Graph cuts and computer vision
an introduction

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Map

- **Introduction :** energie minimization
- **Max flow** and min cut (graph theory)
- **Images as graphs :** an efficient minimization tool
- **Extensions**
- **Discussion**

Introduction : energies

Algorithms and energies

Usual method in computer science :
state the problem $\Longrightarrow$ write an algorithm $\Longrightarrow$ suitable ?

Good case : proof available

$\hookrightarrow$ prove that the algorithm solves the problem
$\hookrightarrow$ ex: to sort a list of words in alphabetical order

Bad case : no proof

$\hookrightarrow$ problem is not precise

$\hookrightarrow$ or it is not clear how the algorithm compares to other ones.

$\Longrightarrow$ there is something wrong !

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Bad case : no proof

$\hookrightarrow$ problem is not precise $\Longrightarrow$ need for more modelisation

$\hookrightarrow$ or it is not clear how the algorithm compares to other ones.

$\Longrightarrow$ need for an objective criterion for quantitative comparison

Energies

Quantitative criterion C related to the current problem:

$\hookrightarrow$ comparison of possible answers : $C(A_1) > C(A_2)$?

$\hookrightarrow$ state the problem mathematically : search for the optimal answer A_0 s.t.:

$$A_0 \in \arg\sup_{A \in \mathcal{X}} C(A)$$

Usually expressed as an energy $E(A)$ to be minimized:

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$\hookrightarrow$ $\mathcal{X}$: search space (including constraints)

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How to minimize energies

- Best case : explicit formula for the solution
  - ↪ ex: search for the center of a cloud of n points P_i
  - ↪ set a mathematical definition : center = mean coordinates
  - ↪ $$\overrightarrow{M} = \frac{1}{n} \sum_i \overrightarrow{P_i}$$
  - ↪ why average ?
- Special cases : ad hoc minimization method suited
  - ↪ Energy and constraints write a particular way
  - ↪ Graph-cuts, loopy belief propagation, kernel methods, dynamic time warping, linear programming, minimum cycles, etc.
- General case : ?
  - ↪ discrete variables : exhaustive search... or stochastic methods (Gibbs...)
  - ↪ continuous variables : gradient descents (possibly stochastic)
    $\Longrightarrow$ local optima, result depends on initialization if non-convex problem

How to minimize energies

- Best case : explicit formula for the solution
  - $\hookrightarrow$ ex: search for the center of a cloud of n points P_i
  - $\hookrightarrow$ set a mathematical definition : center = closest point to all
  - $\hookrightarrow$ $$E(M) = \sum_{1 \leqslant i \leqslant n} \|\overrightarrow{MP_i}\|_2^2$$
  - $\hookrightarrow$ $$\Longrightarrow \overrightarrow{M} = \frac{1}{n} \sum_i \overrightarrow{P_i}$$
  - $\hookrightarrow$ Warning : solution changes with energy design (choice of norm, power, weights, outliers...)
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- Best case : explicit formula for the solution
  - ↪ ex: search for the center of a cloud of n points P_i
  - ↪ set a mathematical definition : center = best fitting Gaussian
  - ↪ $p(P_i) = \frac{1}{(2\pi)^{N/2}|\mathbf{S}|^{1/2}} e^{-\frac{1}{2}\overrightarrow{MP_i}\,\mathbf{S}\,\overrightarrow{MP_i}}$ parameters: M, $\mathbf{S}$
  - ↪ maximize likelihood : $L(M, \mathbf{S}) = \prod_i p(P_i)$
  - ↪ $E = -lnL = \frac{-n}{2}\left(\ln|\mathbf{S}| + M\mathbf{S}M - \frac{2}{n}M\mathbf{S}\sum_i P_i\right) + c^{st} \implies \overrightarrow{M} = \frac{1}{n}\sum_i \overrightarrow{P_i}$
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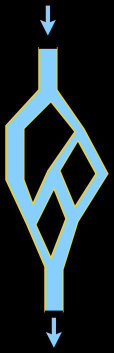
Max flow problem

Water pipes



Max flow problem

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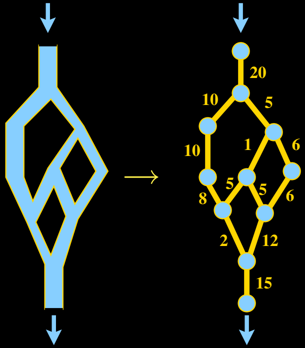

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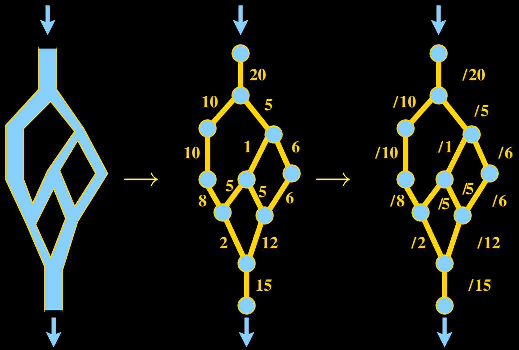

