HDG* methods implemented in HORSE†

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Contents

*Hybridizable Discontinuous Galerkin
†High Order solver for Radar cross Section Evaluation

1 Problem statement and notations

1.1 Boundary value problem

The system of 3D time-harmonic Maxwell's equations is considered

$$\begin{cases} \mathrm{i}\omega\varepsilon_r \mathbf{E} - \mathbf{curl}\,\mathbf{H} = -\mathbf{J}, \text{ in } \Omega, \\ \mathrm{i}\omega\mu_r \mathbf{H} + \mathbf{curl}\,\mathbf{E} = 0, \text{ in } \Omega, \end{cases} \tag{1}$$

where i is the imaginary unit, ω is the angular frequency, ε_r and μ_r are the relative permittivity and permeability, $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{J}$ is a known current source. The boundary of the computational domain Ω is $\partial\Omega = \Gamma_m \cup \Gamma_a$ ($\Gamma_m \cap \Gamma_a = \emptyset$) on which we impose the following boundary conditions

$$\begin{cases} \mathbf{n} \times \mathbf{E} = 0, \text{ on } \Gamma_m, \\ \mathbf{n} \times \mathbf{E} + \mathbf{n} \times (\mathbf{n} \times \mathbf{H}) = \mathbf{n} \times \mathbf{E}^{\mathrm{inc}} + \mathbf{n} \times (\mathbf{n} \times \mathbf{H}^{\mathrm{inc}}) = \mathbf{g}^{\mathrm{inc}}, \text{ on } \Gamma_a, \end{cases} \tag{2}$$

where $\mathbf{n}$ is the outward unit normal vector and $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$ is a given incident electromagnetic wave. The boundary condition on Γ_m indicates a metallic boundary condition (also called perfect electric conductor condition), while the second relation on Γ_a states a Silver-Müller condition (first order absorbing boundary condition). For sake of simplicity, we omit the volume source term $\mathbf{J}$ in what follows but it can be straightforwardly added.

1.2 Notations

We consider a simplicial mesh $\mathcal{T}_h$ (consisting of tetrahedral element K) of the computational domain Ω. We denote by $\mathcal{F}_h^I$ the union of all interior interfaces of $\mathcal{T}_h$, by $\mathcal{F}_h^B$ the union of all boundary interfaces of $\mathcal{T}_h$, and by $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^b$. Note that $\partial\mathcal{T}_h$ is the set of all triangular faces forming the boundary ∂K, for all elements K of $\mathcal{T}_h$. Consequently, an interior face shared by two neighboring elements appears twice in this set $\partial\mathcal{T}_h$, while it appears once in the set $\mathcal{F}_h$. For an interface $F = \overline{K}^+ \cap \overline{K}^- \in \mathcal{F}_h^I$, let $\mathbf{v}^\pm$ be the trace of $\mathbf{v}$ on F from $K^\pm$. We define on this face *mean values* $\{\cdot\}$ and *jumps* $[\![\cdot]\!]$

$$\begin{cases} \{\mathbf{v}\}_F = \dfrac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-), \\ [\![\mathbf{v}]\!]_F = \mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^-, \end{cases}$$

where $\mathbf{n}^\pm$ is the outward unit normal vector to $K^\pm$. For a boundary face $F \in \partial K^+ \cap \partial\Omega$ these expressions are modified as

$$\begin{cases} \{\mathbf{v}\}_F = \mathbf{v}^+, \\ [\![\mathbf{v}]\!]_F = \mathbf{n}^+ \times \mathbf{v}^+. \end{cases}$$

Now we introduce the discontinuous finite element spaces and some basic operations on these spaces for later use. Let $\mathbb{P}_p(D)$ denotes the space of polynomial functions of degree at most p in a domain D. Let $\mathbf{V}_h$ the global approximation space defined by

$$\mathbf{V}_h = \left\{ \mathbf{v} \in \left[L^2(\Omega)\right]^3 \mid \mathbf{v}_{|_{K_e}} \in [\mathbb{P}_{p_e}(K_e)]^3, \quad \forall K_e \in \mathcal{T}_h \right\}. \tag{3}$$

Note that this approximation space is common to DG and HDG methods. It represents the space where we seek an approximation of the electromagnetic field. We also introduce a traced finite element space $\mathbf{M}_h$ defined by

$$\mathbf{M}_h = \left\{ \boldsymbol{\eta} \in \left[L^2(\mathcal{F}_h)\right]^3 \mid \boldsymbol{\eta}_{|_{F_f}} \in \left[\mathbb{P}_{p_f}(F_f)\right]^3, (\boldsymbol{\eta} \cdot \mathbf{n})_{|_{F_f}} = 0, \quad \forall F_f \in \mathcal{F}_h \right\}. \tag{4}$$

This space is specific to the HDG method, it represents the approximation space for traces (particularly for the *hybrid variable* that we will define in the following). For two vectorial functions $\mathbf{u}$ et $\mathbf{v}$ in $[L^2(D)]^3$, we denote $(\mathbf{u},\mathbf{v})_D = \int_D \mathbf{u}\cdot\overline{\mathbf{v}}\,\mathrm{d}\mathbf{x}$ where D is a domain of $\mathbb{R}^3$, and we denote $<\mathbf{u},\mathbf{v}>_F = \int_F \mathbf{u}\cdot\overline{\mathbf{v}}\,\mathrm{d}s$ where F is a two-dimensional face. Accordingly, for the mesh $\mathcal{T}_h$ we write

$$(\cdot,\cdot)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (\cdot,\cdot)_K\,, \qquad \langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} \langle\cdot,\cdot\rangle_{\partial K}\,,$$
$$\langle\cdot,\cdot\rangle_{\mathcal{F}_h} = \sum_{F\in\mathcal{F}_h} \langle\cdot,\cdot\rangle_F\,, \qquad \langle\cdot,\cdot\rangle_{\Gamma_a} = \sum_{F\in\mathcal{F}_h\cap\Gamma_a} \langle\cdot,\cdot\rangle_F\,.$$

Finally we set

$$\gamma_t(\mathbf{v}) = -\mathbf{n}\times(\mathbf{n}\times\mathbf{v})\,, \quad \gamma_n(\mathbf{v}) = \mathbf{n}\,(\mathbf{n}\cdot\mathbf{v})\,,$$

where $\gamma_t(\mathbf{v})$ and $\gamma_n(\mathbf{v})$ denote the tangential and normal components of $\mathbf{v}$ on a face of unit normal $\mathbf{n}$, and $\mathbf{v} = \gamma_t(\mathbf{v}) + \gamma_n(\mathbf{v})$.

2 Principles and general formulation of the HDG method

As in a classical DG method we seek an approximation of the electromagnetic field solution of (1), denoted by $(\mathbf{E}_h,\mathbf{H}_h)$, in the space $\mathbf{V}_h\times\mathbf{V}_h$ such that for all K in $\mathcal{T}_h$

$$\begin{cases} (\mathrm{i}\omega\varepsilon_r\mathbf{E}_h,\mathbf{v})_K - (\mathbf{curl}\,\mathbf{H}_h,\mathbf{v})_K = 0,\ \forall\mathbf{v}\in\mathbf{V}_h,\\ (\mathrm{i}\omega\mu_r\mathbf{H}_h,\mathbf{v})_K + (\mathbf{curl}\,\mathbf{E}_h,\mathbf{v})_K = 0,\ \forall\mathbf{v}\in\mathbf{V}_h. \end{cases} \tag{5}$$

Numerical traces $\widehat{\mathbf{E}}_h$ and $\widehat{\mathbf{H}}_h$ are introduced by applying appropriate Green's formulas

$$\begin{cases} (\mathrm{i}\omega\varepsilon_r\mathbf{E}_h,\mathbf{v})_K - (\mathbf{H}_h,\mathbf{curl}\,\mathbf{v})_K + \left\langle\widehat{\mathbf{H}}_h,\mathbf{n}\times\mathbf{v}\right\rangle_{\partial K} = 0,\ \forall\mathbf{v}\in\mathbf{V}_h,\\ (\mathrm{i}\omega\mu_r\mathbf{H}_h,\mathbf{v})_K + (\mathbf{E}_h,\mathbf{curl}\,\mathbf{v})_K - \left\langle\widehat{\mathbf{E}}_h,\mathbf{n}\times\mathbf{v}\right\rangle_{\partial K} = 0,\ \forall\mathbf{v}\in\mathbf{V}_h. \end{cases} \tag{6}$$

