{x}}\mathscr{F}(W)}(\xi,t)=0\quad\forall(\xi,t)\in\ \Omega_0\times[\mathbf{0},\mathbf{T}] \qquad (1)$$

in which

$$\mathscr{J}\ =(det\ \phi)(\xi,t)$$

$$\hat{W}(\xi,t)=W(\phi(\xi,t),t)$$

$$\widehat{\nabla_{\mathbf{x}}\mathscr{F}(W)}(\xi,t)=(\nabla_{\mathbf{x}}\mathscr{F}(W))(\phi(\xi,t),t)$$

are Lagrangian functions. Then :

$$\int_0^T\int_{\Omega_0}\hat{\varphi}\left(\frac{\partial\,\mathscr{J}\,\hat{W}}{\partial t}+\mathscr{J}\ \widehat{\nabla_{\mathbf{x}}\mathscr{F}(W)}\right)\mathrm{d}\xi\,\mathrm{d}t=0\quad\forall\ \hat{\varphi}\ \in\ \mathscr{H}^1(\Omega_0\times[0,T]). \qquad (2)$$

(...)$\Leftrightarrow$ (with $\hat{\varphi}(\xi,t)=\varphi(\phi(\xi,t),t)$)

$$\frac{d}{dt}\left(\int_{\Omega_t}\varphi(\mathbf{x},t)W(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\right)-\int_{\Omega_0}\hat{W}\,\mathscr{J}\,\frac{d\hat{\varphi}}{dt}\,\mathrm{d}\xi+\int_{\Omega_t}\varphi\ \nabla_{\mathbf{x}}\mathscr{F}(W)\mathrm{d}\mathbf{x}=0\quad\forall\ t.$$

Continuous state equations

We define the subspace $\mathscr{H}^1_{cst}(Q_T)$ of $\mathscr{H}^1(Q_T)$ having as elements the images by the mapping of time-constant functions :

$$\varphi \in \mathscr{H}^1_{cst}(Q_T) \quad \Leftrightarrow \quad \varphi \in \mathscr{H}^1(Q_T)\ ,\ d\hat{\varphi}/dt = 0\ \forall\, t.$$

$$\begin{aligned}
\text{Find } W \in \mathscr{H}^1(Q_T) & \qquad \text{such that } \forall \varphi \in \mathscr{H}^1_{cst}(Q_T), \quad (\Psi(W), \varphi) = 0 \\
\text{with } (\Psi(W), \varphi) &= \int_{\Omega_0} \varphi(0)(W_0 - W(0))\, \mathrm{d}\Omega \\
&+ \int_0^T \frac{\partial}{\partial t} \int_{\Omega_t} \varphi W \,\mathrm{d}\Omega\, \mathrm{d}t \\
&+ \int_0^T \int_{\Omega_t} \varphi\, \nabla \cdot \mathscr{F}(W)\, \mathrm{d}\Omega\, \mathrm{d}t - \int_0^T \int_{\partial\Omega_t} \varphi\, \breve{\mathscr{F}}(W).\mathbf{n}\, \mathrm{d}\Gamma\, \mathrm{d}t\,, \quad (3)
\end{aligned}$$

x-Continuous/time-discrete state equations

Find $W \in \mathscr{H}^1(Q_T)$ such that $\forall \varphi \in \mathscr{H}^1(Q_T)$, $\quad (\Psi(W)\,,\,\varphi) = 0$ with

$$
\begin{aligned}
(\Psi(W)\,,\,\varphi) = (t^{n+1} - t^n) \int_{\Omega_0} \varphi(0)(W_0 - W^1)\,\mathrm{d}\Omega \\
+ \sum_{n=1}^{nmax} \left[\int_{\Omega_{t^{n+1}}} \varphi(.,t^{n+1}) W(.,t^{n+1})\,\mathrm{d}\Omega - \int_{\Omega_{t^n}} \varphi(.,t^n) W(.,t^n)\,\mathrm{d}\Omega \right] \\
+ \sum_{n=1}^{nmax} (t^{n+1} - t^n) \int_{\Omega_{t^n}} \varphi(.,t^n)\, \nabla \cdot \mathscr{F}(W(.,t^n))\,\mathrm{d}\Omega \\
- \sum_{n=1}^{nmax} (t^{n+1} - t^n) \int_{\partial\Omega_{t^n}} \varphi(.,t^n)\, \breve{\mathscr{F}}(W(.,t^n)).\mathbf{n}\,\mathrm{d}\Gamma.
\end{aligned}
$$