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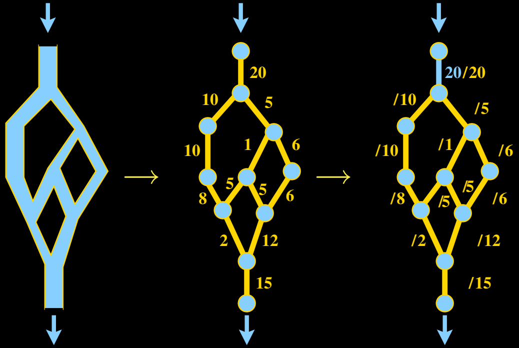

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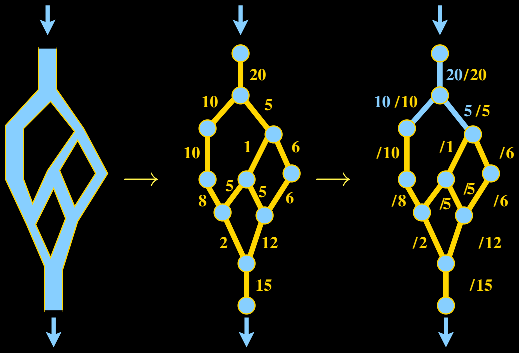

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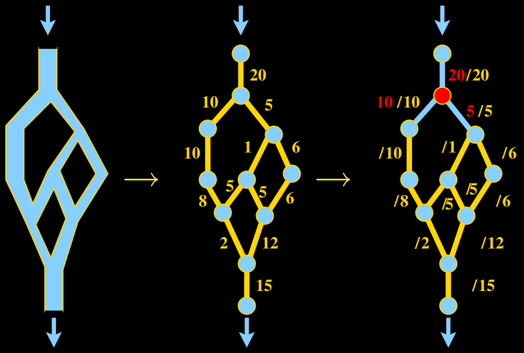

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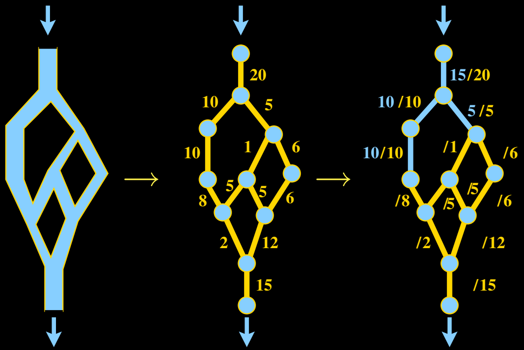

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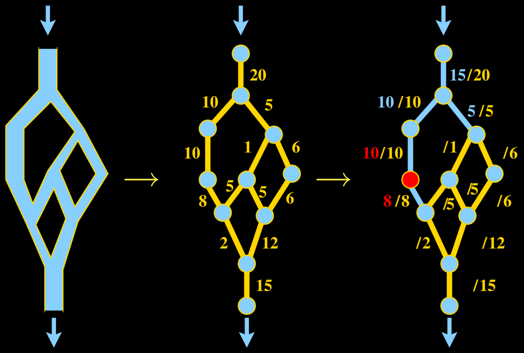

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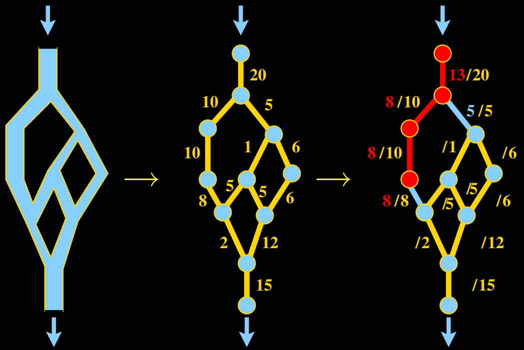

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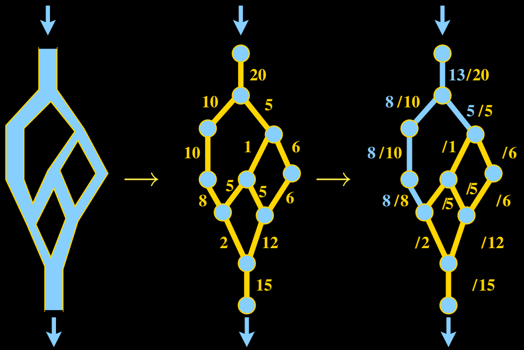

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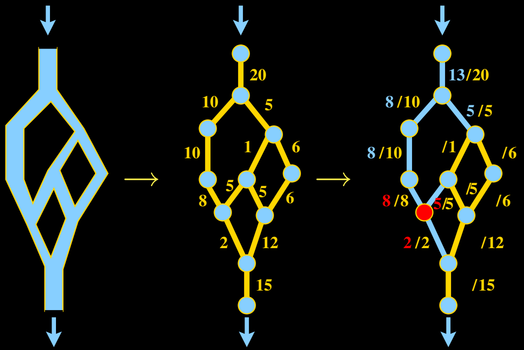