The key to define a DG method lies in the definition of the numerical traces. It enables to weakly enforce the continuity conditions between neighboring elements and thus ensures the consistency of the method. In a classical DG method we couple the local traces of the electromagnetic field $\mathbf{E}_h$ and $\mathbf{H}_h$ between the neighboring elements

$$\widehat{\mathbf{E}}_h = \{\mathbf{E}_h\} + \alpha_H[\![\mathbf{H}_h]\!] \text{ and } \widehat{\mathbf{H}}_h = \{\mathbf{H}_h\} + \alpha_E[\![\mathbf{E}_h]\!],$$

where α_H and α_E are positive penalty parameters. To approximate the electromagnetic field $(\mathbf{E}_h,\mathbf{H}_h)$ for an element K of the mesh $\mathcal{T}_h$ we need the value of the field of each neighboring element of K, i.e. that we need all degrees of freedom of the neighboring elements. Consequently a classical DG method leads to a global linear system to solve and the globally coupled degrees of freedom is

$$\sum_{e=1}^{|\mathcal{T}_h|} 6N_K^e,$$

where N_K^e is the dimension of the space $\mathbb{P}_{p_e}(K_e)$, i.e. $N_K^e = (p_e+1)(p_e+2)(p_e+3)/6$.

The aim of the HDG method is to reduce substantially the number of the globally coupled degrees of freedom. The key to define the HDG method lies in the definition of an hybrid variable which represents an additional unknown on on each face of $\mathcal{F}_h$. A so-called conservativity condition is imposed on the numerical trace, whose definition involved the hybrid variable, at the interface between neighboring elements. As result, the HDG method leads to a linear system in terms of the degrees of freedom of the hybrid variable only. In this way, the number of globally coupled degrees of freedom is reduced. The local values of the electromagnetic fields can be obtained by solving local problems element-by-element. For the proposed HDG method (formulated in next section) the number of globally coupled degrees of freedom is

$$\sum_{f=1}^{|\mathcal{F}_h|} 2N_F^f,$$

where N_F^f is the dimension of the space $\mathbb{P}_{p_f}(F_f)$, i.e. $N_F^f = (p_f+1)(p_f+2)/2$.

Remark 1 Assuming the interpolation degrees are $p_e = p_f = p$, $\forall K_e \in \mathcal{T}_h$, $\forall F_f \in \mathcal{F}_h$, the number of globally coupled degrees of freedom is then

$$\begin{array}{ll} \textit{DGM :} & (p+1)(p+2)(p+3)|\mathcal{T}_h|, \\ \textit{HDGM :} & (p+1)(p+2)|\mathcal{F}_h|. \end{array}$$

For a simplicial mesh $|\mathcal{F}_h| \approx 2|\mathcal{T}_h|$, the ratio of the globally coupled degrees of freedom is roughly $2/(p+3)$ for HDG method over DG method.

Now we are interested in the formulation of the HDG method. First note that $\mathbf{n}\times\mathbf{v} = \mathbf{n}\times\gamma_t(\mathbf{v})$, thus the system (6) is equivalent to

$$\begin{cases} (\mathrm{i}\omega\varepsilon_r\mathbf{E}_h,\mathbf{v})_K - (\mathbf{H}_h,\mathbf{curl}\,\mathbf{v})_K + \left\langle \gamma_t(\widehat{\mathbf{H}}_h), \mathbf{n}\times\mathbf{v}\right\rangle_{\partial K} = 0,\ \forall \mathbf{v}\in\mathbf{V}_h, \\ (\mathrm{i}\omega\mu_r\mathbf{H}_h,\mathbf{v})_K + (\mathbf{E}_h,\mathbf{curl}\,\mathbf{v})_K - \left\langle \gamma_t(\widehat{\mathbf{E}}_h), \mathbf{n}\times\mathbf{v}\right\rangle_{\partial K} = 0,\ \forall \mathbf{v}\in\mathbf{V}_h. \end{cases} \tag{7}$$

We introduce the hybrid variable $\mathbf{\Lambda}_h$ defined by

$$\mathbf{\Lambda}_h := \gamma_t(\widehat{\mathbf{H}}_h), \quad \forall F\in\mathcal{F}_h. \tag{8}$$

In the HDG method we want to formulate the local fields in K through (7) assuming that $\mathbf{\Lambda}_h$ is known on all faces of an element K. In order to achieve this, we consider a numerical trace $\gamma_t(\widehat{\mathbf{E}}_h)$ of the form

$$\gamma_t(\widehat{\mathbf{E}}_h) = \gamma_t(\mathbf{E}_h) + \tau^K\mathbf{n}\times(\mathbf{\Lambda}_h - \gamma_t(\mathbf{H}_h)) \text{ on } \partial K, \tag{9}$$

where τ^K is a local stabilization parameter.

Remark 2 Once the hybrid variable $\mathbf{\Lambda}_h$ is obtained on all the faces of an element K the electromagnetic field inside this element can be solved through the associated local linear system (7) *using the numerical traces defined by* (8) *and* (9)*.*

Adding all contributions of (7) over all elements and enforcing the continuity of the tangential component of $\widehat{\mathbf{E}}_h$, we can formulate the following problem: find $(\mathbf{E}_h,\mathbf{H}_h,\mathbf{\Lambda}_h)\in\mathbf{V}_h\times\mathbf{V}_h\times\mathbf{M}_h$ such that

$$\begin{cases} (\mathrm{i}\omega\varepsilon_r\mathbf{E}_h,\mathbf{v})_{\mathcal{T}_h} - (\mathbf{H}_h,\mathbf{curl}\,\mathbf{v})_{\mathcal{T}_h} + \langle \mathbf{\Lambda}_h, \mathbf{n}\times\mathbf{v}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \mathbf{v}\in\mathbf{V}_h, \\ (\mathrm{i}\omega\mu_r\mathbf{H}_h,\mathbf{v})_{\mathcal{T}_h} + (\mathbf{E}_h,\mathbf{curl}\,\mathbf{v})_{\mathcal{T}_h} - \left\langle \gamma_t(\widehat{\mathbf{E}}_h), \mathbf{n}\times\mathbf{v}\right\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \mathbf{v}\in\mathbf{V}_h, \\ \left\langle [\![\gamma_t(\widehat{\mathbf{E}}_h)]\!], \boldsymbol{\eta}\right\rangle_{\mathcal{F}_h} - \langle\mathbf{\Lambda}_h,\boldsymbol{\eta}\rangle_{\Gamma_a} = \left\langle \mathbf{g}^{\mathrm{inc}},\boldsymbol{\eta}\right\rangle_{\Gamma_a},\ \forall\boldsymbol{\eta}\in\mathbf{M}_h, \end{cases} \tag{10}$$

where the last equation is called the conservativity condition, with which we ask the tangential component of $\widehat{\mathbf{E}}_h$ to be weakly continuous across any interfaces between neighboring elements. With the definition of $\gamma_t(\widehat{\mathbf{E}}_h)$ we employ again a Green formula in the second equation of (10) to obtain the following problem: find $(\mathbf{E}_h,\mathbf{H}_h,\mathbf{\Lambda}_h)\in\mathbf{V}_h\times\mathbf{V}_h\times\mathbf{M}_h$ such that

$$\begin{cases} (\mathrm{i}\omega\varepsilon_r\mathbf{E}_h,\mathbf{v})_{\mathcal{T}_h} - (\mathbf{H}_h,\mathbf{curl}\,\mathbf{v})_{\mathcal{T}_h} + \langle \mathbf{\Lambda}_h, \mathbf{n}\times\mathbf{v}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \mathbf{v}\in\mathbf{V}_h, \\ (\mathrm{i}\omega\mu_r\mathbf{H}_h,\mathbf{v})_{\mathcal{T}_h} + (\mathbf{curl}\,\mathbf{E}_h,\mathbf{v})_{\mathcal{T}_h} + \langle \tau\mathbf{n}\times(\mathbf{H}_h-\mathbf{\Lambda}_h), \mathbf{n}\times\mathbf{v}\rangle_{\partial\mathcal{T}_h} = 0,\ \forall \mathbf{v}\in\mathbf{V}_h, \\ \langle\mathbf{n}\times\mathbf{E}_h,\boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h} + \langle\tau(\gamma_t(\mathbf{H}_h)-\mathbf{\Lambda}_h),\boldsymbol{\eta}\rangle_{\partial\mathcal{T}_h} - \langle\mathbf{\Lambda}_h,\boldsymbol{\eta}\rangle_{\Gamma_a} = \left\langle \mathbf{g}^{\mathrm{inc}},\boldsymbol{\eta}\right\rangle_{\Gamma_a},\ \forall\boldsymbol{\eta}\in\mathbf{M}_h. \end{cases} \tag{11}$$

Note that we have used

$$\mathbf{n}\times\gamma_t(\mathbf{v}) = \mathbf{n}\times\mathbf{v} \text{ and } \mathbf{n}\times(\mathbf{n}\times\gamma_t(\mathbf{v})) = -\gamma_t(\mathbf{v}),$$

to obtain (11).

