Anisotropic Norm-Oriented Mesh Adaptation for Compressible Inviscid Flows

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Some Mesh Adaptation Strategies

Feature-based anisotropic mesh adaptation

Deriving the **best mesh** to compute the characteristics of a given solution $\mathbf{w}$

[Tam et al., CMAME 2000], [Pain et al., CMAME 2001], [Picasso, SIAMJSC 2003], [Formaggia et al., ANM 2004], [Bottasso, IJNME 2004], [Li et al., CMAME 2005], [Frey and Alauzet, CMAME 2005], [Gruau and Coupez, CMAME 2005], [Huang, JCP 2005], [Compere et al., 2007], [Loseille and Alauzet, IMR 2009], ...

Goal-oriented anisotropic mesh adaptation

Deriving the **best mesh** to observe a given scalar functional $\mathbf{j(w) = (g, w)}$

[Venditti and Darmofal, JCP 2003], [Jones et al., AIAA 2006], [Power et al., CMA 2006], [Wintzer et al., AIAA 2008], [Leicht and Hartmann, JCP 2010], [Loseille et al., JCP 2010], ...

Norm-oriented anisotropic mesh adaptation

Deriving the **best mesh** to observe a given norm $\|\mathbf{L(w) - L(w_h)}\|_{\mathbf{L^2}}$

- Observe multiple functionals simultaneously
- Explicit control of the implicit error $\mathbf{\Pi_h w - w_h}$:
  $$\|\mathbf{w} - \mathbf{w_h}\|_{\mathbf{L^2}} \leq \|\mathbf{w} - \mathbf{\Pi_h w}\|_{\mathbf{L^2}} + \|\mathbf{\Pi_h w} - \mathbf{w_h}\|_{\mathbf{L^2}}$$

[Arian and Salas, AIAA Journal, 1999], [Hartmann, SIAM 2008], [Dervieux et Brettes, Eccomas 2014], , ...

Outline

Feature-based Anisotropic Mesh Adaptation

We propose a multiscale anisotropic mesh adaptation

[Loseille et al., AIAA 2007], [Alauzet, IJNMF 2008], [Loseille and Alauzet, IMR 2009]

- Optimal control of the interpolation error in L^p norm :

$$\|W - \Pi_h W\|_{L^p(\Omega_h)}$$

- Develops for analytical function
- W solution of the problem is unknown
- W_h the numerical solution piecewise linear by element
- Assume it exists a reconstruction $R_h(W_h)$ such that

$$\begin{cases} \|W - R_h(W_h)\| \leq \alpha \|W - W_h\| \quad \text{where} \;\; 0 \leq \alpha < 1 \\ \Pi_h R_h \phi_h = \phi_h \,, \;\; \forall \phi_h \in V_h^1 \end{cases}$$

We deduce: $\|W - W_h\| \leq \frac{1}{1-\alpha} \|R_h(W_h) - \Pi_h R_h(W_h)\|$

Apply interpolation error estimate to $R_h W_h$:

$$\mathcal{M}_{\mathbf{L}^p} = \mathcal{K}_{L^1}(|H_{R_h(W_h)}|) = D_{\mathbf{L}^p} \left(\det |H_{R_h(W_h)}|\right)^{\frac{-1}{2p+3}} |H_{R_h(W_h)}|$$

Feature-based Mesh Adaptation: Algorithm

1. Compute state W_h on mesh $\mathcal{H}$
2. Compute sensor $u_h = u(W_h)$ and $R_h(u_h)$
3. Compute optimal metric $\mathcal{M}_{L^1}^{opt}(R_h(u_h))$
4. Generate a new adapted mesh $\mathcal{H}$ which is unit for metric $\mathcal{M}_{L^1}^{opt}(R_h(u_h))$

- If not converge, goto 1.