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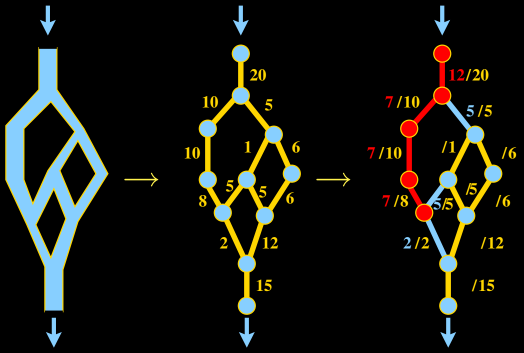

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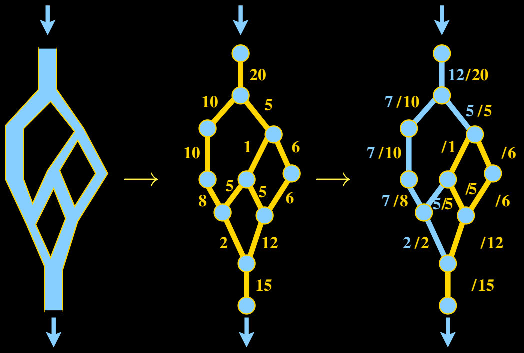

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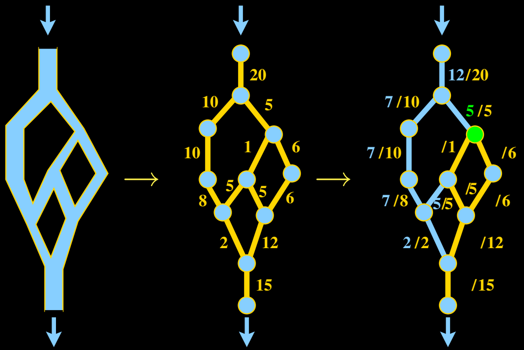

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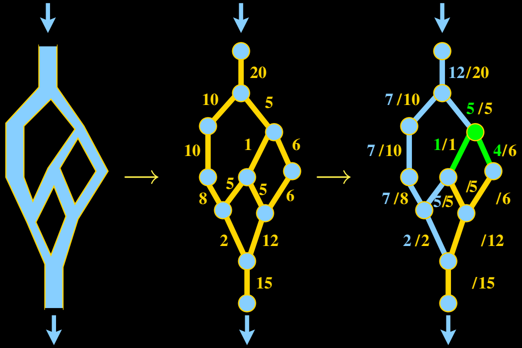

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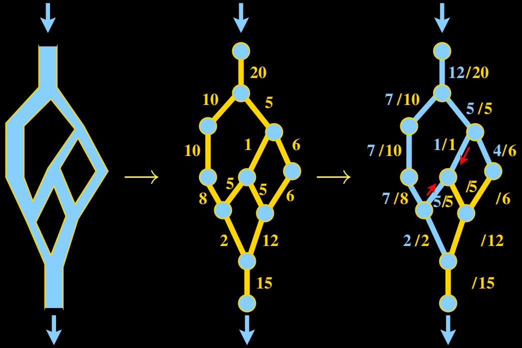

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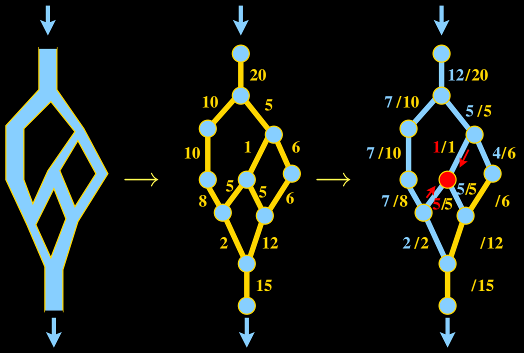

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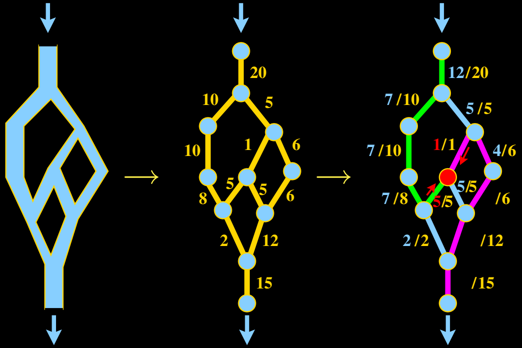

