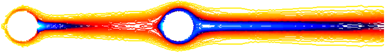
Curved mesh generation via linear elasticity. Applications to fluid dynamics problems.

Rémi Abgrall*, Cécile Dobrzynski*, Algiane Froehly*

* IMB - Université de Bordeaux and
Team Bacchus - INRIA Bordeaux Sud-Ouest, Bordeaux, France

Motivation

- Many researches to design method for fluid problems with accuracy higher than 2nd order
- True accuracy of those method requires that the mesh boundary are represented with at least the same accuracy



Subsonic flow around 2 cylinders: spurious entropy production

$\Rightarrow$ Curved meshes fitting well the boundaries:
Bézier/NURBS approximation with triangular patches

Guideline

Bézier and NURBS approximation

Curvilinear mesh generation

- Mesh checking
- Computing curved mesh via linear elasticity analogy

Numerical simulations

Isogeometric analysis for unstructured meshes

Bézier basis functions of order k over a simplex

- Lets λ_i the barycentric coordinates,
- α a multi-index of length k,
- the Bézier polynomial is:

$$B_\alpha(M) := c(\alpha)\Pi_{i=1}^{d+1}\lambda_i(M)^{\alpha_i}$$

$\mathcal{NURBS}$ basis function of order k

$$N_\alpha = \frac{\omega_\alpha B_\alpha}{\sum\limits_{\alpha', |\alpha'|=k} \omega_{\alpha'} B^k_{\alpha'}}.$$

$$\begin{cases} N_\alpha(M) & \geq 0 \text{ for any } \alpha, |\alpha| = k \text{ and } M \in K \\ \sum\limits_{\alpha, |\alpha|=k} & N_\alpha(M) = 1 \end{cases}$$

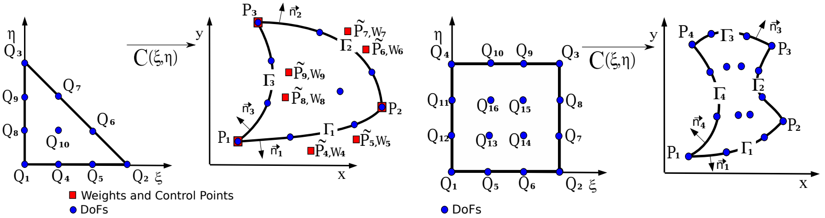
Isogeometric analysis for unstructured meshes

Lets $\mathcal{P} = \{P_\alpha \in \mathbb{R}^p, \alpha \in \mathcal{I}\}$, a family of control points, we approximate a function ψ by

$$\psi(M) \approx \sum_{\alpha, \alpha \in \mathcal{I}} P_\alpha \varphi_\alpha^n(M).$$

- The basis functions can be defined on triangles/tetra and quadrangulars,
- they allow to define conics.

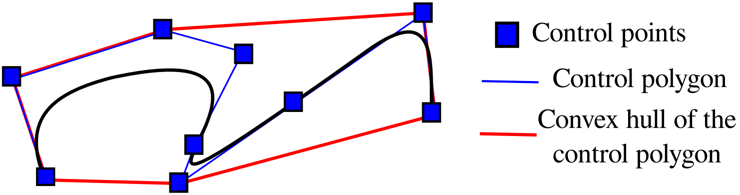
Example of elements



Property of curved element

Convexity of control polygon

A $\mathcal{NURBS}$ curve is contained in the convexe hull of its control polygon



Recalls

- extrapolation of control point
- the end points are always interpolated

Curved mesh generation

Problematic

- How to generate $\mathcal{NURBS}$ meshes ?