Properties:

- Highly anisotropic meshes
- Capture all scales of the observed variable
- Global 2^{nd} of mesh convergence for the observed variable
- Early capturing property: asymptotic convergence is reached faster

Outline

Goal-Oriented Anisotropic Mesh Adaptation

We propose a goal-oriented error estimate for compressible Euler equations [Loseille et al., JCP 2010]

Problem statement and solving

- Minimize the approximation error of a given scalar output functional J:

$$|J(W) - J(W_h)|$$

- State equations:

$$A(W) \approx div\left(\mathcal{F}(W)\right) = 0, \qquad A \cdot W_h \approx div\left(\mathcal{F}_h(W_h)\right) = 0.$$

- Taylor expansion:

$$J(W) - J(W_h) \approx (\frac{\partial J}{\partial W}(W), W - W_h) = (g_{go}, W - W_h).$$

Goal-Oriented Anisotropic Mesh Adaptation

We propose a goal-oriented error estimate for compressible Euler equations [Loseille et al., JCP 2010]

- Optimal control of the functional approximation error in L^1 norm

$$|J(W) - J(W_h)| \approx (|\nabla W_{go}^*|, |\mathcal{F}(W) - \Pi_h \mathcal{F}(W)|)$$

$$\text{with } A^* W_{go}^* = \frac{\partial J}{\partial W}(W)$$

Solve this problem in the continuous framework

$$\Longrightarrow \quad \mathcal{M}_{go}^{opt}(W) = \mathcal{K}_{L^1}(|\nabla W_{go}^*| \cdot |H_{\mathcal{F}(W)}|)$$

- Highly anisotropic meshes
- Optimal distribution of DOF to observe j
- 2^{nd} order of mesh convergence for the output functional is not guaranteed
- Early capturing property: asymptotic convergence is reached faster

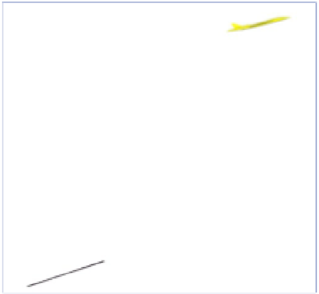
Goal-Oriented Mesh Adaptation: Algorithm

1. Compute state W_h on mesh $\mathcal{H}$
2. Compute adjoint state $W^*_{go,h}$ using $g_{go,h} = \dfrac{\partial j}{\partial W}(W_h)$ and $R_h(\mathcal{F}_h(W_h))$
3. Compute optimal metric $\mathcal{M}^{opt}_{go}(W_h) = \mathcal{K}_{L^1}(|\nabla W^*_{go,h}| \cdot |H_{R_h(\mathcal{F}_h(W_h))}|)$
4. Generate a new adapted mesh $\mathcal{H}$ which is unit for metric $\mathcal{M}^{opt}_{go}(W_h)$

- If not converge, goto 1.

Application to sonic boom :

$$j(W) = \int_\gamma \left(\frac{p - p_\infty}{p_\infty} \right)^2 \mathrm{d}\gamma$$



Goal-Oriented Mesh Adaptation: Algorithm

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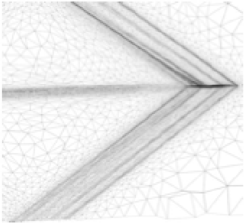
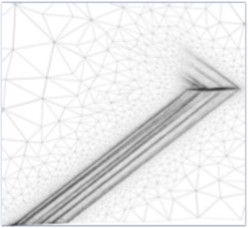


Multiscale mesh adaptation



Goal-oriented mesh adaptation

Outline

Norm-oriented mesh adaptation

Problem statement

- Minimize a semi-norm of the approximation error:

$$\|L(W) - L(W_h)\|^2_{L^2(\Omega_h)}$$

- State equations:

$$div\,(\mathcal{F}(W)) = 0, \qquad div\,(\mathcal{F}_h(W_h)) = 0.$$

Recast the problem within the goal-oriented strategy

- Taylor expansion of L:

$$\begin{aligned}
(L(W) - L(W_h), L(W) - L(W_h)) &\approx \left(\tfrac{\partial L}{\partial W}(W)(W - W_h), \tfrac{\partial L}{\partial W}(W)(W - W_h)\right) \\
&\approx \left(\left[\tfrac{\partial L}{\partial W}^*(W)\,\tfrac{\partial L}{\partial W}(W)\right](W - W_h), W - W_h\right) \\
&= (g_{no}, W - W_h)
\end{aligned}$$

$\rightarrow$A corrector is needed to estimate $W - W_h = W - \Pi_h W + \Pi_h W - W_h$

