CALCULS RECENTS EN NORM-ORIENTED

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April 7, 2015

Motivations

$$-div(\frac{1}{\rho}\nabla u) = f$$

- Norm-oriented adaptation
- Adaptative FMG
- Smooth test case: boundary layer
- Singular test case: discontinuous coefficient

Norm-Oriented mesh adaptation

Minimize:

$$j(\Omega_h) = ||u - u_h||^2_{L^2(\Omega_h)}$$

Term $u - u_h$ is approximated by a *corrector* u'.

A priori Corrector:

$$a(\bar{u}'_{prio}, \phi_h) \;=\; \sum_{\partial T_{ij}} [\nabla \phi_h]^{jump}_{T_i,T_j} \cdot \mathbf{n}_{ij} \;\int_{\partial T_{ij}} (\Pi_h u - u)\, \mathrm{d}\sigma$$
$$+(f, \phi_h) - (f_h, \phi_h)$$

with u approximated by a quadratic function.
$\bar{u}'_{prio}$ approximates $\Pi_h u - u_h$:

$$u'_{prio} = \bar{u}'_{prio} - (\pi_h u_h - u_h).$$

Norm-Oriented mesh adaptation

Novel corrector : Defect Correction:
Ω_h,$\Omega_{h/2}$:

$$u_h = A_h^{-1} f_h \quad , \quad u_{h/2} = A_{h/2}^{-1} f_{h/2} \quad \Rightarrow \quad u - u_{h/2} \approx \frac{1}{4}(u - u_h)$$

Finer-grid Defect-Correction (DC) corrector:

$$A_h \bar{u}'_{DC} = \frac{4}{3} R_{h/2 \to h}(A_{h/2} P_{h \to h/2} u_h - f_{h/2})$$

$\bar{u}'_{DC}$ approximates $\Pi_h u - u_h$:

$$u'_{DC} = \bar{u}'_{DC} - (\pi_h u_h - u_h).$$

Norm-Oriented Adaptation

We introduce an adjoint:

$$\psi \in \Omega, a(\psi, u^*) = (u', \psi)$$

$$a(u', u^*) = (u', u')$$

$$a(u - u_h, u^*) = j(\Omega_h)$$

$$\mathcal{M}_{opt,norm} = \mathcal{K}_1(|\rho(H(u^*))|, u_{\mathcal{M}})$$

Step 1:

$$a(\bar{u}'_{prio}, \phi_h) = \sum_{\partial T_{ij}} [\nabla \phi_h]^{jump}_{T_i, T_j} \cdot \mathbf{n}_{ij} \int_{\partial T_{ij}} (\Pi_h u - u_h) \, \mathrm{d}\sigma$$

$$+(f, \phi_{\mathcal{M}}) - (f_h, \phi_h)$$

or

$$A_h \bar{u}'_{DC} = \frac{4}{3} R_{h/2 \to h}(A_{h/2} P_{h \to h/2} u_h - f_{h/2})$$

$$u' = \bar{u}' - (\pi_h u_h - u_h).$$

Step 2:

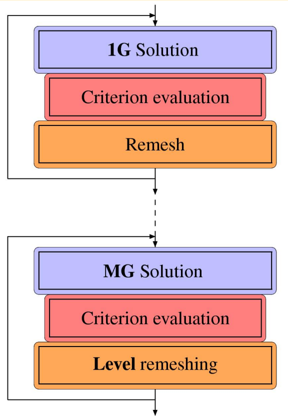
$$a(\psi, u^*) = (u', \psi)$$

Step 3:

$$\mathcal{M}_{opt,norm} = \mathcal{K}_1(|\rho(H(u^*))|, u_{\mathcal{M}})$$

Adaptative FMG

The mesh adaptation loop is applied as an **intermediate loop** between:
- the FMG phases loop and
- the MG cycling.