Our approach

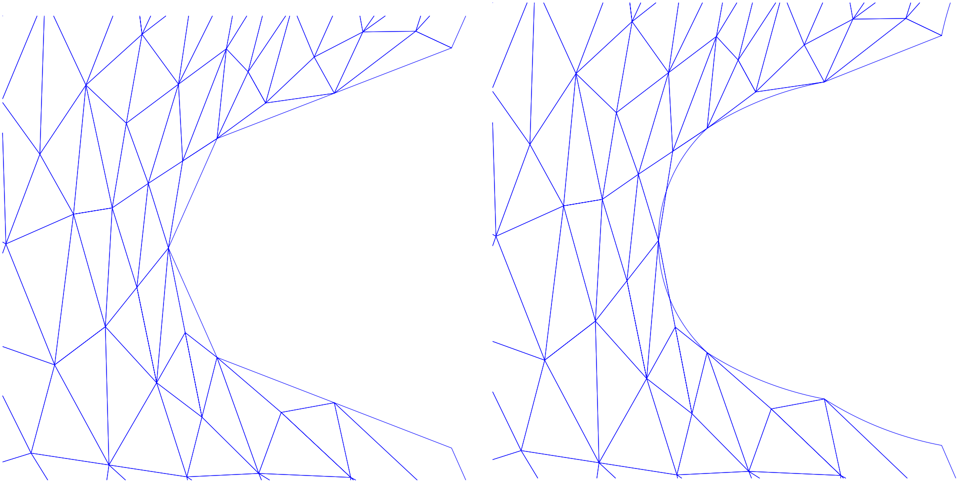
- Starting from a classical mesh of the domain (piecewise linear), we modify it to generate $\mathcal{NURBS}$ mesh.

2 main steps

1. Have or compute the geometrical information: control point, weight.. $\iff$ curve the boundary mesh
2. Check validity of volumic elements and if needed curve the interior mesh

From a piecewise linear mesh to a $\mathcal{NURBS}$ mesh

Straight mesh and curved boundary.

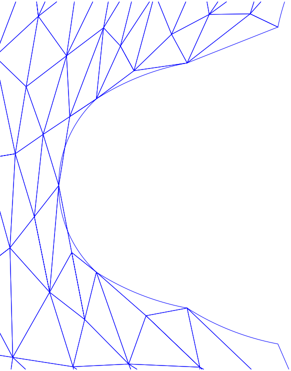
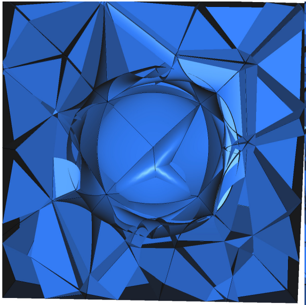
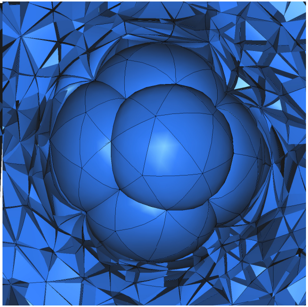


How to curve a piecewise linear mesh ?

Given

a simplicial mesh + the definition of boundary geometry

- Is the curved mesh conformal?

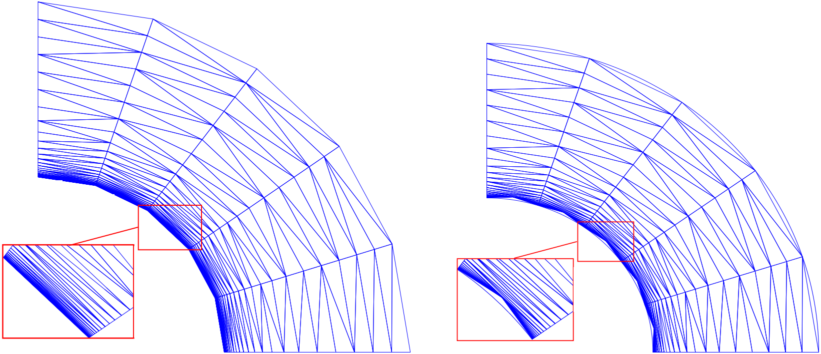




How to curve a piecewise linear mesh ?

Given

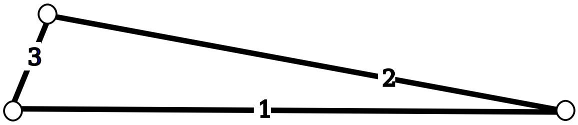
a simplicial mesh + the definition of boundary geometry

- Is the curved mesh conformal?
- How to curve the volumic mesh with keep as much as possible the structure of the mesh ?

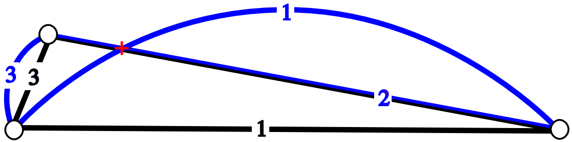


How to detect automatically invalid element?