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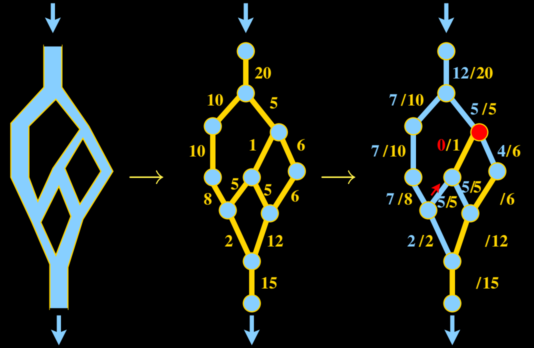

push - relabel

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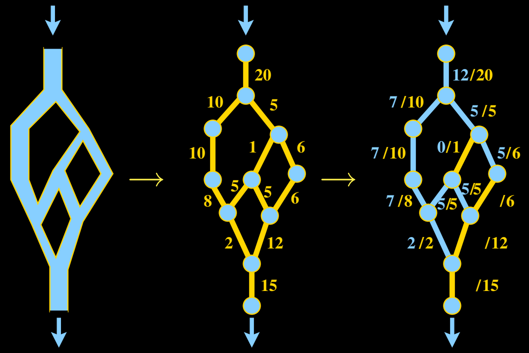

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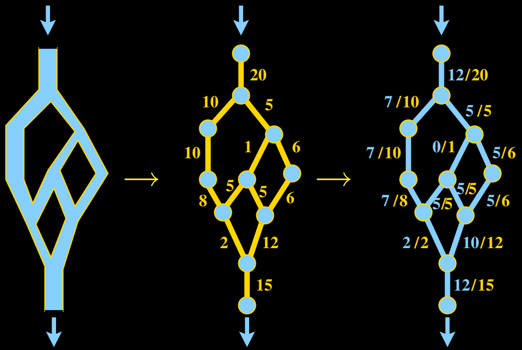

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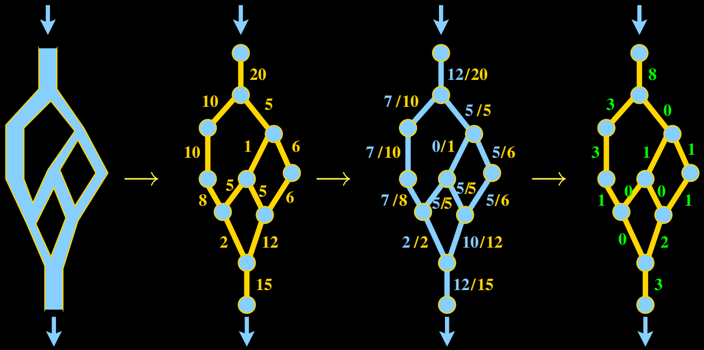
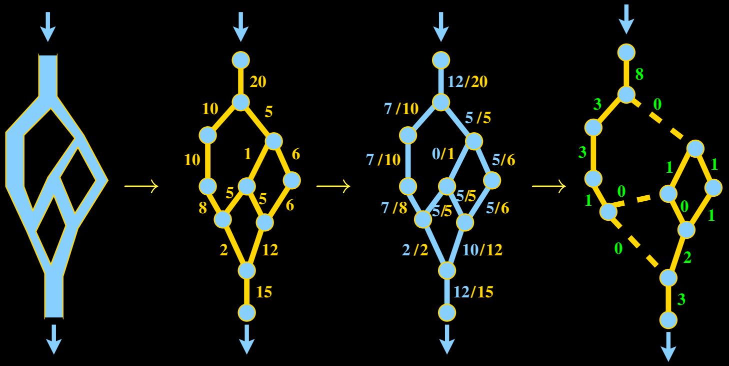


Theorem : maximum flow $\Longleftrightarrow$ no augmenting path in the residual graph

Max flow problem

Water pipes



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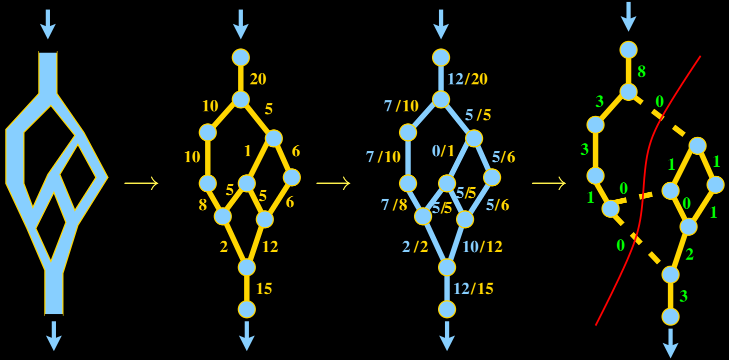