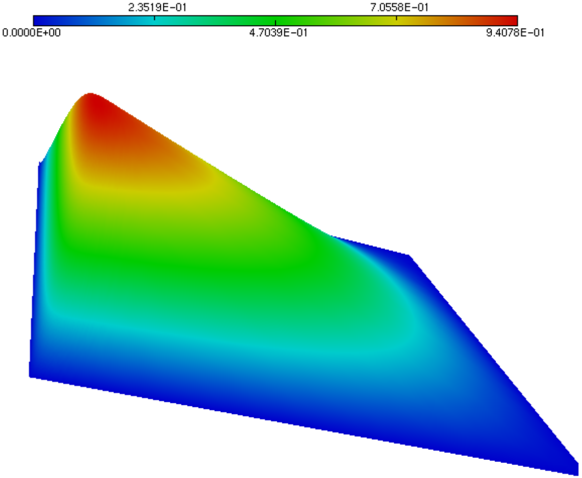


Adaptative FMG

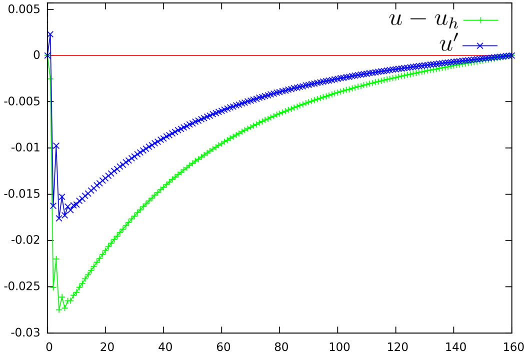
The Hessian case relies only on the solution of the discret PDE.
We apply a stopping MG criterion using mesh convergence and residual.
The stopping criterion used for the Hessian case cannot directly be applied to the case of norm-oriented since the two other states,
- corrector and
- adjoint

tend to zero when N increases.
New stopping criteria are being tested.

Smooth test case: boundary layer

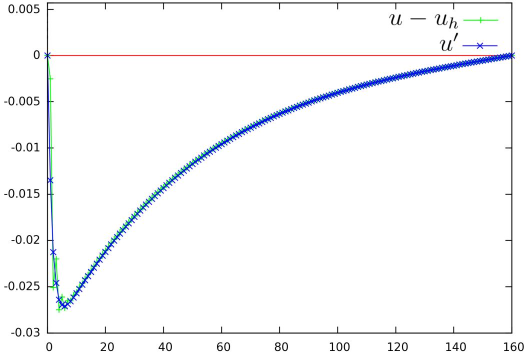


Smooth test case: boundary layer

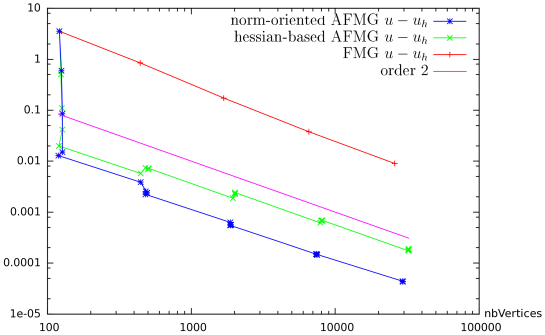


Fully 2D Boundary layer test case : comparison of error cuts for $y = 0.5$: plus signs $(+)$ depict the approximation error $u - u_h$ and crosses $(\times)$ depict the *a priori* corrector u'_{prio}. The corrector is able to correct about 60% of the approximation error.

Smooth test case: boundary layer

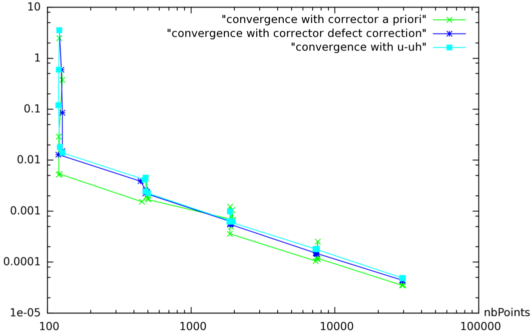


Fully 2D Boundary layer test case : comparison of error cuts for $y = 0.5$: plus signs $(+)$ depict the approximation error $u - u_h$ and crosses $(\times)$ depict the Defect-Correction corrector u'_{DC}. The corrector is able to correct about 95% of the approximation error.



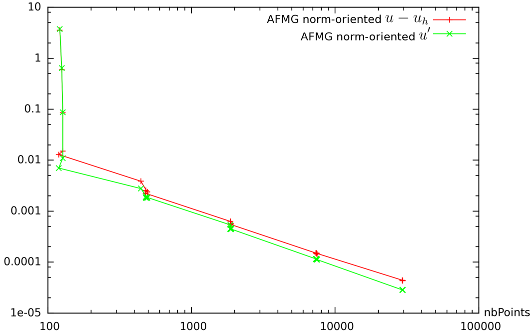
Fully 2D Boundary layer test case: convergence of the error norm $|u - u_h|_{L^2}$ as a function of number of vertices in the mesh for $(+)$ non-adaptative FMG, $(\times)$ Hessian-based adaptative FMG, $(*)$ norm-oriented adaptative FMG.

Smooth test case: boundary layer

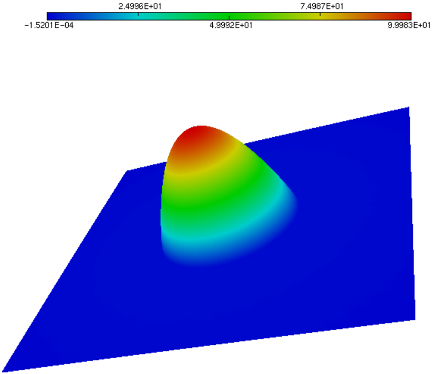


Fully 2D Boundary layer test case: convergence based on ($\times$) the *a priori* corrector or on the ($*$) Defect-Correction one, compared with ($\square$) a virtual adaptation controlled by $u - u_h$.