A valid straight element

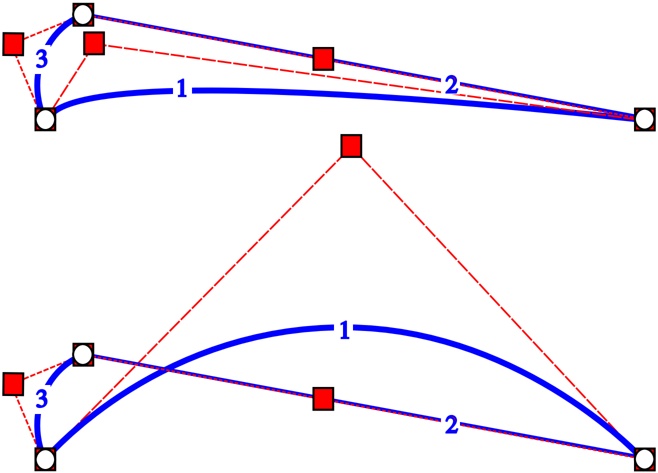


but.... a non-valid curved element

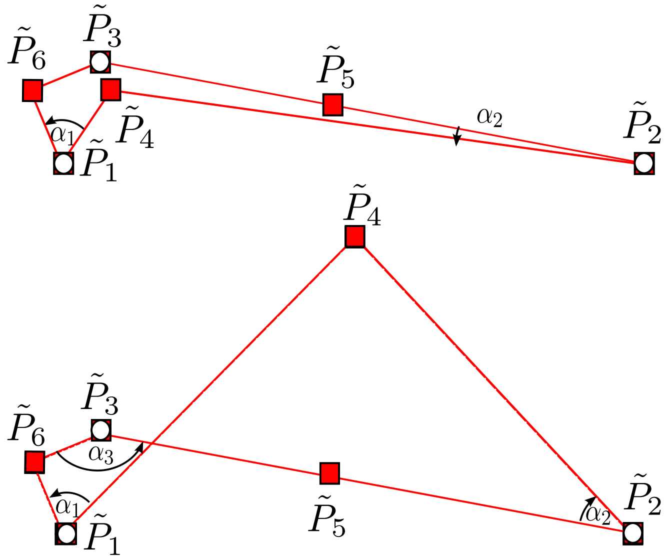


How to detect automatically invalid element?

Convex hull property



How to detect automatically invalid element?



$\alpha_2 \leq 0 \Rightarrow$ invalid element

How to curve a piecewise linear mesh ?

Given

a simplicial mesh + the definition of boundary geometry

1. Is the curved mesh conformal?
   Compute signed area for each triangle which have a boundary edge:
   - if all area ≤ 0, the mesh is ok
   - else .....
2. How to curve the volumic mesh with keep as much as possible the structure of the mesh ?

How to curve straight mesh ?

Given

A piecewise linear mesh + $\mathcal{NURBS}$ curves on the boundary

Idea

Using linear analogy on control points subdivided mesh.

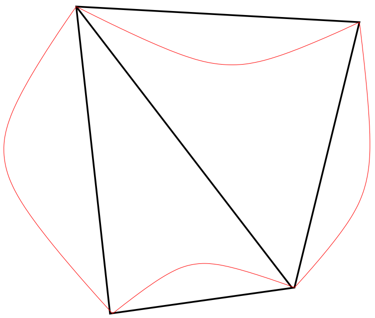
Thanks to convex hull property,

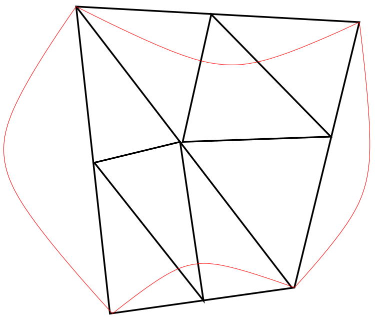
$$\text{Subdivided mesh legal} \Rightarrow \text{NURBS mesh valid.}$$

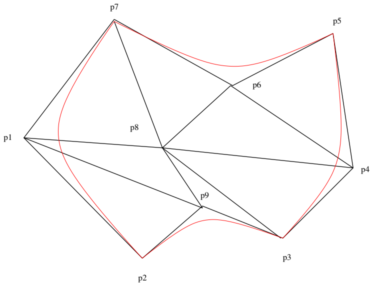
Steps:

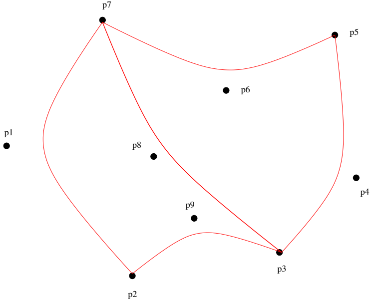
- create mid-edge points and consider them as control points,
- subdivide the mesh with those points,
- deform the subdivided mesh using elasticity analogy to fit with $\mathcal{NURBS}$ boundary curves.

How to curve straight mesh ?









Linear elasticity analogy

Solving on the initial control points mesh $\mathcal{T}_h^0$:

$$\operatorname{div}\left(\lambda \operatorname{tra}(\nabla u) + \mu(\nabla u + \nabla u^T)\right) = 0 \text{ on } \Omega$$
$$u = g \text{ on } \partial\Omega$$

with $\lambda > 0$ and $\mu > 0$.

The deformed control points mesh $\mathcal{T}_h^D$:

$$M^D = M^0 + u(M^0)$$

with M^D vertex of $\mathcal{T}_h^D$ and M^0 vertex of $\mathcal{T}_h^0$

$\Rightarrow$ Dirichet BD : $g(M^0) := M^D - M^0$.

Linear elasticity analogy

Is the deformed mesh legal ?

No if the boundary deformation is too large.
To overcome this problem, we notice that the previous problem is linear.

- Let an element $K^0 = (M_1^0, \ldots, M_{d+1}^0)$ of $\mathcal{T}_h^0$ and

$$K^D = (M_1^0 + u(M_1^0), \ldots, M_{d+1}^0 + u(M_{d+1}^0))$$

  the same element in $\mathcal{T}_h^D$
- A mesh is legal if, for any element K^0,

$$vol(K^0) \text{ and } vol(K^D) \text{ have the same sign.}$$

Linear elasticity analogy

- Considering

$$\omega_{K_0} : \theta \mapsto \mathrm{vol}\Big((M_1^0 + \theta u(M_1^0), \ldots, M_{d+1}^0 + \theta u(M_{d+1}^0)) \Big).$$

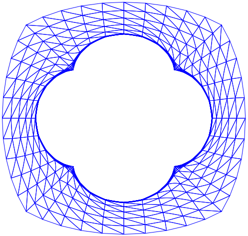
- Find the smallest θ such as there exists one K^0 for which $\omega_{K_0}(\theta) = 0$ is enough to know if resulting mesh is valid.

- This amounts to solving a quadratic (in 2D) or a cubic (in 3D) polynomial on all the simplices of $\mathcal{T}^0$ and look for the smallest root.

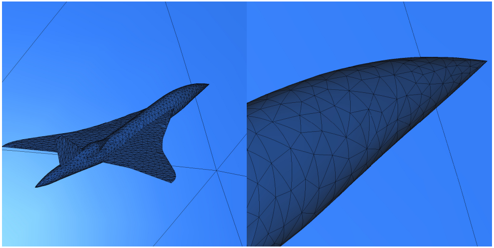
Linear elasticity analogy algorithm

1. Solve linear elasticity equations for the BC g,
2. Look for the smallest θ, say θ_0 such that for any $\theta \in [0, \theta_0[$, $\omega_{K^0}(\theta) > 0$ for any K^0 in $\mathcal{T}^0$, then
   - if $\theta_0 \geq 1$, the final mesh is obtained,
   - if $\theta_0 < 1$, then
     - update the initial mesh with
       $$M^D = M^0 + u(\theta\, M^0),$$
     - goto step 1 with $\mathcal{T}^0 = \mathcal{T}^D$ and $g = (1 - \theta_0)g$