In summary we can decomposed the HDG method in two steps:

1. A conservativity condition (third equation of (11)) is imposed on the numerical trace, whose definition involved the hybrid variable at the interface between neighboring elements. As result we obtain a global linear system in terms of the degrees of freedom of the hybrid variable.

2. Once the degrees of freedom of the hybrid variable are known, the local values of the electromagnetic fields can be obtained by solving local linear systems element-by-element from the first and the second equation of (11).

3 Application of the HDG method

In this section we discretize the HDG method leading:

1. Matrix formulations of local solvers to approximate the local values (i.e. the degree of freedom) of the electromagnetic fields element-by-element.

2. Matrix formulation of the global solver to approximate the values of the numerical trace, i.e. the degree of freedom of the hybrid variable, on each face of $\mathcal{F}_h$.

First we introduce some notations and definitions for later use. We denote the restriction of the electromagnetic field on an element K_e in $\mathcal{T}_h$ by $(\mathbf{E}_h|_{K_e}, \mathbf{H}_h|_{K_e}) = (\mathbf{E}^e, \mathbf{H}^e) : K_e \times K_e \longrightarrow \mathbb{C}^3 \times \mathbb{C}^3$, where $\mathbf{E}^e(\mathbf{x}) = \left[E^e_x(\mathbf{x}), E^e_y(\mathbf{x}), E^e_z(\mathbf{x})\right]^T$ and $\mathbf{H}^e(\mathbf{x}) = \left[H^e_x(\mathbf{x}), H^e_y(\mathbf{x}), H^e_z(\mathbf{x})\right]^T$. As in a classical DG method for each element K_e we seek an approximation of the components of the electromagnetic fields by a linear combination of basis functions $\varphi^e_j(\mathbf{x})$ of the space $\mathbb{P}_{p_e}(K_e)$, i.e.

$$E^e_\xi(\mathbf{x}) = \sum_{j=1}^{N^e_K} \underline{E}^e_\xi[j]\,\varphi^e_j(\mathbf{x}), \qquad H^e_\xi(\mathbf{x}) = \sum_{j=1}^{N^e_K} \underline{H}^e_\xi[j]\,\varphi^e_j(\mathbf{x}) \qquad (\xi \in \{x, y, z\}), \tag{12}$$

where $\underline{E}^e_\xi[j]$, $\underline{H}^e_\xi[j]$ represent the degrees of freedom of the electromagnetic field in K^e and N^e_K is the dimension of the space $\mathbb{P}_{p_e}(K_e)$. Similarly for a face F_f in $\mathcal{F}_h$ we denote $\mathbf{\Lambda}_h|_{F_f} = \mathbf{\Lambda}^f : F_f \longrightarrow \mathbb{C}^3$ and we set

$$\mathbf{\Lambda}^f(\mathbf{x}) = \Lambda^f_{\mathbf{u}}(\mathbf{x})\,\mathbf{u}^f + \Lambda^f_{\mathbf{w}}(\mathbf{x})\,\mathbf{w}^f, \tag{13}$$

where $\mathbf{u}^f$ and $\mathbf{w}^f$ are coordinate axis (not necessarily orthogonal). We seek an approximation of $\Lambda^f_{\mathbf{u}}$ and $\Lambda^f_{\mathbf{w}}$ by a linear combination of basis functions $\psi^f_j(\mathbf{x})$ of the space $\mathbb{P}_{p_f}(F_f)$, i.e.

$$\Lambda^f_{\mathbf{u}}(\mathbf{x}) = \sum_{j=1}^{N^f_F} \underline{\Lambda}^f_{\mathbf{u}}[j]\,\psi^f_j(\mathbf{x}), \qquad \Lambda^f_{\mathbf{w}}(\mathbf{x}) = \sum_{j=1}^{N^f_F} \underline{\Lambda}^f_{\mathbf{w}}[j]\,\psi^f_j(\mathbf{x}), \tag{14}$$

where $\underline{\Lambda}^f_{\mathbf{u}}[j]$, $\underline{\Lambda}^f_{\mathbf{w}}[j]$ are the degrees of freedom of the components of $\mathbf{\Lambda}^f$ associated to the face F^f, and N^f_F is the dimension of the space $\mathbb{P}_{p_f}(F_f)$.

Remark 3 For computation we set $\mathbf{u}^f = \overline{\mathbf{u}}^f/\|\overline{\mathbf{u}}^f\|_2$ *and* $\mathbf{w}^f = \overline{\mathbf{w}}^f/\|\overline{\mathbf{w}}^f\|_2$ *with* $\overline{\mathbf{u}}^f = n^f_2 - n^f_1$ *and* $\overline{\mathbf{w}}^f = n^f_3 - n^f_1$*, where* n^f_1, n^f_2 *and* n^f_3 *are the three nodes of the face* F_f*. Since the outward normal* $\mathbf{n}$ *can be computed through* $\mathbf{u}^f \times \mathbf{w}^f$ *we can easily show that* $(\mathbf{n} \cdot \mathbf{\Lambda}^f_h)\big|_{F_f} = 0$ *in accordance with the definition of the space* $\mathbf{M}_h$ *given by* (4)*.*

We denote by ν_e the set of indices of the elements which are neighbors of K_e (having an interface in common). Thus for each element of the mesh $K_{e_i} \in \mathcal{T}_h$ ($i \in \{1, \cdots, |\mathcal{T}_h|\}$) we associate $|\nu_{e_i}|$ faces, denoted $\partial K^l_{e_i} \in \partial\mathcal{T}_h$, defined by

$$\partial K^l_{e_i} = \overline{K_{e_i}} \cap \overline{K_{e_j}}, \quad l \in \{1, \cdots, |\nu_{e_i}|\}, \quad j \in \nu_{e_i}.$$

As it will be useful later, let us define an index mapping function (local to global), denoted by σ, graphically depicted in Fig. 3 and defined as follow

$$\begin{cases} \forall F_f \in \mathcal{F}_h^I \text{ such that } F_f = \partial K_e^l \cap \partial K_g^k, \quad \sigma(e,l) = \sigma(g,k) = f, \\ \forall F_f \in \mathcal{F}_h^B \text{ such that } F_f = \partial K_e^l \cap (\Gamma_m \cup \Gamma_m), \quad \sigma(e,l) = f. \end{cases}$$

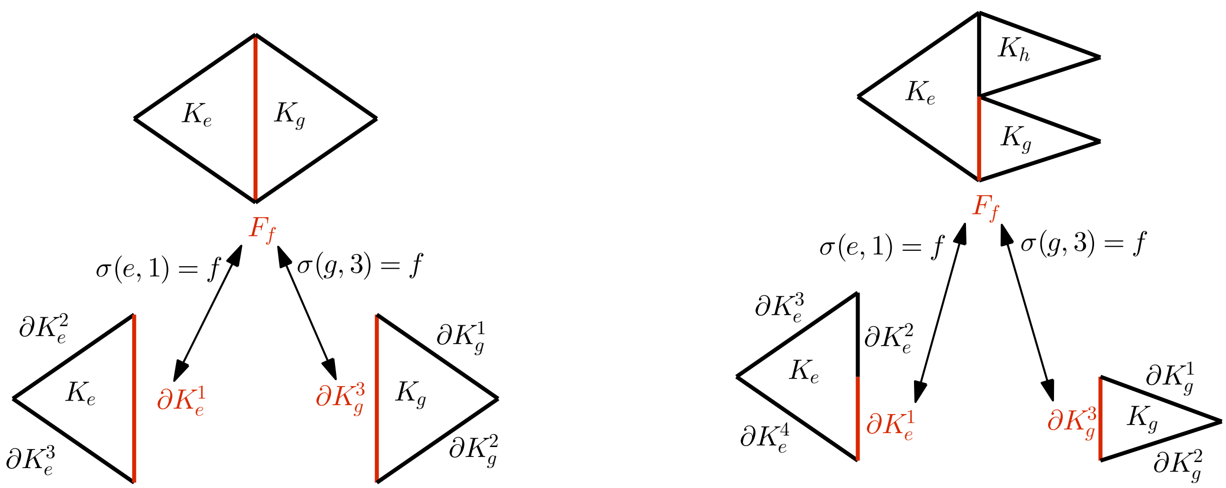

Figure 1: Diagram (2D) demonstrating the use of the index mapping function (local to global) σ described in the text (conforming case on left / non-conforming case on right).