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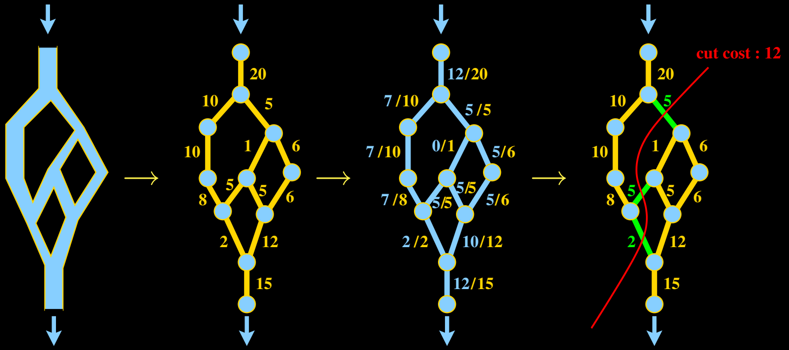


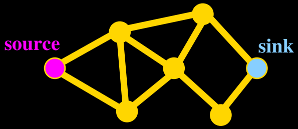
Theorem : maximum flow $\iff$ minimal cut

From graphs to images

Energy minimized on graphs

Graph :

- nodes N_i (including *source* and *sink*)
- weights w_{ij} between nodes N_i and N_j



A cut :

- a partition of the nodes
- a binary function L which associates to each node N_i a label $L(i)$: *source* or *sink*
- cost of a cut : $\sum_{ij} \delta_{L(i) \neq L(j)} w_{ij}$; min cut found e.g. by *Push relabel*

$\Longrightarrow$ **graph cuts give the optimal solution to any binary problem written this way !**

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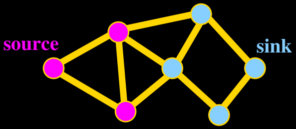

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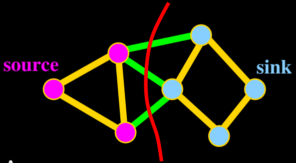

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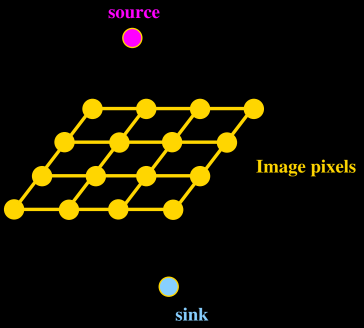
Images as graphs

Build a graph :

- one node for each pixel N_i
- edges between adjacent pixels
- two more nodes : the source A and the sink B
- edges from A and B to all pixels



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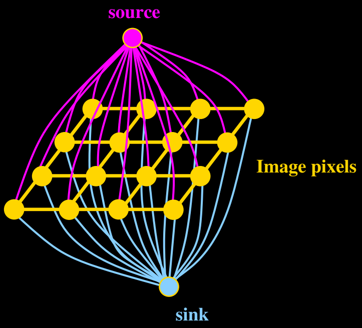

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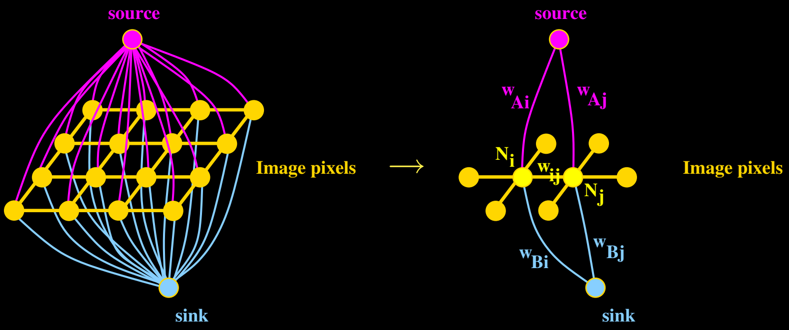
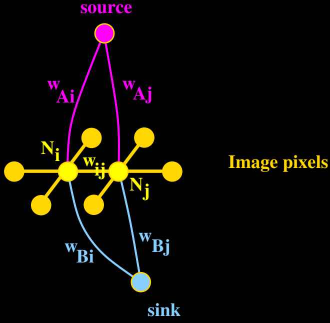


Choosing costs to design energies

Cut cost : sum over edges cut

- vertical edges : for each pixel, either Ai or Bi is cut :
  if $L(i) = A$: w_{Bi}, if $L(i) = B$: w_{Ai}
  $\Longrightarrow \sum_i w_{\neg L(i),\, i}$
- horizontal edges : sum over edges between nodes of different labels
  $\Longrightarrow \sum_{ij} \delta_{L(i) \neq L(j)} w_{ij}$
- Total : $\sum_i w_{\neg L(i),\, i} + \sum_{ij} \delta_{L(i) \neq L(j)} w_{ij}$



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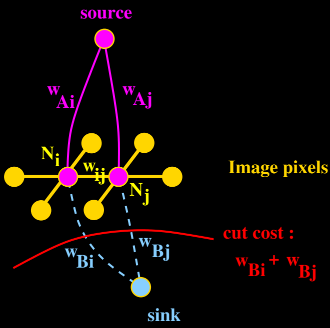