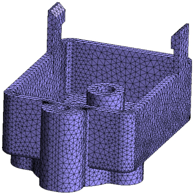
Curved mesh examples



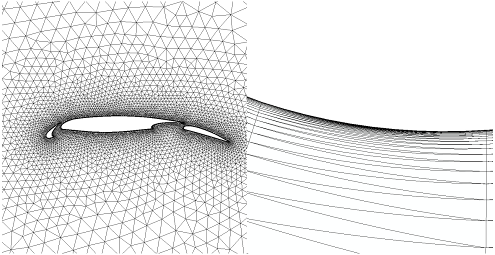
(a) 2d analytic



(b) Airplane



(c) Mechanical part



(d) Three components wing

Isogeometric numerical simulations

Euler or Navier-Stokes equations

Problem to solve: $\begin{cases} \overrightarrow{\nabla}.\overrightarrow{\mathcal{F}}(\boldsymbol{U}, \overrightarrow{\nabla}(\boldsymbol{U})) = 0, \quad \forall x \in \Omega \\ \text{Boundary Conditions } (\mathcal{BC}) + \text{Initial Conditions} \end{cases}$

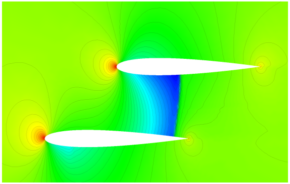
Isogeometric numerical method:

- Residual distributed scheme : generalization of SUPG finite element (Abgrall and al)

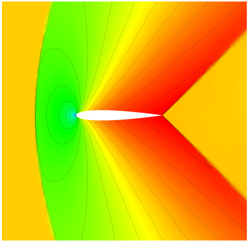
Differences between isogeometrical analysis and classical one:

- Derivates of basis functions not constant on an element
- Degrees of freedom not necessary interpolated