Norm-oriented mesh adaptation: Corrector

Non-linear corrector

- Minimize a defect/residual
- A posteriori analysis

$$div(\mathcal{F}(W_h)) = \delta \approx 0$$

- A priori analysis

$$div(\mathcal{F}_h(\Pi_h W)) = \delta_h \approx 0$$

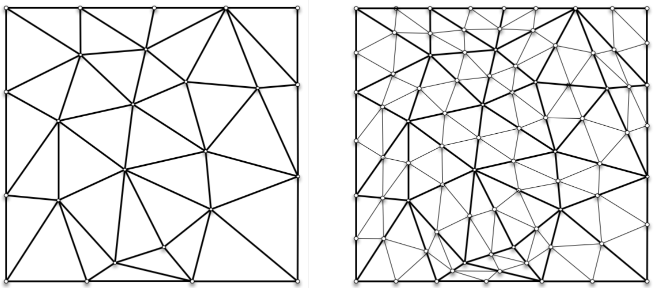
- Create a source term accounting for the defect

$$\begin{aligned} div(\mathcal{F}_h(W_h)) &= div(\mathcal{F}_h(\Pi_h W)) \\ &\approx div(\mathcal{F}_h(\Pi_h W_{h/2})) \end{aligned}$$

Norm-oriented mesh adaptation: Accumulation

A posteriori source term : $div(\mathcal{F}(W_h)) = \delta \approx 0$

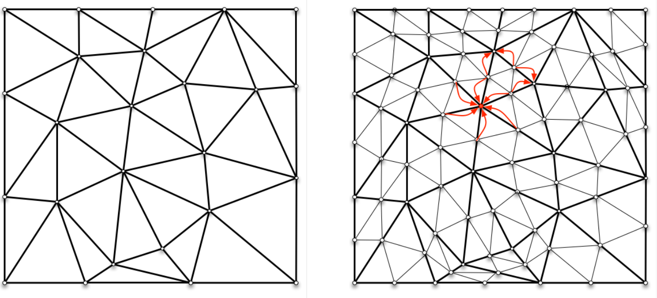
- Compute $\mathcal{F}_{h/2}\left(\Pi_{h/2} W_h\right)$: one solver flux evaluation



Norm-oriented mesh adaptation: Accumulation

A posteriori source term : $div(\mathcal{F}(W_h)) = \delta \approx 0$

- Compute $\mathcal{F}_{h/2}\left(\Pi_{h/2} W_h\right)$: one solver flux evaluation
- Accumulate back $S_{post} = A_{h/2 \rightarrow h}(\mathcal{F}_{h/2}(\Pi_{h/2} W_h))$ on the current mesh
- Solve $div(\mathcal{F}_h(W_c)) = S_{post}$ starting with $W_c = W_h$



Norm-oriented mesh adaptation: Accumulation

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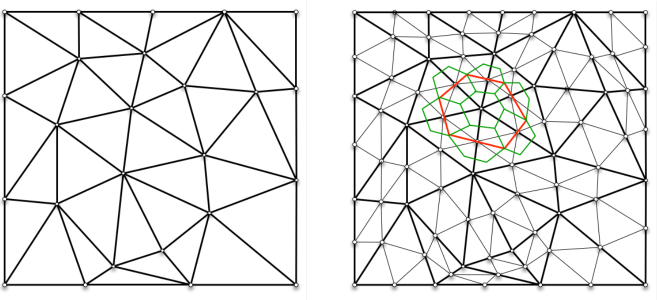


Norm-oriented mesh adaptation: Algorithms

1. Compute state W_h on mesh $\mathcal{H}$
2. Compute an approximation $W_c - W_h$ of $W - W_h$
3. Compute adjoint state $W^*_{no,h}$ and $R_h(\mathcal{F}_h(W_h))$
4. Compute optimal metric $\mathcal{M}^{opt}_{no}(W_h) = \mathcal{K}_{L^1}(|\nabla W^*_{no,h}| \cdot |H_{R_h(\mathcal{F}_h(W_h))}|)$
5. Generate a new adapted mesh $\mathcal{H}$ which is unit for metric $\mathcal{M}^{opt}_{no}(W_h)$

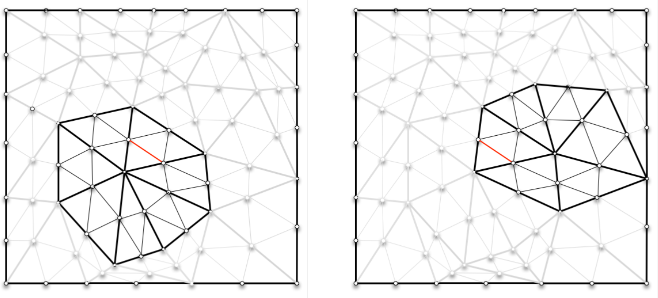
- If not converge, goto 1.