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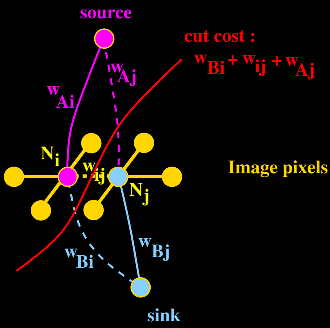

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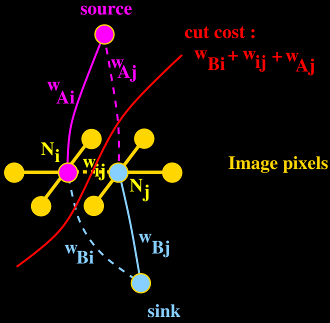

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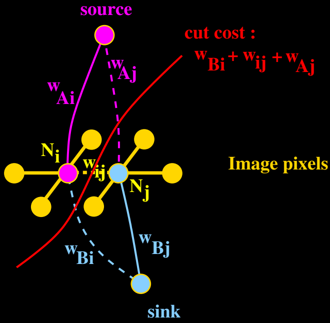
Choose the weights that suit your problem

- vertical edges : individual label preferences for each pixel
  ↪ rename $V_i(\ L(i)\ ) = w_{\neg L(i),\, i}$
  ↪ only constraint : $V_i(L(i))$ should be $\geqslant 0$
- horizontal edges : pairwise interaction between neighboring pixels, 0 if same labels
  ↪ rename $D_{ij}(\ L(i),\ L(j)\ ) = \delta_{L(i) \neq L(j)} w_{ij}$
  ↪ constraints : $D_{ij}(A,A) = D_{ij}(B,B) = 0$ and $D_{ij}(A,B) = D_{ij}(B,A) \geqslant 0$
- Total : $E(L) = \sum_i V_i(L(i)) + \sum_{ij} D_{ij}(L(i), L(j))$
  + constants : $+\sum_i K_i \quad + \sum_{ij} K_{ij}$



- real constraints :
  ↪ no constraint on potentials V_i
  ↪ no constraint on neighborhood choices
  ↪ for each interaction ij : $D_{ij}(A,A) = D_{ij}(B,B) \leqslant D_{ij}(A,B) = D_{ij}(B,A)$ i.e.locally, labels are preferred to be homogeneous

- Total : $\sum_i w_{\neg L(i),\, i} + \sum_{ij} \delta_{L(i) \neq L(j)} w_{ij}$

Choose the weights that suit your problem

- vertical edges : individual label preferences for each pixel
  - ↪ rename $V_i(\ L(i)\ ) = w_{\neg L(i),\, i}$
  - ↪ only constraint : $V_i(L(i))$ should be $\geqslant 0$
- horizontal edges : pairwise interaction between neighboring pixels, 0 if same labels
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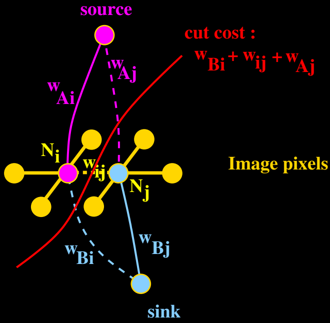

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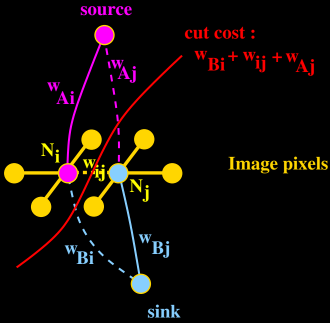

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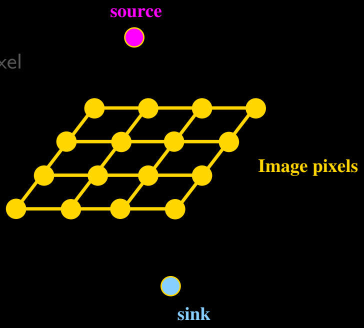

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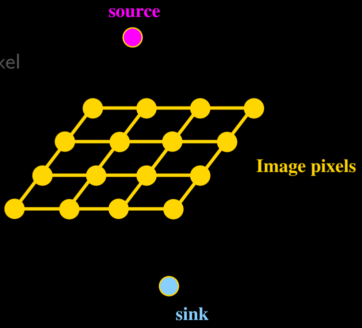

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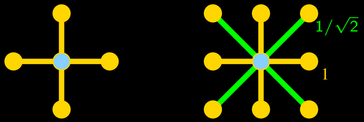
$\Longrightarrow$ graph cuts give the global optimum to any binary problem written this way !