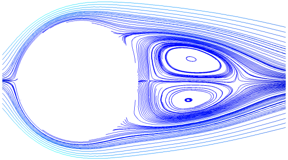
Isogeometric numerical results



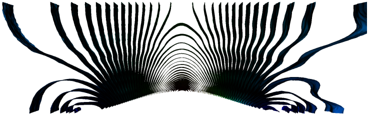
(e) Euler subsonic



(f) Euler supersonic

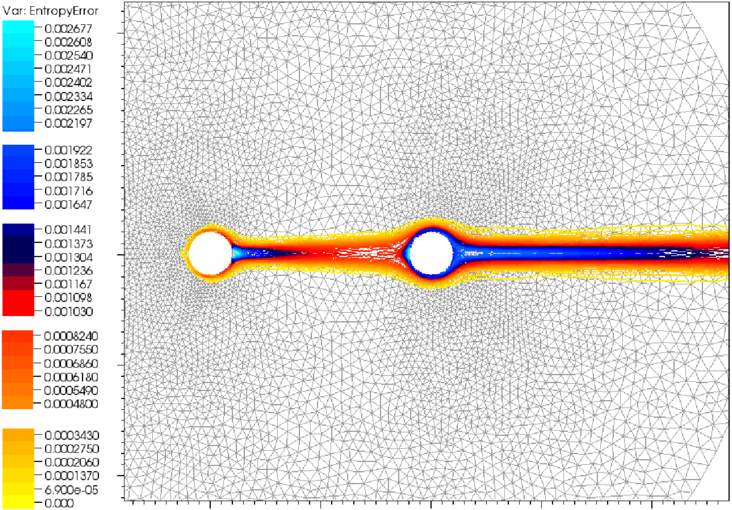


(g) Navier Stokes subsonic



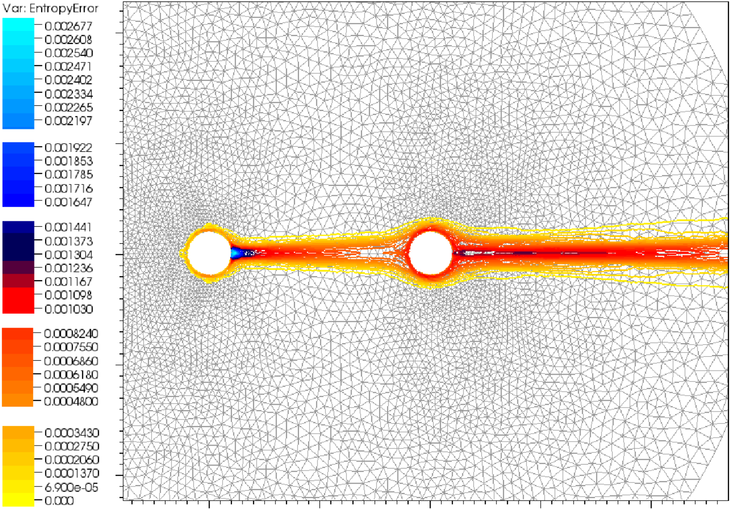
(h) $3d$ test case

Subsonic flow around 2 cylinders: $Ma = 0.38$



Entropy error for P2 simulation

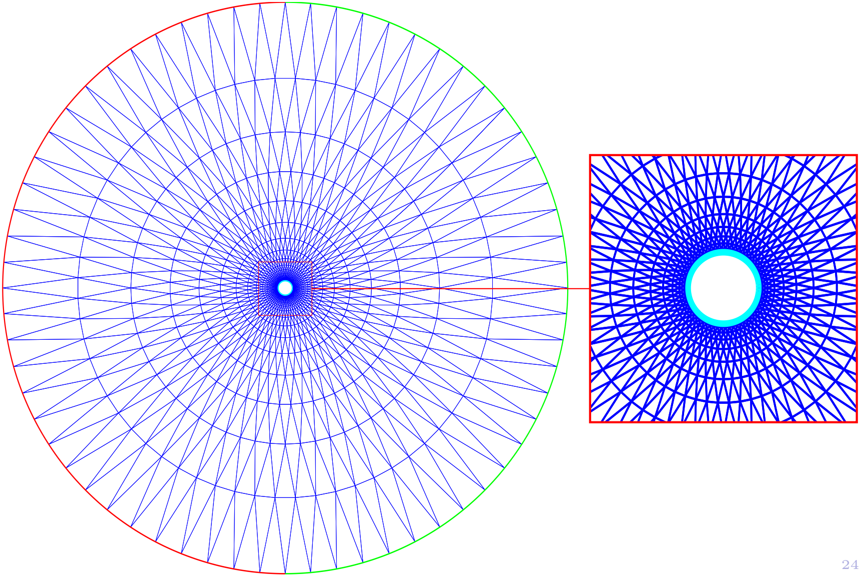
Subsonic flow around 2 cylinders: $Ma = 0.38$



Entropy error for NURBS simulation

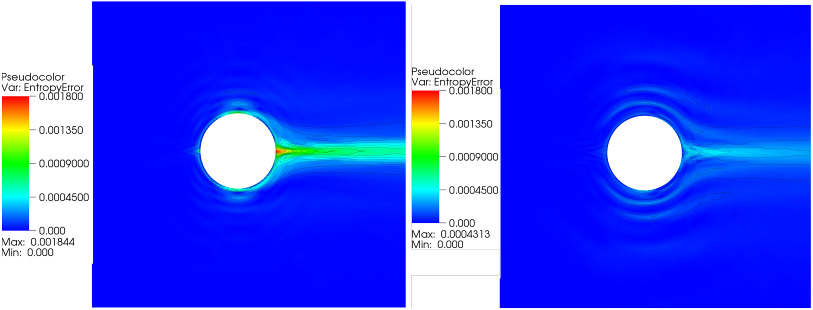
Subsonic flow around a cylinder: $Ma = 0.38$

17×64 vertices mesh



Subsonic flow around a cylinder: $Ma = 0.38$

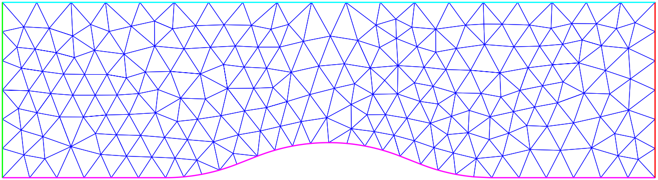
Entropy error: P2 simulation vs NURBS simulation