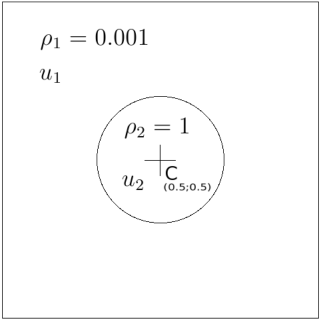
Smooth test case: boundary layer



Fully 2D Boundary layer test case: comparison of the predicted error norm proposed by the method with deffect correction, curve with (□), with the exact error norm (×).

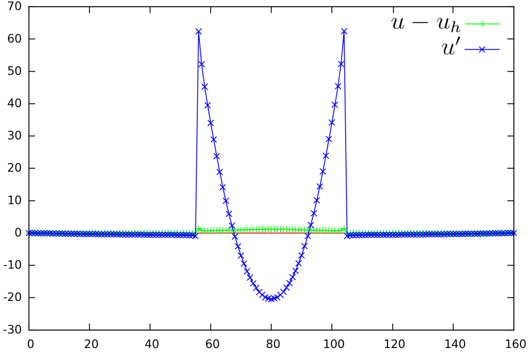


$$u_1 = 1.236 - 2.471((0.5-x)^2 + (0.5-y)^2)$$
$$u_2 = 100 - 2471((0.5-x)^2 + (0.5-y)^2)$$



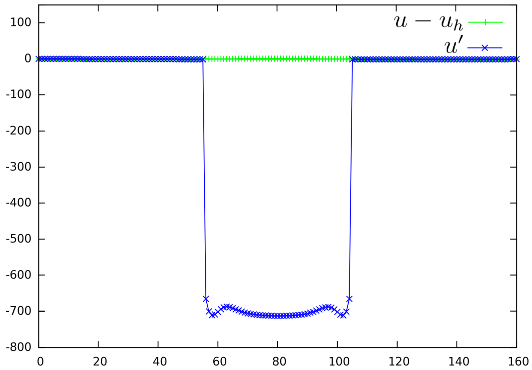
Poisson problem with discontinuous coefficient: sketch of exact solution definition and a typical computation of it.

Singular test case: discontinuous coefficient



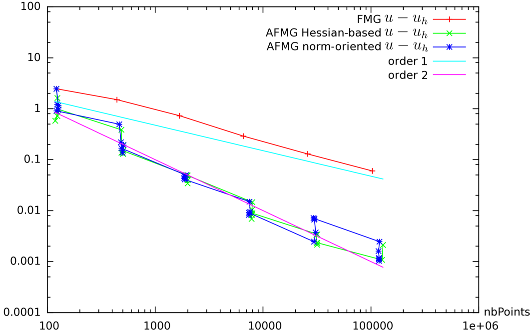
Singular test case : comparison of error cuts for $y = 0.5$: plus signs $(+)$ depict the approximation error $u - u_h$ and crosses $(\times)$ depict the *a priori* corrector u'_{prio}.

Singular test case: discontinuous coefficient



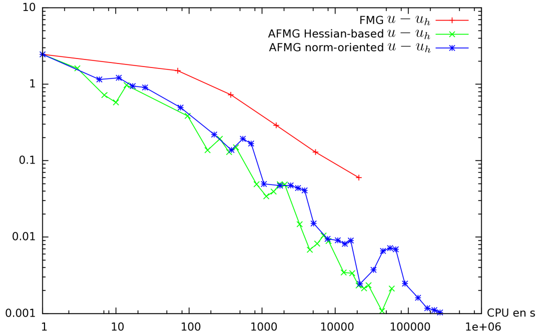
Singular test case : comparison of error cuts for $y = 0.5$: plus signs $(+)$ depict the approximation error $u - u_h$ and crosses $(\times)$ depict the Defect-Correction corrector u'_{DC}.

Singular test case: discontinuous coefficient



Poisson problem with discontinuous coefficient: convergence of the error norm $|u - u_h|_{L^2}$ as a function of number of vertices in the mesh for $(+)$ non-adaptative FMG, $(\times)$ Hessian-based adaptative FMG and $(*)$ norm-oriented adaptative FMG.

Singular test case: discontinuous coefficient



Poisson problem with discontinuous coefficient: efficiency analysis : CPU time for the norm-oriented adaptative FMG based on ($\times$) the *a priori* corrector or on the ($*$) Defect-Correction one, compared with ($\Box$) a virtual adaptation controlled by $u - u_h$.

Conclusions

The norm-oriented is improving:
- new corrector, more accurate,
- first test for FMG,
- new applications: the discontinuous case produces second-order convergence are predicted by theory.

Open problems remain:
- solve the FMG efficiency issue,
- validate a good measure of the approximation error.