Neighborhoods

Any neighborhood can be chosen
but
the choice will influence the shape of the cut

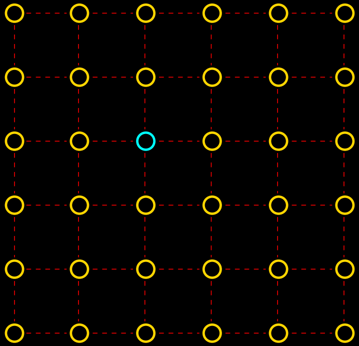
$\hookrightarrow$ 4-neighborhood $\Longrightarrow$ vertical and horizontal segments

$\hookrightarrow$ 8-neighborhood $\Longrightarrow$ $\approx$ ok in practice



Example : binary segmentation

- For each pixel i of image A associate a node
- define 8-neighborhood
- two possible labels : black (B) and white (W)
- set potentials: $V_i(B) = A(i), \quad V_i(W) = 256 - A(i)$
- set spatial coherency: $D_{ij}(L_i, L_j) = K\, \delta_{L_i \neq L_j} \left(\frac{1}{\varepsilon + |A(i) - A(j)|} \right)$



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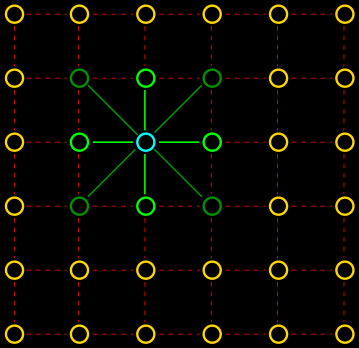


Example : binary segmentation

original image

$+$ noise

threshold

graph cut

Extension to multi-label problems

Previously : binary problems : $L(i) \in \{0, 1\}$
Now : multi-label problems : $L(i) \in \mathcal{X}$ (discrete set, indep. of i)

In some cases : possible to build a graph to get the global optimum
Most often : use (α, β)-swap or α-expansions

α-expansions :

$\hookrightarrow$ $D_{ij}(\cdot, \cdot)$: required to be distance on labels

$\hookrightarrow$ iteratively : choose one particular label α, and consider the binary problem : for each pixel i, is it better to keep current label $L(i)$, or to move to label α ?

$\hookrightarrow$ each step solved by graph-cuts

$\hookrightarrow$ repeat until no evolution

$\hookrightarrow$ convergence and good local optimum guaranteed

Energies that can be minimized

- Minimizing $E(L) = \sum_i V_i(\, L(i)\,) + \sum_{ij} D_{ij}(\, L(i),\, L(j)\,)$
  is NP-hard in the general case
- The sub-modularity condition $D_{ij}(A, A) = D_{ij}(B, B) < D_{ij}(A, B) = D_{ij}(B, A)$ makes it minimizable by graph-cuts
- If labels are ordered : $D_{ij}(\, L(i),\, L(j)\,) = g_{ij}(\, L(i) - L(j)\,)$ with g_{ij} convex $\Longrightarrow$ global optimum
- (α, β) swap : if D_{ij} is a semi-metric on labels
- α expansion : if D_{ij} is a metric : good local minimum guaranteed theoretically

Probabilistic / Markovian rewriting

$$
\begin{aligned}
p(L) &\propto \exp(-E(L)) \\
&\propto \exp\Big(-\sum_i V_i(\,L(i)\,)\Big)\ \exp\Big(-\sum_{ij} D_{ij}(\,L(i),\,L(j)\,)\Big) \\
&\propto \prod_i e^{-V_i(\,L(i)\,)} \prod_{i\sim j} e^{-D_{ij}(\,L(i),\,L(j)\,)} \\
&\propto \prod_i p_i(\,L(i)\,) \prod_{i\sim j} q_{ij}(\,L(i),\,L(j)\,)
\end{aligned}
$$

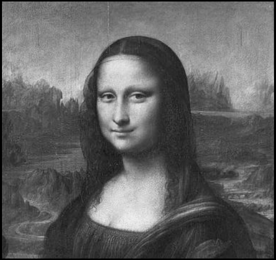
$$
p(\,L(i)\,|\,L(k)\,\forall k \neq i) \;\propto\; p_i(\,L(i)\,) \prod_{j\sim i} q_{ij}(\,L(i),\,L(j)\,) \;\propto\; p(\,L(i)\,|\,L(j)\,\forall j \in \mathcal{N}_i)
$$

$$
p(\,L(i)\,|\,L(j)\,) \;\propto\; p_i(\,L(i)\,)\ q_{ij}(\,L(i),\,L(j)\,)
$$

An example in the probabilistic setting : colorization

- Learning how to color greyscale images
- training set : one or several color images
- main idea : copy colors from patches with similar greyscale texture

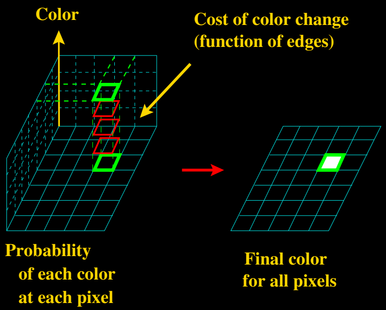
$+$



$= ?$

An other example

- pixel $\mapsto$ greyscale patch $\mapsto$ texture features (SURF)
- SVR : features $\mapsto$ proba(each color)
- SVR : features $\mapsto$ norm of the gradient of the color
- graph cut : proba(each color) $\times$ cost of color change $\mapsto$ color