Maximum error reduced by 76%

Subsonic flow over a bump: $Ma = 0.5$

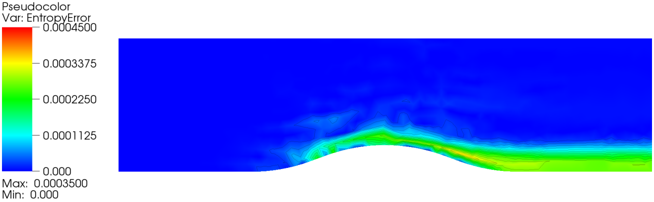
Mesh



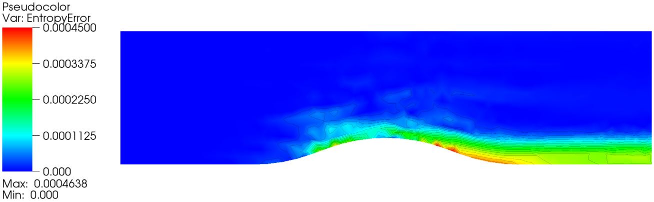
- 240 vertices
- order 3 : 840 degrees of freedom
- order 4 : 1969 degrees of freedom

Subsonic flow over a bump: $Ma = 0.5$

Entropy error



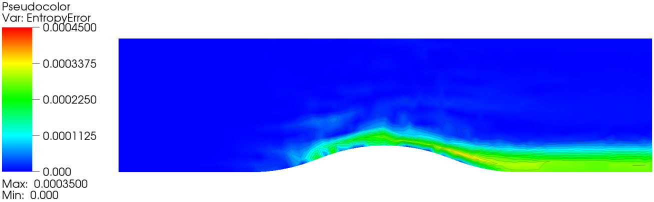
(i) Isogeometric scheme of order 3



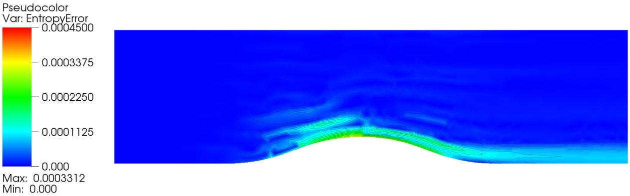
(j) Classic scheme of order 3

Subsonic flow over a bump: $Ma = 0.5$

Entropy error



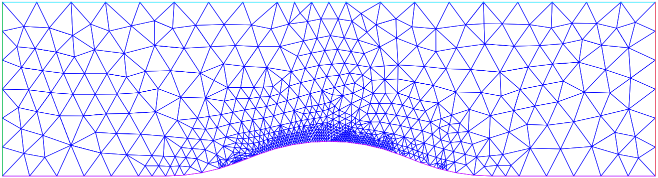
(k) Isogeometric scheme of order 3



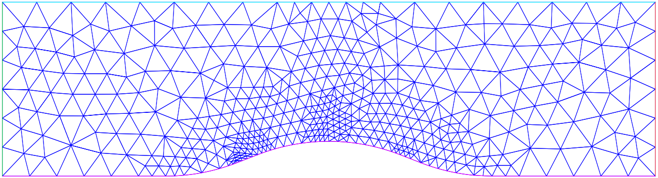
(l) Isogeometric scheme of order 4

Subsonic flow over a bump: $Ma = 0.5$

Mesh refinement



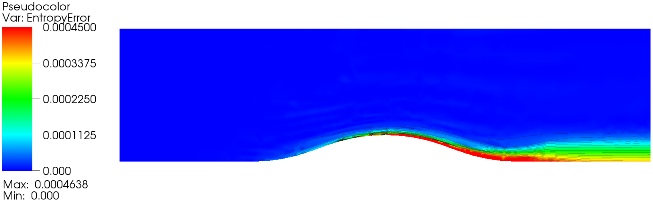
(m) Classical scheme



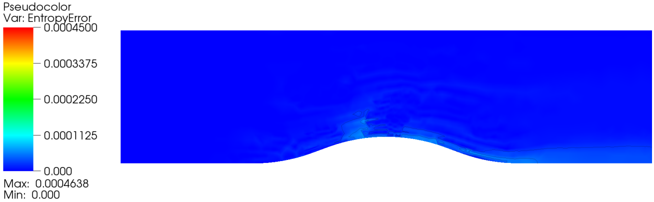
(n) Isogeometric scheme of order 3

Subsonic flow over a bump: $Ma = 0.5$

Entropy error



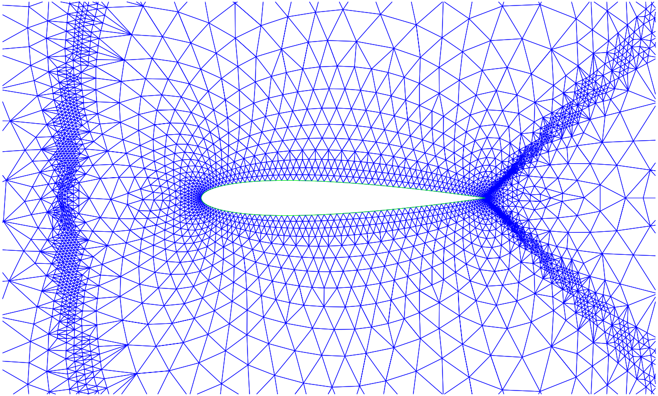
(o) Classical scheme



(p) Isogeometric scheme

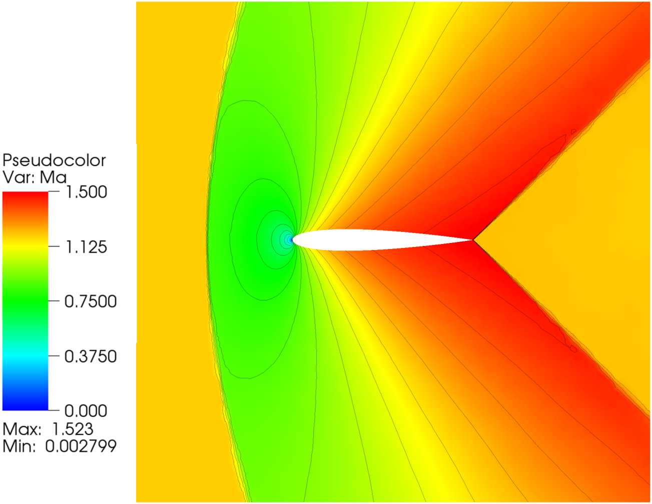