3.1 Discretization of local problems

To get the discretization of the first and second equation of (11) we do some preliminary computations. For all $\mathbf{v} \in \mathbf{V}_h$ we have

$$\begin{aligned}
\langle \mathbf{\Lambda}_h, \mathbf{n} \times \mathbf{v} \rangle_{\partial K_e^l} &= \int_{\partial K_e^l} \left[\left(n_z^e u_y^{\sigma(e,l)} - n_y^e u_z^{\sigma(e,l)} \right) \Lambda_{\mathbf{u}}^{\sigma(e,l)} + \left(n_z^e w_y^{\sigma(e,l)} - n_y^e w_z^{\sigma(e,l)} \right) \Lambda_{\mathbf{w}}^{\sigma(e,l)} \right] \overline{v}_x \\
&+ \left[\left(n_x^e u_z^{\sigma(e,l)} - n_z^e u_x^{\sigma(e,l)} \right) \Lambda_{\mathbf{u}}^{\sigma(e,l)} + \left(n_x^e w_z^{\sigma(e,l)} - n_z^e w_x^{\sigma(e,l)} \right) \Lambda_{\mathbf{w}}^{\sigma(e,l)} \right] \overline{v}_y \\
&+ \left[\left(n_y^e u_x^{\sigma(e,l)} - n_x^e u_y^{\sigma(e,l)} \right) \Lambda_{\mathbf{u}}^{\sigma(e,l)} + \left(n_y^e w_x^{\sigma(e,l)} - n_x^e w_y^{\sigma(e,l)} \right) \Lambda_{\mathbf{w}}^{\sigma(e,l)} \right] \overline{v}_z \mathrm{d}\mathbf{s}, \\
\langle \mathbf{n} \times \mathbf{\Lambda}_h, \mathbf{n} \times \mathbf{v} \rangle_{\partial K_e^l} &= \langle -\mathbf{n} \times (\mathbf{n} \times \mathbf{\Lambda}_h), \mathbf{v} \rangle_{\partial K_e^l} \\
&= \langle \mathbf{\Lambda}_h, \mathbf{v} \rangle_{\partial K_e^l} \\
&= \int_{\partial K_e^l} \left(u_x^{\sigma(e,l)} \Lambda_{\mathbf{u}}^{\sigma(e,l)} + w_x^{\sigma(e,l)} \Lambda_{\mathbf{w}}^{\sigma(e,l)} \right) \overline{v}_x + \left(u_y^{\sigma(e,l)} \Lambda_{\mathbf{u}}^{\sigma(e,l)} + w_y^{\sigma(e,l)} \Lambda_{\mathbf{w}}^{\sigma(e,l)} \right) \overline{v}_y \\
&+ \left(u_z^{\sigma(e,l)} \Lambda_{\mathbf{u}}^{\sigma(e,l)} + w_z^{\sigma(e,l)} \Lambda_{\mathbf{w}}^{\sigma(e,l)} \right) \overline{v}_z \mathrm{d}\mathbf{s}.
\end{aligned}$$

Using (12) - (13) and the basis functions of the space $\mathbf{V}_h$ as test functions in the first and the second equation of the system (11), we obtain

$$
\left\{
\begin{aligned}
&\mathrm{i}\omega\varepsilon_r\mathbb{M}^e\underline{E}_x^e - \mathbb{D}_z^e\underline{H}_y^e + \mathbb{D}_y^e\underline{H}_z^e + \sum_{l=l}^{|\nu_e|}\Big[\Big(n_z^{(e,l)}u_y^{\sigma(e,l)} - n_y^{(e,l)}u_z^{\sigma(e,l)}\Big)\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)}\\
&\qquad + \Big(n_z^{(e,l)}w_y^{\sigma(e,l)} - n_y^{(e,l)}w_z^{\sigma(e,l)}\Big)\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)}\Big] = 0,\\
&\mathrm{i}\omega\varepsilon_r\mathbb{M}^e\underline{E}_y^e + \mathbb{D}_z^e\underline{H}_x^e - \mathbb{D}_x^e\underline{H}_z^e + \sum_{l=1}^{|\nu_e|}\Big[\Big(n_x^{(e,l)}u_z^{\sigma(e,l)} - n_z^{(e,l)}u_x^{\sigma(e,l)}\Big)\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)}\\
&\qquad + \Big(n_x^{(e,l)}w_z^{\sigma(e,l)} - n_z^{(e,l)}w_x^{\sigma(e,l)}\Big)\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)}\Big] = 0,\\
&\mathrm{i}\omega\varepsilon_r\mathbb{M}^e\underline{E}_z^e - \mathbb{D}_y^e\underline{H}_x^e + \mathbb{D}_x^e\underline{H}_y^e + \sum_{l=1}^{|\nu_e|}\Big[\Big(n_y^{(e,l)}u_x^{\sigma(e,l)} - n_x^{(e,l)}u_y^{\sigma(e,l)}\Big)\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)}\\
&\qquad + \Big(n_y^{(e,l)}w_x^{\sigma(e,l)} - n_x^{(e,l)}w_y^{\sigma(e,l)}\Big)\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)}\Big] = 0,\\
&\mathrm{i}\omega\mu_r\mathbb{M}^e\underline{H}_x^e - (\mathbb{D}_z^e)^T\underline{E}_y^e + (\mathbb{D}_y^e)^T\underline{E}_z^e - \sum_{l=1}^{|\nu_e|}\tau^{(e,l)}\Big(u_x^{\sigma(e,l)}\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)} + w_x^{\sigma(e,l)}\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)}\Big)\\
&\qquad + \sum_{l=1}^{|\nu_e|}\tau^{(e,l)}\Big[\Big(1 - (n_x^{(e,l)})^2\Big)\mathbb{E}^{(e,l)}\underline{H}_x^e - n_x^{(e,l)}n_y^{(e,l)}\mathbb{E}^{(e,l)}\underline{H}_y^e - n_x^{(e,l)}n_z^{(e,l)}\mathbb{E}^{(e,l)}\underline{H}_z^e\Big] = 0,\\
&\mathrm{i}\omega\mu_r\mathbb{M}^e\underline{H}_y^e + (\mathbb{D}_z^e)^T\underline{E}_x^e - (\mathbb{D}_x^e)^T\underline{E}_z^e - \sum_{l=1}^{|\nu_e|}\tau^{(e,l)}\Big(u_y^{\sigma(e,l)}\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)} + w_y^{\sigma(e,l)}\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)}\Big)\\
&\qquad + \sum_{l=1}^{|\nu_e|}\tau^{(e,l)}\Big[\Big(1 - (n_y^{(e,l)})^2\Big)\mathbb{E}^{(e,l)}\underline{H}_y^e - n_x^{(e,l)}n_y^{(e,l)}\mathbb{E}^{(e,l)}\underline{H}_x^e - n_y^{(e,l)}n_z^{(e,l)}\mathbb{E}^{(e,l)}\underline{H}_z^e\Big] = 0,\\
&\mathrm{i}\omega\mu_r\mathbb{M}^e\underline{H}_z^e - (\mathbb{D}_y^e)^T\underline{E}_x^e + (\mathbb{D}_x^e)^T\underline{E}_y^e - \sum_{l=1}^{|\nu_e|}\tau^{(e,l)}\Big(u_z^{\sigma(e,l)}\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)} + w_z^{\sigma(e,l)}\mathbb{F}^{(e,l)}\underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)}\Big)\\
&\qquad + \sum_{l=1}^{|\nu_e|}\tau^{(e,l)}\Big[\Big(1 - (n_z^{(e,l)})^2\Big)\mathbb{E}^{(e,l)}\underline{H}_z^e - n_x^{(e,l)}n_z^{(e,l)}\mathbb{E}^{(e,l)}\underline{H}_x^e - n_y^{(e,l)}n_z^{(e,l)}\mathbb{E}^{(e,l)}\underline{H}_y^e\Big] = 0,
\end{aligned}
\right. \tag{15}
$$

where $\tau^{(e,l)}$, $n_\xi^{(e,l)}$ are the local stabilization parameters and the components of the outward unit normal vector of the face ∂K_e^l, respectively ; $\underline{E}_\xi^e$, $\underline{H}_\xi^e$ and $\underline{\Lambda}_\zeta^{\sigma(e,l)}$ are the column vectors of degrees of freedom, i.e.