Discrete state equations

Find $W_h \in \mathcal{V}_h$ such that $\forall \varphi_h \in \mathcal{V}_{h,cst}, \quad (\Psi_h(W_h)\,,\,\varphi_h) = 0$

where Ψ_h is extended to continuous fields :

$$\forall W \in \mathcal{H}^1(Q_T), \forall \varphi \in \mathcal{H}^1_{cst}(Q_T)\,,$$

$$(\Psi_h(W)\,,\,\varphi) = (t^{n+1} - t^n) \int_{\Omega_0} \varphi(.,t^0)(\Pi_h W_0 - \Pi_h W(.,t^1))\,\mathrm{d}\Omega$$

$$+ \sum_{n=1}^{nmax} \frac{1}{t^{n+1} - t^n} \int_{\Omega_{t^n}} \left[\widehat{\Pi_h \varphi}(.,t^n) \widehat{\Pi_h W}(.,t^{n+1})\ \mathcal{J}^n\ - \Pi_h \varphi(.,t^n) \Pi_h W(.,t^n)) \right] \mathrm{d}\Omega$$

$$+ \sum_{n=1}^{nmax} \int_{\Omega_{t^n}} \Pi_h \varphi(.,t^n))\, \nabla \cdot \Pi_h \mathcal{F}(W(.,t^n)))\,\mathrm{d}\Omega$$

$$- \sum_{n=1}^{nmax} \int_{\partial\Omega_{t^n}} \Pi_h \varphi(.,t^n))\, \Pi_h \breve{\mathcal{F}}(W(.,t^n)).\mathbf{n}\,\mathrm{d}\Gamma.$$

Principle of analysis

$$j(W) = (g, W)_{L^2(Q_T)} \;\; ; \;\; Min_{mesh} \;\; \delta j = j(W) - j(W_h)$$

$$(\Psi_h(W), \varphi_h) - (\Psi_h(W_h), \varphi_h) \; = \; (\Psi_h(W) - \Psi(W), \varphi_h)$$

$$W^* \;\; \in \;\; \mathscr{V}, \forall \psi \in \mathscr{V}, \;\; \left(\frac{\partial \Psi}{\partial W}(W)\psi, W^* \right) = (g, \psi), \qquad (6)$$

$$W_h^* \;\; \in \;\; \mathscr{V}_h, \forall \psi_h \in \mathscr{V}_h, \;\; \left(\frac{\partial \Psi_h}{\partial W}(W_h)\psi_h, W_h^* \right) = (g, \psi_h). \qquad (7)$$

Then assuming that W^*, $\Pi_h W^*$ and W_h^* and their gradients are close to each other :

$$\delta j \; \approx \; (\Psi_h(W) - \Psi(W), W^*) \; \approx \; (\Psi_h(W) - \Psi(W), \Pi_h W^*) \, . \qquad (8)$$

Error estimate

$$
\begin{aligned}
(\Psi_h(W)\,,\Pi_h W^*) &= \int_{\Omega_0} \varphi(.,t^0)(\Pi_h W_0 - \Pi_h W(.,t^1))\,\mathrm{d}\Omega \\
&+ \sum_{n=1}^{nmax} \frac{1}{t^{n+1}-t^n} \int_{\Omega_{t^n}} \left[\widehat{\Pi_h W^*}(.,t^n)\widehat{\Pi_h W}(.,t^{n+1})\ \mathscr{J}^n\ -\ \Pi_h W^*(.,t^n)\Pi_h W(.,t^n)\right] \mathrm{d}\Omega \\
&+ \sum_{n=1}^{nmax} \int_{\Omega_{t^n}} \Pi_h W^*(.,t^n)\,\nabla\cdot\Pi_h \mathscr{F}(W(.,t^n)))\,\mathrm{d}\Omega \\
&- \sum_{n=1}^{nmax} \int_{\partial\Omega_{t^n}} \Pi_h W^*(.,t^n)\,\Pi_h \breve{\mathscr{F}}(W(.,t^n)).\mathbf{n}\,\mathrm{d}\Gamma.
\end{aligned}
$$

Analysis of steady terms

- Boundary integrals are neglected.