Observations:

- A finer mesh $h/2$ yields $4N$ nodes in 2D and $8N$ nodes in 3D
  $\rightarrow$ A local solution is proposed
- Local submesh requires no more memory than the h problem
- Requires to be able to create several substructures and solvers
- Already parrallelized, reduces memory accesses

Sub Mesh Generation

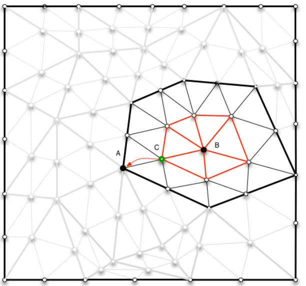
- A local submesh is sufficient to compute fluxes
- Each edges flux is computed three times in 2D, four times in 3D

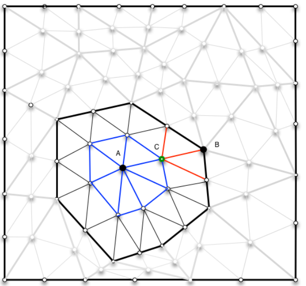


Norm-oriented mesh adaptation: Optimisation

Edge partitioning

- Total flux at vertex C is the sum of edges flux, it can be computed separately
- Each edges flux is computed with it's closest "master node" *ie.* the node of the coarse mesh
- Edges fluxes are computed only once
- Can be extended as well to Navier-Stokes and RANS computations by treating nodes and triangles

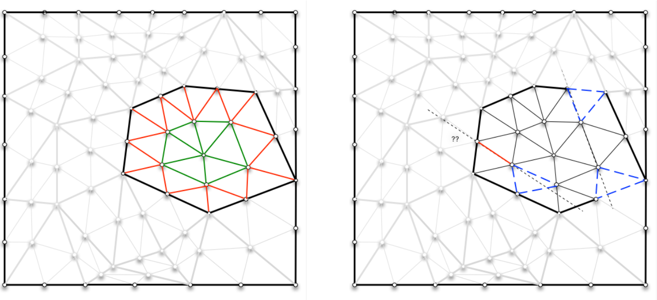




Norm-oriented mesh adaptation: Optimisation

Order two computation

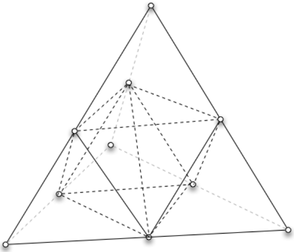
- Outer edges does'nt have anymore access to upwind or downwind elements to compute gradients
- Inner edges still have access to this information : a bigger submesh is not mandatory

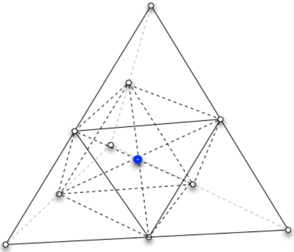


Norm-oriented mesh adaptation: Optimisation

3D mesh generation

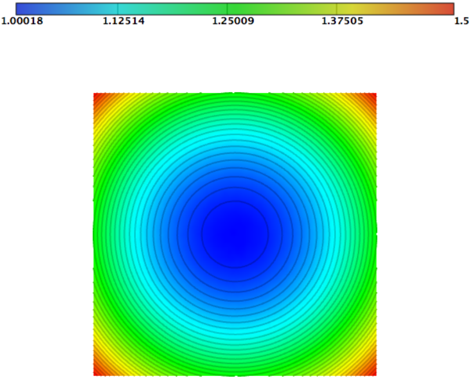
- The usual $h/2$ tetrahedra generates an edge that can't be included in any 1^{st} order elements ball
- Adding an inner point solves this issue by creating new inner edges
- Edges must be appropriately shared