Minimize energy: $\prod_i \psi(color\ c_i, patch\ p_i) \times \prod_{i \sim j} \Psi(c_i, c_j | p_i, p_j)$

An other example

Examples of results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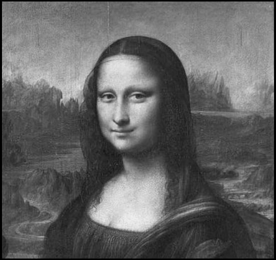
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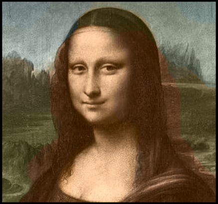

Examples of results

An other example

Examples of results

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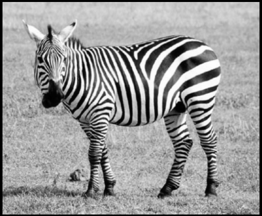

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An other example

Examples of results

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Variations

- directed vs. undirected graph
- higher-order interaction, with m variables : $V_{i,j,k\ldots}(\,L(i),\,L(j),\,L(k)\ldots)$
- dynamic cut : knowing a solution to a close problem $\Longrightarrow$ iterative
- active graph cut : knowing a solution on a part of the graph $\Longrightarrow$ multi-scale
- multiple sources or sinks within the image

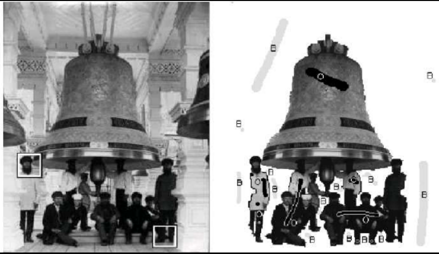
Complexity

- Theoretical complexity, worse case : about $(\#\text{pixels})^3 \times (\#\text{labels})^2$ (depends on the algorithm)
- In practice : worse case never reached, much faster
- GPU implementation possible $\Longrightarrow$ incredibly fast

Some applications

Any problem that can be written as a multi-labelling problem with simple local interaction terms (Markov fields)

- image denoising
- image segmentation, knowing color histograms of objects
- segmentation knowing seeds (points inside and outside the object)



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- multi-scale approach for segmentation
- iterative segmentation with parameter estimations (e.g. color histograms)

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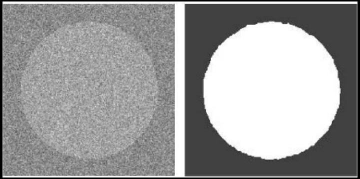
- EM algorithms : iterative clustering / parameter-estimation



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- ...
- iterative segmentation with parameter estimations (e.g. color histograms)
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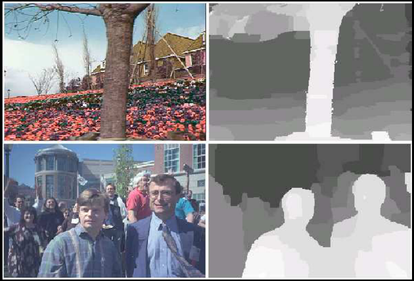
- stereovison



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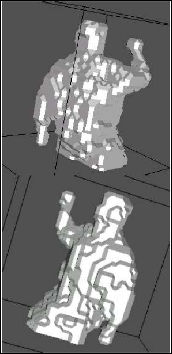
- ...
- stereovison
- 3D-reconstruction



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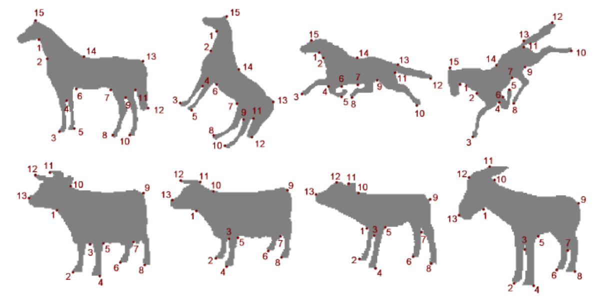
- ...
- stereovison
- 3D-reconstruction
- video segmentation : based on motion
- texture synthesis

Some applications

Any problem that can be written as a multi-labelling problem with simple local interaction terms (Markov fields)

- ...
- texture synthesis
- shape matching (different kind of graph)
- segmentation with rigid shape prior (insanely huge graph)



Discussion

Pros:

↪ gives the global optimum of certain types of energies

↪ gives a very good local optimum of all Markov-like energies with discrete values

↪ practical way to bring spatial coherency

↪ it's fast

Cons :

↪ only those kinds of simple energies

↪ tends to make people do simplisitic modelings

Competitor :

↪ loopy belief propagation

References

Tutorials

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