Supersonic flow over a NACA0012: $Ma = 1.2$

Mesh refinement



Supersonic flow over a NACA0012: $Ma = 1.2$

Mach number



P2 turbulent NS simulation

ONERA M6 wing

- inflow mach number: $M_\infty = 0.8395$,
- Angle of Attack: $\alpha = 3.06°$,
- Reynolds number: $\mathrm{Re} = 11.72 \times 10^6$

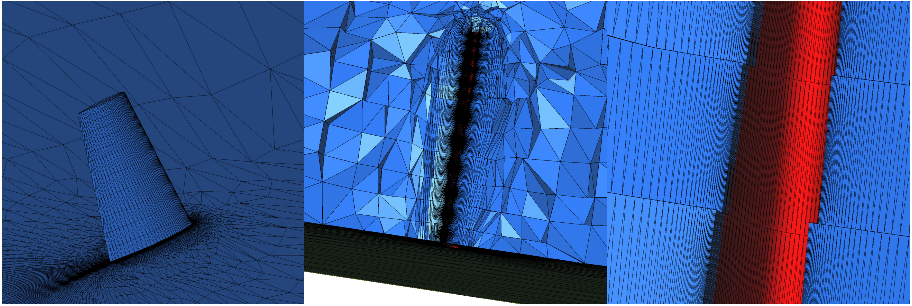
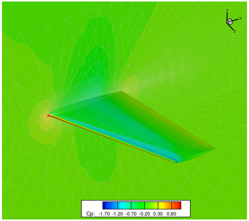
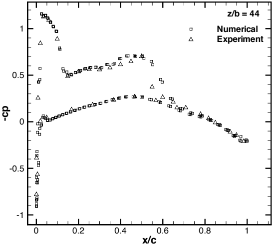


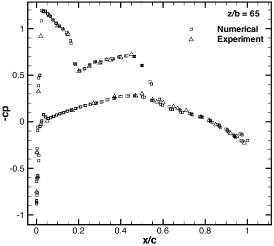
Figure: meshes for Dassault wing test case.

P2 turbulent NS simulation



(a) C_p coefficient.





(b) Cross section of the pressure coefficient at 44%, 65% of the chord compared to experimental results.

Conclusion

Done work

- Génération automatique de maillage Bézier/$\mathcal{NURBS}$ meshes.
- Génération de maillages courbes avec couches limites.
- Simulation Euler and Navier-Stokes avec schémas aux rédidus en formulation P2 Lagrange et isogéométrique.

Perspectives

- Traitement des frontières courbes,
- Modification de maillages courbes (adaptation),
- Ordre très élevé.