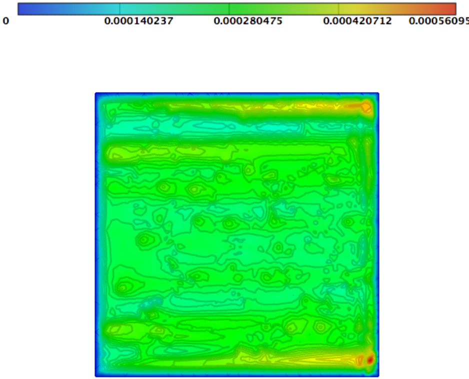




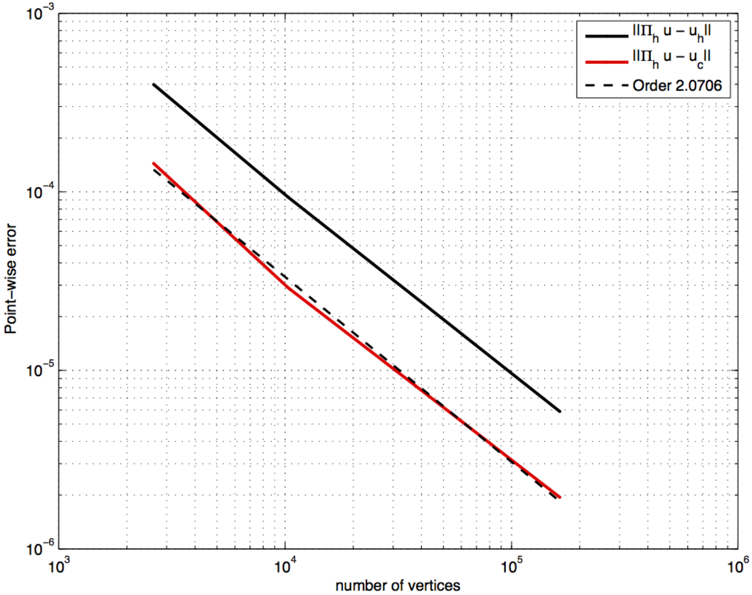
Validation on a manufactured solution

- Quadratic density on $[0,1] \times [0,1]$ with Dirichlet boundary condition
- $\rho(x,y) = (x-0.5)^2 + (y-0.5)^2 + 1,$
  $u = 1, v = 0, p = 1$

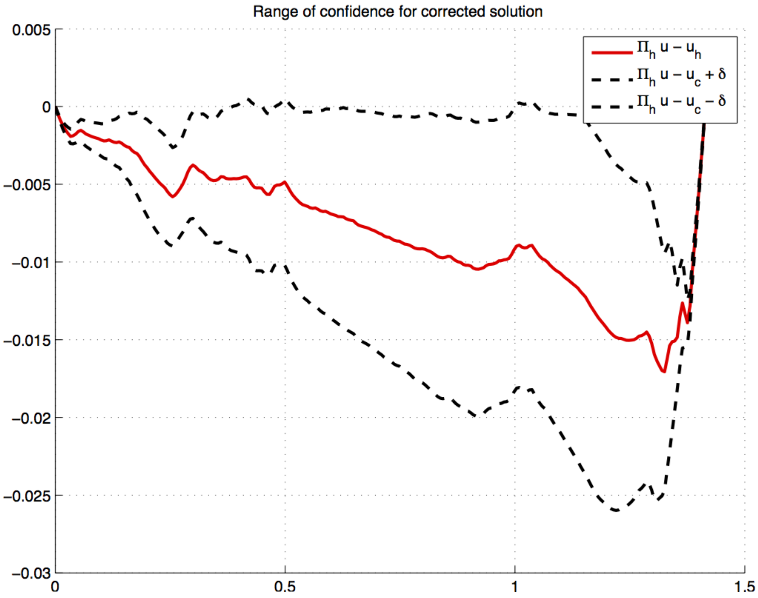




Norm-oriented mesh adaptation: Validation



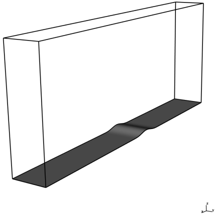
Norm-oriented mesh adaptation: Validation



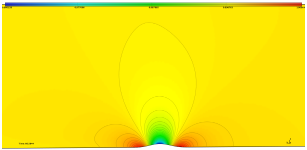
Norm-oriented mesh adaptation: Validation