$$
\begin{aligned}
\underline{E}_\xi^e &= \left[\,\underline{E}_\xi^e\,[1]\,, \cdots, \underline{E}_\xi^e\,[N_K^e]\,\right]^T, \quad \xi \in \{x,y,z\},\\
\underline{H}_\xi^e &= \left[\,\underline{H}_\xi^e\,[1]\,, \cdots, \underline{H}_\xi^e\,[N_K^e]\,\right]^T, \quad \xi \in \{x,y,z\},\\
\underline{\Lambda}_{\boldsymbol{\nu}}^{\sigma(e,l)} &= \left[\underline{\Lambda}_{\boldsymbol{\nu}}^{\sigma(e,l)}[1], \cdots, \underline{\Lambda}_{\boldsymbol{\nu}}^{\sigma(e,l)}[N_F^{\sigma(e,l)}]\right]^T, \quad \boldsymbol{\nu} \in \{\mathbf{u},\mathbf{w}\}.
\end{aligned}
$$

Finally the entries of the local matrices are given by

$$\begin{cases} \mathbb{M}^e[i,j] = \displaystyle\int_{K_e} \varphi_i^e \varphi_j^e \,\mathrm{d}\mathbf{x}, \quad 1 \le i,j \le N_K^e, \\ \mathbb{D}_\xi^e[i,j] = \displaystyle\int_{K_e} (\partial_\xi \varphi_i^e)\varphi_j^e \,\mathrm{d}\mathbf{x}, \quad 1 \le i,j \le N_K^e \text{ et } \xi \in \{x,y,z\}, \\ \mathbb{E}^{(e,l)}[i,j] = \displaystyle\int_{\partial K_e^l} \varphi_i^e \varphi_j^e \,\mathrm{d}\mathbf{s}, \quad 1 \le i,j \le N_K^e, \\ \mathbb{F}^{(e,l)}[i,j] = \displaystyle\int_{\partial K_e^l} \varphi_i^e \psi_j^{\sigma(e,l)} \,\mathrm{d}\mathbf{s}, \quad 1 \le i \le N_e^K \text{ et } 1 \le j \le N_F^{\sigma(e,l)}. \end{cases}$$

Now we can write the local linear system associated to the element K_e as

$$\mathbb{A}^e \begin{bmatrix} \underline{E}_x^e \\ \underline{E}_y^e \\ \underline{E}_z^e \\ \underline{H}_y^e \\ \underline{H}_x^e \\ \underline{H}_z^e \end{bmatrix} + \sum_{l=1}^{|\nu_e|} \mathbb{C}^{(e,l)} \begin{bmatrix} \underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)} \\ \underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)} \end{bmatrix} = 0, \tag{16}$$

where

- $\mathbb{A}^e$ matrix of size $6N_K^e \times 6N_K^e$, defined by

$$\mathbb{A}^e = \begin{bmatrix} \mathrm{i}\omega\varepsilon_r \mathbb{M}^e & 0 & 0 & 0 & -\mathbb{D}_z^e & \mathbb{D}_y^e \\ 0 & \mathrm{i}\omega\varepsilon_r \mathbb{M}^e & 0 & \mathbb{D}_z^e & 0 & -\mathbb{D}_x^e \\ 0 & 0 & \mathrm{i}\omega\varepsilon_r \mathbb{M}^e & -\mathbb{D}_y^e & \mathbb{D}_x^e & 0 \\ 0 & -\left[\mathbb{D}_z^e\right]^T & \left[\mathbb{D}_y^e\right]^T & \mathrm{i}\omega\mu_r \mathbb{M}^e + \mathbb{E}_x^e & -\mathbb{E}_{xy}^e & -\mathbb{E}_{xz}^e \\ \left[\mathbb{D}_z^e\right]^T & 0 & -\left[\mathbb{D}_x^e\right]^T & -\mathbb{E}_{xy}^e & \mathrm{i}\omega\mu_r \mathbb{M}^e + \mathbb{E}_y^e & -\mathbb{E}_{yz}^e \\ -\left[\mathbb{D}_y^e\right]^T & \left[\mathbb{D}_x^e\right]^T & 0 & -\mathbb{E}_{xz}^e & -\mathbb{E}_{yz}^e & \mathrm{i}\omega\mu_r \mathbb{M}^e + \mathbb{E}_z^e \end{bmatrix},$$

  with

$$\begin{cases} \mathbb{E}_\xi^e = \displaystyle\sum_{l=1}^{|\nu_e|} \tau^{(e,l)}((1-(n_\xi^{(e,l)})^2))\mathbb{E}^{(e,l)}, \\ \mathbb{E}_{\xi\zeta}^e = \displaystyle\sum_{l=1}^{|\nu_e|} \tau^{(e,l)} n_\xi^{(e,l)} n_\zeta^{(e,l)} \mathbb{E}^{(e,l)}, \end{cases} \qquad \xi,\zeta \in \{x,y,z\},$$

- $\mathbb{C}^{(e,l)}$ matrix of size $6N_K^e \times 2N_F^{\sigma(e,l)}$, defined by

$$\mathbb{C}^{(e,l)} = \begin{bmatrix} (n_z^{(e,l)} u_y^{\sigma(e,l)} - n_y^{(e,l)} u_z^{\sigma(e,l)})\mathbb{F}^{(e,l)} & (n_z^{(e,l)} w_y^{\sigma(e,l)} - n_y^{(e,l)} w_z^{\sigma(e,l)})\mathbb{F}^{(e,l)} \\ (n_x^{(e,l)} u_z^{\sigma(e,l)} - n_z^{(e,l)} u_x^{\sigma(e,l)})\mathbb{F}^{(e,l)} & (n_x^{(e,l)} w_z^{\sigma(e,l)} - n_z^{(e,l)} w_x^{\sigma(e,l)})\mathbb{F}^{(e,l)} \\ (n_y^{(e,l)} u_x^{\sigma(e,l)} - n_x^{(e,l)} u_y^{\sigma(e,l)})\mathbb{F}^{(e,l)} & (n_y^{(e,l)} w_x^{\sigma(e,l)} - n_x^{(e,l)} w_y^{\sigma(e,l)})\mathbb{F}^{(e,l)} \\ -\tau^{(e,l)} u_x^{\sigma(e,l)} \mathbb{F}^{(e,l)} & -\tau^{(e,l)} w_x^{\sigma(e,l)} \mathbb{F}^{(e,l)} \\ -\tau^{(e,l)} u_y^{\sigma(e,l)} \mathbb{F}^{(e,l)} & -\tau^{(e,l)} w_y^{\sigma(e,l)} \mathbb{F}^{(e,l)} \\ -\tau^{(e,l)} u_z^{\sigma(e,l)} \mathbb{F}^{(e,l)} & -\tau^{(e,l)} w_z^{\sigma(e,l)} \mathbb{F}^{(e,l)} \end{bmatrix}.$$

3.2 Global discretization for the hybrid variable

In this section we discretize the third equation of (11), i.e. the conservativity condition, to obtain a global linear system in terms of the degrees of freedom of the hybrid variable Λ_h.