- Flux terms are analysed after an integration by parts :

$$\int_{\Omega_{t^n}} \Pi_h W^*(.,t^n)\,\nabla\cdot\Pi_h\mathscr{F}(W(.,t^n)\,\mathrm{d}\Omega$$

$$-\int_{\Omega_{t^n}} \Pi_h W^*(.,t^n)\,\nabla\cdot\mathscr{F}(W(.,t^n)\,\mathrm{d}\Omega =$$

$$-\int_{\Omega_{t^n}} \nabla\Pi_h W^*(.,t^n)\,[\Pi_h\mathscr{F}(W(.,t^n) - \mathscr{F}(W(.,t^n)]\,\mathrm{d}\Omega$$

$$\preceq \frac{1}{8}\int_{\Omega_{t^n}} \nabla\Pi_h W^*(.,t^n) \quad \mathrm{trace}(\mathscr{M}^{-\frac{1}{2}}(\mathbf{x})\,|H_{\mathscr{F}}(\mathbf{x})|\,\mathscr{M}^{-\frac{1}{2}}(\mathbf{x})) \quad \mathrm{d}\Omega$$

Analysis of time-derivative

$$\widehat{\Pi_h W^*}(.,t^n)\left[\widehat{\Pi_h W}(.,t^{n+1})-\widehat{W}(.,t^{n+1})\right]\ \mathscr{J}^n\ -\ \Pi_h W^*(.,t^n)\left[\Pi_h W(.,t^n)-W(.,t^n)\right]$$

$$=\left[\ \mathscr{J}^n\ \widehat{\Pi_h W^*}(.,t^n)-\Pi_h W^*(.,t^n)\right]\left[\widehat{\Pi_h W}(.,t^{n+1})-\widehat{W}(.,t^{n+1})\right]$$

$$-\Pi_h W^*(.,t^n)\left[\widehat{\Pi_h W}(.,t^{n+1})-\widehat{W}(.,t^{n+1})-\Pi_h W(.,t^n)+W(.,t^n)\right]$$

$$\widehat{\Pi_h W}(.,t^{n+1})-\widehat{W}(.,t^{n+1}))\preceq\frac{1}{8}\mathrm{trace}(\mathscr{M}^{-\frac{1}{2}}(\mathbf{x})\,|H_{\widehat{W}}(\mathbf{x})|\,\mathscr{M}^{-\frac{1}{2}}(\mathbf{x}))$$

$$\widehat{\Pi_h W}(.,t^{n+1})-\widehat{W}(.,t^{n+1})-\Pi_h W(.,t^n)+W(.,t^n)\preceq$$
$$\frac{(t^{n+1}-t^n)}{8}\mathrm{trace}(\mathscr{M}^{-\frac{1}{2}}(\mathbf{x})\,|H_{DW/Dt}(\mathbf{x})|\,\mathscr{M}^{-\frac{1}{2}}(\mathbf{x}))$$

Synthèse

$$\delta j \approx \sum_{n=1}^{nmax} \int_{\Omega_{t^n}} \left[\mathrm{trace}(\mathcal{M}^{-\frac{1}{2}}(\mathbf{x}) \, |H_{goal}(\mathbf{x})| \, \mathcal{M}^{-\frac{1}{2}}(\mathbf{x})) \right] \mathrm{d}\Omega$$

from which we deduce $\mathcal{M}_{opt}$ for each time sub-interval.