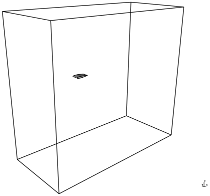
Impact of the choice of $\mathcal{F}_{h/k}$ to estimate $\mathcal{F}(u_h)$

		$\mathcal{F}_{h/2}$	$\mathcal{F}_{h/4}$
$\#vertices$	$\|\Pi_u - u_h\|_2$	$\|\Pi_u - u_c\|_2$	$\|\Pi_u - u_c\|_2$
2625	$4.014663e^{-4}$	$1.512386e^{-4}$	$1.097588e^{-4}$
10333	$9.481249e^{-5}$	$3.182181e^{-5}$	$2.236964e^{-5}$
41001	$2.378164e^{-5}$	$8.213737e^{-6}$	$5.927686e^{-6}$
163345	$5.900477e^{-6}$	$2.060095e^{-6}$	$1.498703e^{-6}$
		improvement	*improvement*
		62.3%	72.6%
		66.4%	76.4%
		65.4%	75.0%
		65.0%	74.6%

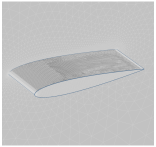


Mach 0.3



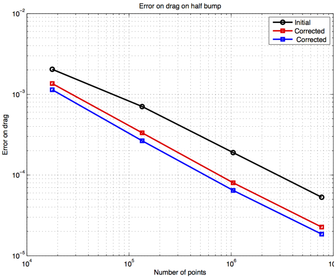


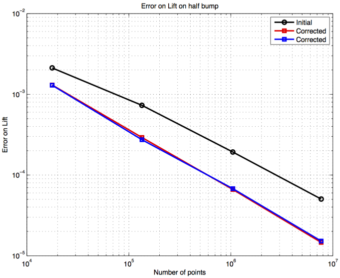
Mach 0.4 with $\alpha = 4^o$
Mach 0.8 with $\alpha = 3^o$

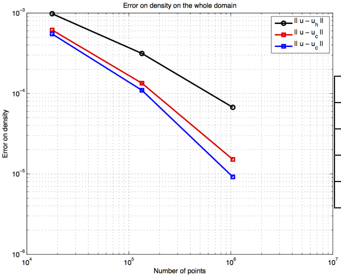


- Sequence of meshes of size $h, h/2, \ldots$
- Estimation of the convergence on the density, lift, drag
- A posteriori source term $\mathcal{F}_{\frac{h}{2}}(W_h)$ (red) and $\mathcal{F}_{\frac{h}{2}}(R_h(W_h))$ (blue)

Validations of the corrector: 3D bump

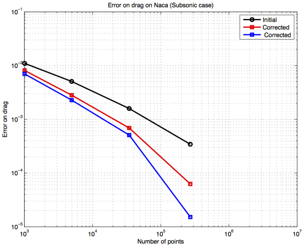


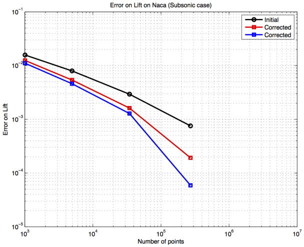


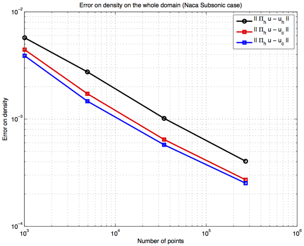


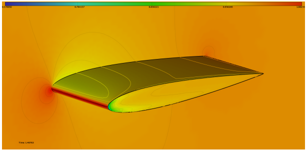
# vertices	Error on lift	Error on corrected lift
17723	66%	39%
134381	22 %	8.6%
1044943	6%	1.9%
7682230	1.5%	0.4%

- Similar observations hold for pressure, Mach number, velocity, ...



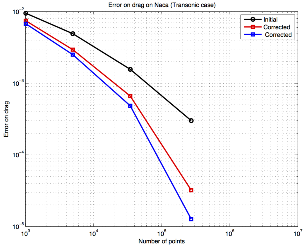


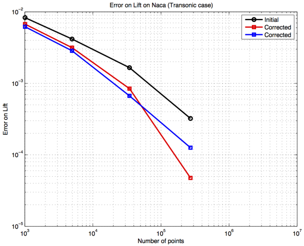


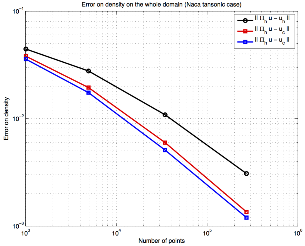


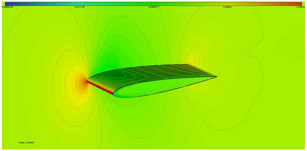
- Similar observations hold for pressure, Mach number, velocity, . . .