Let $F_f \in \mathcal{F}_h^I$ an internal face shared by the elements K_e and K_g with local indices l and k, respectively, i.e. $f = \sigma(e,l) = \sigma(g,k)$. The conservativity condition for F_f and for all $\boldsymbol{\eta} \in \mathbf{M}_h$ can be written as

$$\begin{aligned}&\langle \mathbf{n}\times\mathbf{E}_h, \boldsymbol{\eta}\rangle_{\partial K_e^l} - \tau^{(e,l)}\langle \mathbf{n}\times(\mathbf{n}\times\mathbf{H}_h), \boldsymbol{\eta}\rangle_{\partial K_e^l} - \tau^{(e,l)}\langle \boldsymbol{\Lambda}_h, \boldsymbol{\eta}\rangle_{\partial K_e^l}\\ &+ \langle \mathbf{n}\times\mathbf{E}_h, \boldsymbol{\eta}\rangle_{\partial K_g^k} - \tau^{(g,k)}\langle \mathbf{n}\times(\mathbf{n}\times\mathbf{H}_h), \boldsymbol{\eta}\rangle_{\partial K_g^k} - \tau^{(g,k)}\langle \boldsymbol{\Lambda}_h, \boldsymbol{\eta}\rangle_{\partial K_g^k} = 0.\end{aligned} \tag{17}$$

For a boundary face $F_f \in \Gamma_a$ such that $F_f \in \partial K_e^l \cap \Gamma_a$, the conservativity condition for all $\boldsymbol{\eta} \in \mathbf{M}_h$ is given by

$$\langle \mathbf{n}\times\mathbf{E}_h, \boldsymbol{\eta}\rangle_{\partial K_e^l} - \tau^{(e,l)}\langle \mathbf{n}\times(\mathbf{n}\times\mathbf{H}_h), \boldsymbol{\eta}\rangle_{\partial K_e^l} - (1+\tau^{(e,l)})\langle \boldsymbol{\Lambda}_h, \boldsymbol{\eta}\rangle_{\partial K_e^l} = \left\langle \mathbf{g}^{inc}, \boldsymbol{\eta}\right\rangle_{\partial K_e^l} \tag{18}$$

As it will be useful for the discretization of these latter equations, let us do some preliminary computations. For all $\boldsymbol{\eta} \in \mathbf{M}_h$, we write

$$\begin{aligned}\langle \mathbf{n}\times\mathbf{E}_h, \boldsymbol{\eta}\rangle_{\partial K_e^l} &= \int_{\partial K_e^l} \Big[\left(n_z^e u_y^{\sigma(e,l)} - n_y^e u_z^{\sigma(e,l)}\right) E_x^e + \left(n_x^e u_z^{\sigma(e,l)} - n_z^e u_x^{\sigma(e,l)}\right) E_y^e\\ &\quad + \left(n_y^e u_x^{\sigma(e,l)} - n_x^e u_y^{\sigma(e,l)}\right) E_z^e\Big]\overline{\eta}_{\mathbf{u}}^{\sigma(e,l)} + \Big[\left(n_z^e w_y^{\sigma(e,l)} - n_y^e w_z^{\sigma(e,l)}\right) E_x^e\\ &\quad + \left(n_x^e w_z^{\sigma(e,l)} - n_z^e w_x^{\sigma(e,l)}\right) E_y^e + \left(n_y^e w_x^{\sigma(e,l)} - n_x^e w_y^{\sigma(e,l)}\right) E_z^e\Big]\overline{\eta}_{\mathbf{w}}^{\sigma(e,l)}\mathrm{d}\mathbf{s},\\ \langle -\mathbf{n}\times(\mathbf{n}\times\mathbf{H}_h), \boldsymbol{\eta}\rangle_{\partial K_e^l} &= \int_{\partial K_e^l} \left(u_x^{\sigma(e,l)} H_x^e + u_y^{\sigma(e,l)} H_y^e + u_z^{\sigma(e,l)} H_z^e\right)\overline{\eta}_{\mathbf{u}}^{\sigma(e,l)}\\ &\quad + \left(w_x^{\sigma(e,l)} H_x^e + w_y^{\sigma(e,l)} H_y^e + w\sigma(e,l)_z H_z^e\right)\overline{\eta}_{\mathbf{w}}^{\sigma(e,l)}\mathrm{d}\mathbf{s},\\ \langle \boldsymbol{\Lambda}_h, \boldsymbol{\eta}\rangle_{\partial K_e^l} &= \int_{\partial K_e^l} \left[\Lambda_{\mathbf{u}}^{\sigma(e,l)} + \left(\mathbf{u}^{\sigma(e,l)}\cdot\mathbf{w}^{\sigma(e,l)}\right)\Lambda_{\mathbf{w}}^{\sigma(e,l)}\right]\overline{\eta}_{\mathbf{u}}^{\sigma(e,l)}\\ &\quad + \left[\Lambda_{\mathbf{w}}^{\sigma(e,l)} + \left(\mathbf{u}^{\sigma(e,l)}\cdot\mathbf{w}^{\sigma(e,l)}\right)\Lambda_{\mathbf{u}}^{\sigma(e,l)}\right]\overline{\eta}_{\mathbf{w}}^{\sigma(e,l)}\mathrm{d}\mathbf{s}.\end{aligned} \tag{19}$$

Note that we have used the following equality for the second equation of (19)

$$\langle -\mathbf{n}\times(\mathbf{n}\times\mathbf{H}_h), \boldsymbol{\eta}\rangle_F = \langle \mathbf{n}\times\mathbf{H}_h, \mathbf{n}\times\boldsymbol{\eta}\rangle_F = \langle \mathbf{H}_h, -\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\eta})\rangle_F = \left\langle \mathbf{H}_h, \boldsymbol{\eta}^t\right\rangle_F, \quad \forall F \in \mathcal{F}_h,$$

Furthermore $\boldsymbol{\eta} \in \mathbf{M}_h$ then we have $(\boldsymbol{\eta}\cdot\mathbf{n})\,|_F = 0$ and

$$\langle -\mathbf{n}\times(\mathbf{n}\times\mathbf{H}_h), \boldsymbol{\eta}\rangle_F = \langle \mathbf{H}_h, \boldsymbol{\eta}\rangle_F, \quad \forall F \in \mathcal{F}_h, \quad \forall \boldsymbol{\eta} \in \mathbf{M}_h.$$

To discretize (17) and (18) we use (12) - (13) and the basis functions of the space $\mathbf{M}_h$ as test functions. Let $F_f \in \mathcal{F}_h^I$ such that $F_f = \partial K_e^l \cap \partial K_g^k$ from (17) we write

$$\left\{\begin{aligned}&\left(n_z^{(e,l)} u_y^{\sigma(e,l)} - n_y^{(e,l)} u_z^{\sigma(e,l)}\right)\left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_x^e + \left(n_x^{(e,l)} u_z^{\sigma(e,l)} - n_z^{(e,l)} u_x^{\sigma(e,l)}\right)\left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_y^e\\ &+ \left(n_z^{(e,l)} u_y^{\sigma(e,l)} - n_y^{(e,l)} u_z^{\sigma(e,l)}\right)\left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_z^e + \tau_x^{(e,l)}\left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_x^e + \tau^{(e,l)} u_y^{\sigma(e,l)}\left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_y^e\\ &+\tau^{(e,l)} u_z^{\sigma(e,l)}\left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_z^e - \tau^{(e,l)}\mathbb{G}^{(e,l)}\underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)} - \tau^{(e,l)}\left(\mathbf{u}^{\sigma(e,l)}\cdot\mathbf{w}^{\sigma(e,l)}\right)\mathbb{G}^{(e,l)}\underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)} + R_{\mathbf{u}}^{(g,k)} = 0,\\ \\ &\left(n_z^{(e,l)} w_y^{\sigma(e,l)} - n_y^{(e,l)} w_z^{\sigma(e,l)}\right)\left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_x^e + \left(n_x^{(e,l)} w_z^{\sigma(e,l)} - n_z^{(e,l)} w_x^{\sigma(e,l)}\right)\left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_y^e\\ &+ \left(n_z^{(e,l)} w_y^{\sigma(e,l)} - n_y^{(e,l)} w_z^{\sigma(e,l)}\right)\left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_z^e + \tau^{(e,l)} w_x^{\sigma(e,l)}\left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_x^e + \tau^{(e,l)} w_y^{\sigma(e,l)}\left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_y^e\\ &+\tau^{(e,l)} w_z^{\sigma(e,l)}\left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_z^e - \tau^{(e,l)}\left(\mathbf{u}^{\sigma(e,l)}\cdot\mathbf{w}^{\sigma(e,l)}\right)\mathbb{G}^{(e,l)}\underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)} - \tau^{(e,l)}\mathbb{G}^{(e,l)}\underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)} + R_{\mathbf{w}}^{(g,k)} = 0,\end{aligned}\right. \tag{20}$$