Bump ALE

Numerical computation

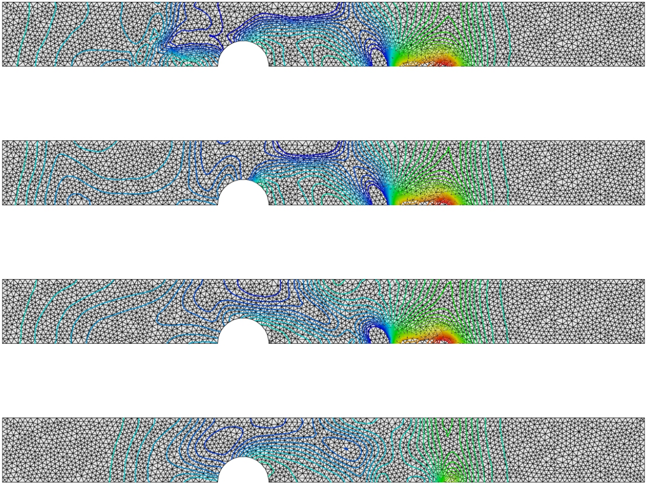
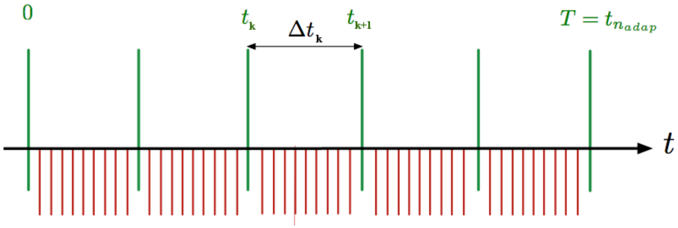


FIGURE: Adjoint ALE

Global fixed point for mesh adaptation

$$[0,T] = [t_0 = 0, t_1] \cup [t_1, t_2] \cup \ldots [t_{kmax}, T]$$

$$t \in [t_k, t_{k+1}] \Rightarrow \Omega_t = \{x = \phi_k(\xi, t), \xi \in \Omega_{t_k}\}$$



Numerical computation

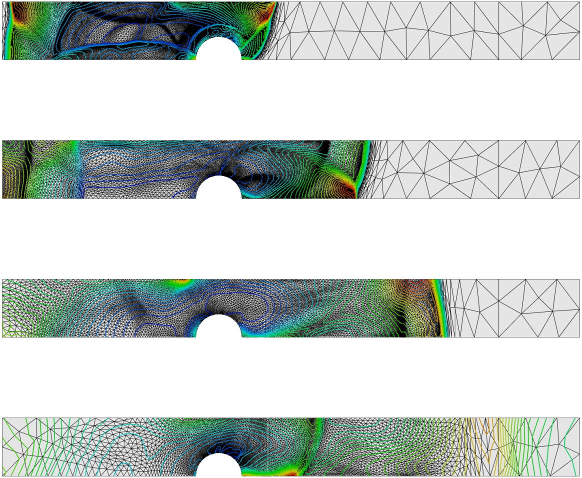


FIGURE: Goal-oriented mesh adaptation : State

Numerical computation

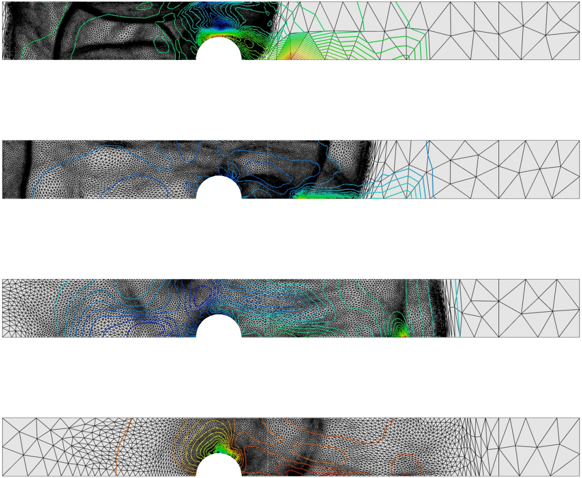


FIGURE: Goal-oriented mesh adaptation : Adjoint