Validations of the corrector: Transonic Naca



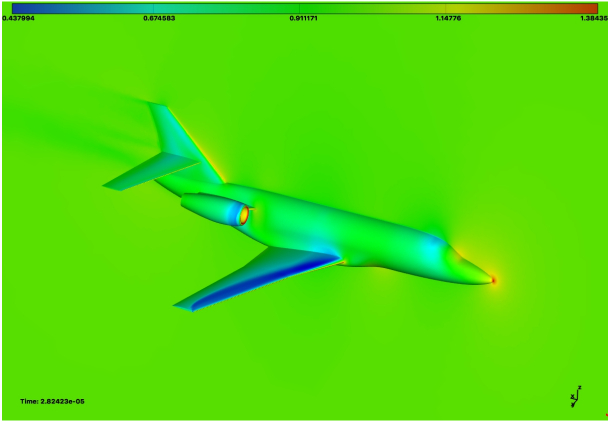






Outline

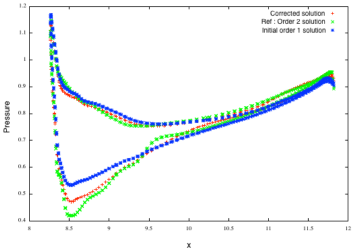
Turbulent Transonic Falcon

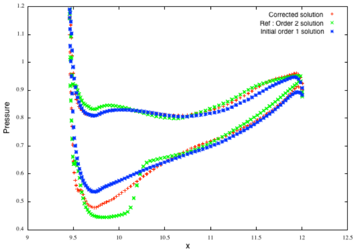


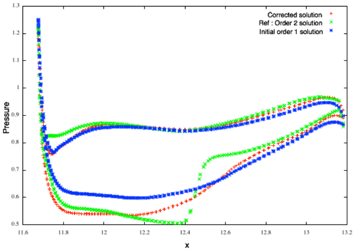
Mach 0.8 with $\alpha = 2^o$
Spalart-Allmaras RANS turbulence model

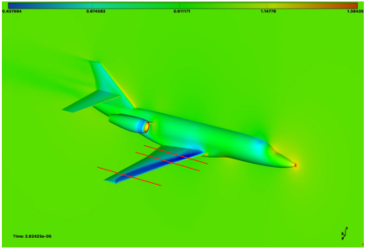
- Order 1 numerical scheme
- Local source term computation
- Initial mesh : 3.10^6 nodes

Turbulent Transonic Falcon









- Similar observations hold for pressure, Mach number, velocity, ...

Conclusion

- Norm-oriented mesh anisotropic adaptation
  - Use of previous strategies : feature-based and goal-oriented
  - Natural anisotropy : not given by a specific field
  - Based on the dual control of :

Interpolation error and implicit error

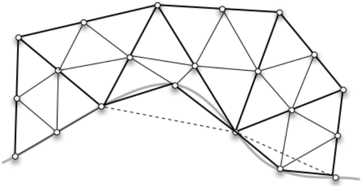
- Source-term-based corrector
  - Simple source-term evaluation : multi-grid techniques
  - Light modification of the flow solver
  - Use of local meshes : no memory over cost
  - The corrector fits numerical scheme and dissipation order
  - Non linear correction

$\Longrightarrow$ The whole flow field is corrected

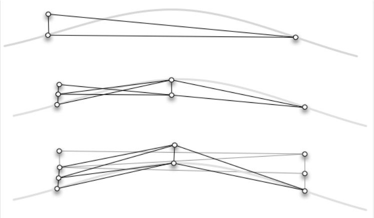
$\Longrightarrow$ Multiple functionals may be corrected simultaneously

- RANS equations:
  - Kernel for goal-oriented Laminar equations [Belme Thesis 2013]
  - Kernel for goal-oriented RANS equations $\rightarrow$ source terms
- Mesh generation:

CAD Interaction



Boundary layer treatment



- Validation