where

$$\mathbb{G}^{(e,l)}[i,j] = \int_{\partial K_e^l} \psi_i^{\sigma(e,l)} \psi_j^{\sigma(e,l)}\, \mathrm{d}s, \quad 1 \le i,j \le N_F^{\sigma(e,l)},$$

and $R_{\mathbf{u}}^{(g,k)}$, $R_{\mathbf{w}}^{(g,k)}$ collect the counterparts from the k-th face of the element with index g. Similarly for (18)„ i.e. for an absorbing boundary face $F_f \in \Gamma_a$ such that $F_f = \partial K_e^l \cap \Gamma_a$, we obtain

$$\begin{cases} \left(n_z^{(e,l)} u_y^{\sigma(e,l)} - n_y^{(e,l)} u_z^{\sigma(e,l)}\right) \left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_x^e + \left(n_x^{(e,l)} u_z^{\sigma(e,l)} - n_z^{(e,l)} u_x^{\sigma(e,l)}\right) \left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_y^e \\ + \left(n_z^{(e,l)} u_y^{\sigma(e,l)} - n_y^{(e,l)} u_z^{\sigma(e,l)}\right) \left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_z^e + \tau_x^{(e,l)} \left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_x^e + \tau^{(e,l)} u_y^{\sigma(e,l)} \left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_y^e \\ + \tau^{(e,l)} u_z^{\sigma(e,l)} \left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_z^e - \left(1+\tau^{(e,l)}\right) \mathbb{G}^{(e,l)} \underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)} - \left(1+\tau^{(e,l)}\right) \left(\mathbf{u}^{\sigma(e,l)} \cdot \mathbf{w}^{\sigma(e,l)}\right) \mathbb{G}^{(e,l)} \underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)} \\ = \mathbb{G}^{(e,l)} \underline{\mathbf{g}}_{\mathbf{u}}^{\mathrm{inc},\sigma(e,l)}, \\ \\ \left(n_z^{(e,l)} w_y^{\sigma(e,l)} - n_y^{(e,l)} w_z^{\sigma(e,l)}\right) \left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_x^e + \left(n_x^{(e,l)} w_z^{\sigma(e,l)} - n_z^{(e,l)} w_x^{\sigma(e,l)}\right) \left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_y^e \\ + \left(n_z^{(e,l)} w_y^{\sigma(e,l)} - n_y^{(e,l)} w_z^{\sigma(e,l)}\right) \left[\mathbb{F}^{(e,l)}\right]^T \underline{E}_z^e + \tau^{(e,l)} w_x^{\sigma(e,l)} \left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_x^e + \tau^{(e,l)} w_y^{\sigma(e,l)} \left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_y^e \\ + \tau^{(e,l)} w_z^{\sigma(e,l)} \left[\mathbb{F}^{(e,l)}\right]^T \underline{H}_z^e - \left(1+\tau^{(e,l)}\right) \left(\mathbf{u}^{\sigma(e,l)} \cdot \mathbf{w}^{\sigma(e,l)}\right) \mathbb{G}^{(e,l)} \underline{\Lambda}_{\mathbf{u}}^{\sigma(e,l)} - \left(1+\tau^{(e,l)}\right) \mathbb{G}^{(e,l)} \underline{\Lambda}_{\mathbf{w}}^{\sigma(e,l)} \\ = \mathbb{G}^{(e,l)} \underline{\mathbf{g}}_{\mathbf{w}}^{\mathrm{inc},\sigma(e,l)}, \end{cases} \tag{21}$$

with

$$\underline{\mathbf{g}}_{\boldsymbol{\nu}}^{\mathrm{inc},\sigma(e,l)} = \left[\underline{\mathbf{g}}_{\boldsymbol{\nu}}^{\mathrm{inc},\sigma(e,l)}[1], \cdots, \underline{\mathbf{g}}_{\boldsymbol{\nu}}^{\mathrm{inc},\sigma(e,l)}[N_F^{\sigma(e,l)}]\right]^T, \ \boldsymbol{\nu} \in \{\mathbf{u}, \mathbf{w}\}.$$

The efficiency of the HDG implementation arises from obtaining an element-wise matrix system to construct a global system for the trace space degrees of freedom. Let $\underline{\Lambda}^f = [\underline{\Lambda}_{\mathbf{u}}^f, \underline{\Lambda}_{\mathbf{w}}^f]^T$ denote the column vector of degrees of freedom on the face $F_f \in \mathcal{F}_h$, of size $2N_F^f$. We define the global trace vector of degrees of freedom, denoted by $\underline{\Lambda}$, as the concatenation of these vectors for all faces F_f in $\mathcal{F}_h$. We define the trace spreading operator $\mathcal{A}_{HDG}$ which spreads or scatters the unique trace space values to their local face vectors. We can represent the operator $\mathcal{A}_{HDG}$ as a matrix of size

$$\sum_{e=1}^{|\mathcal{T}_h|} \left(\sum_{l=1}^{|\nu_e|} 2N_F^{\sigma(e,l)} \right) \times \sum_{f=1}^{|\mathcal{F}_h|} 2N_F^f.$$

Furthermore we organize this matrix by elements such that

$$\mathcal{A}_{HDG} = \left[\mathcal{A}_{HDG}^{e_1}, \cdots, \mathcal{A}_{HDG}^{e_{|\mathcal{T}_h|}}\right]^T,$$

where the action of $\mathcal{A}_{HDG}^e$ is to copy global trace space information into local (elemental) storage. Then for each element $K_e \in \mathcal{T}_h$ we define a matrix $\mathcal{A}_{HDG}^e$ of size

$$\sum_{l=1}^{|\nu_e|} 2N_F^{\sigma(e,l)} \times \sum_{f=1}^{|\mathcal{F}_h|} 2N_F^f,$$

such that

$$\mathcal{A}_{HDG}^e \underline{\Lambda} = \left[\underline{\Lambda}^{\sigma(e,1)}, \cdots, \underline{\Lambda}^{\sigma(e,|\nu_e|)}\right]^T.$$

With these notations in place, adding all equations involving interior face (20) and every boundary face (21) element-by-element we have

$$\sum_{e=1}^{|\mathcal{T}_h|} \left[\mathcal{A}^e_{HDG}\right]^T \left(\mathbb{B}^e \underline{W}^e + \mathbb{G}^e \mathcal{A}^e_{HDG} \underline{\Lambda}\right) = \sum_{e=1}^{|\mathcal{T}_h|} \left[\mathcal{A}^e_{HDG}\right]^T \underline{\mathbf{g}}^e, \tag{22}$$

where

- $\underline{W}^e$ the column vector of size $6N_K^e$, defined by $\underline{W}^e = \left[\underline{E}_x^e, \underline{E}_y^e, \underline{E}_z^e, \underline{H}_x^e, \underline{H}_y^e, \underline{H}_z^e\right]^T$,
- $\mathbb{B}^e$ the matrix of size $\displaystyle\sum_{l=1}^{|\nu_e|} 2N_F^{\sigma(e,l)} \times 6N_K^e$, defined by

  $\mathbb{B}^e =$

$$\begin{bmatrix} \mathbb{F}^{(e,1)}_{zy,u} & \mathbb{F}^{(e,1)}_{xz,u} & \mathbb{F}^{(e,1)}_{yx,u} & \tau^{(e,1)} u_x^{\sigma(e,1)} \left[\mathbb{F}^{(e,1)}\right]^T & \tau^{(e,1)} u_y^{\sigma(e,1)} \left[\mathbb{F}^{(e,1)}\right]^T & \tau^{(e,1)} u_z^{\sigma(e,1)} \left[\mathbb{F}^{(e,1)}\right]^T \\ \mathbb{F}^{(e,1)}_{zy,w} & \mathbb{F}^{(e,1)}_{xz,w} & \mathbb{F}^{(e,1)}_{yx,w} & \tau^{(e,1)} w_x^{\sigma(e,1)} \left[\mathbb{F}^{(e,1)}\right]^T & \tau^{(e,1)} w_y^{\sigma(e,1)} \left[\mathbb{F}^{(e,1)}\right]^T & \tau^{(e,1)} w_z^{\sigma(e,1)} \left[\mathbb{F}^{(e,1)}\right]^T \\ \mathbb{F}^{(e,2)}_{zy,u} & \mathbb{F}^{(e,2)}_{xz,u} & \mathbb{F}^{(e,2)}_{yx,u} & \tau^{(e,2)} u_x^{\sigma(e,2)} \left[\mathbb{F}^{(e,2)}\right]^T & \tau^{(e,2)} u_y^{\sigma(e,2)} \left[\mathbb{F}^{(e,2)}\right]^T & \tau^{(e,2)} u_z^{\sigma(e,2)} \left[\mathbb{F}^{(e,2)}\right]^T \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbb{F}^{(e,|\nu_e|)}_{zy,w} & \mathbb{F}^{(e,|\nu_e|)}_{xz,w} & \mathbb{F}^{(e,|\nu_e|)}_{yx,w} & \tau^{(e,|\nu_e|)} w_x^{\sigma(e,|\nu_e|)} \left[\mathbb{F}^{(e,|\nu_e|)}\right]^T & \tau^{(e,|\nu_e|)} w_y^{\sigma(e,|\nu_e|)} \left[\mathbb{F}^{(e,|\nu_e|)}\right]^T & \tau^{(e,|\nu_e|)} w_z^{\sigma(e,|\nu_e|)} \left[\mathbb{F}^{(e,|\nu_e|)}\right]^T \end{bmatrix},$$

  with

$$\mathbb{F}^{(e,l)}_{\xi\zeta,\nu} = \left(n_\xi^{(e,l)} \nu_\zeta^{\sigma(e,l)} - n_\zeta^{(e,l)} \nu_\xi^{\sigma(e,l)}\right) \left[\mathbb{F}^{(e,l)}\right]^T,\ l = 1, \cdots, |\nu_e|,\ \xi, \zeta \in \{x, y, z\},\ \nu \in \{u, w\},$$

- $\mathbb{G}^e$ the matrix of size $\displaystyle\sum_{l=1}^{|\nu_e|} 2N_F^{\sigma(e,l)} \times \sum_{l=1}^{|\nu_e|} 2N_F^{\sigma(e,l)}$, defined by

$$\mathbb{G}^e = \begin{bmatrix} -\kappa^{(e,1)}\mathbb{G}^{(e,1)} & -\kappa^{(e,1)}(\mathbf{u}^{\sigma(e,1)} \cdot \mathbf{w}^{\sigma(e,1)})\mathbb{G}^{(e,1)} & \cdots & 0 \\ -\kappa^{(e,1)}(\mathbf{u}^{\sigma(e,1)} \cdot \mathbf{w}^{\sigma(e,1)})\mathbb{G}^{(e,1)} & -\kappa^{(e,1)}\mathbb{G}^{(e,1)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -\kappa^{(e,|\nu_e|)}(\mathbf{u}^{\sigma(e,|\nu_e|)} \cdot \mathbf{w}^{\sigma(e,|\nu_e|)})\mathbb{G}^{(e,|\nu_e|)} \\ 0 & 0 & \cdots & -\kappa^{(e,|\nu_e|)}\mathbb{G}^{(e,|\nu_e|)} \end{bmatrix},$$

  with

$$\kappa^{(e,l)} = \begin{cases} \tau^{(e,l)}, \text{ si la face } F_{\sigma(e,l)} \in \mathcal{F}_h \setminus \Gamma_a, \\ 1 + \tau^{(e,l)}, \text{ si la face } F_{\sigma(e,l)} \in \mathcal{F}_h^B \cap \Gamma_a, \end{cases} \quad l = 1, \cdots, |\nu_e|,$$

- $\underline{\mathbf{g}}^e$ the column vector of size $\displaystyle\sum_{l=1}^{|\nu_e|} 2N_F^{\sigma(e,l)}$, defined by

$$\underline{\mathbf{g}}^e = \left[\underline{\mathbf{g}}^{\sigma(e,1)}, \cdots, \underline{\mathbf{g}}^{\sigma(e,|\nu_e|)}\right]^T \text{ with } \underline{\mathbf{g}}^{\sigma(e,l)} = \left[\underline{\mathbf{g}}_{\mathbf{u}}^{\sigma(e,l)}, \underline{\mathbf{g}}_{\mathbf{w}}^{\sigma(e,l)}\right]^T, l = 1, \cdots, |\nu_e|,$$

  where

$$\underline{\mathbf{g}}^{\sigma(e,l)} = \begin{cases} \mathbb{G}^(e,l)\, \underline{\mathbf{g}}_{\mathbf{u}}^{\sigma(e,l)} \\ \mathbb{G}^(e,l)\, \underline{\mathbf{g}}_{\mathbf{w}}^{\sigma(e,l)} \end{cases} \text{ and } \underline{\mathbf{g}}_{\boldsymbol{\nu}}^{\sigma(e,l)} = \begin{cases} 0 & \text{if } F_{\sigma(e,l)} \in \mathcal{F}_h \setminus \Gamma_a \\ \underline{\mathbf{g}}_{\boldsymbol{\nu}}^{\text{inc},\sigma(e,l)} & \text{if } F_{\sigma(e,l)} \in \mathcal{F}_h^B \cap \Gamma_a \end{cases} \quad \boldsymbol{\nu} \in \{\mathbf{u}, \mathbf{w}\}.$$

Now we can rewrite the equation for the local solver (16) as

$$\mathbb{A}^e \underline{W}^e + \mathbb{C}^e \mathcal{A}^e_{HDG} \underline{\Lambda} = 0, \tag{23}$$

where $\mathbb{C}^e$ is the matrix of size $6N_K^e \times \displaystyle\sum_{l=1}^{|\nu_e|} N_F^{\sigma(e,l)}$, defined by $\mathbb{C}^e = \left[\mathbb{C}^{(e,1)} \quad \cdots \quad \mathbb{C}^{(e,|\nu_e|)}\right]$.

Finally we substitute $\underline{W}^e$ by the solution of the local system (23) in (22) to obtain

$$\sum_{e=1}^{|\mathcal{T}_h|} [\mathcal{A}^e_{HDG}]^T \left(-\mathbb{B}^e [\mathbb{A}^e]^{-1} \mathbb{C}^e \mathcal{A}^e_{HDG} \underline{\Lambda} + \mathbb{G}^e \mathcal{A}^e_{HDG} \underline{\Lambda} \right) = \sum_{e=1}^{|\mathcal{T}_h|} [\mathcal{A}^e_{HDG}]^T \underline{\mathbf{g}}^e.$$

Thus we write the following linear system for the global trace $\underline{\Lambda}$

$$\mathbb{K} \underline{\Lambda} = \underline{\mathbf{g}}, \tag{24}$$

where

- $\mathbb{K}$ the matrix of size $\sum_{f=1}^{|\mathcal{F}_h|} 2N_F^f \times \sum_{f=1}^{|\mathcal{F}_h|} 2N_F^f$, defined by

$$\mathbb{K} = \sum_{e=1}^{|\mathcal{T}_h|} [\mathcal{A}^e_{HDG}]^T \mathbb{K}^e \mathcal{A}^e_{HDG} = \sum_{e=1}^{|\mathcal{T}_h|} [\mathcal{A}^e_{HDG}]^T \left(\mathbb{G}^e - \mathbb{B}^e [\mathbb{A}^e]^{-1} \mathbb{C}^e \right) \mathcal{A}^e_{HDG},$$

- $\underline{\mathbf{g}}$ the column vector $\sum_{f=1}^{|\mathcal{F}_h|} 2N_F^f$, defined by $\underline{\mathbf{g}} = \sum_{e=1}^{|\mathcal{T}_h|} [\mathcal{A}^e_{HDG}]^T \underline{\mathbf{g}}^e$.

In HORSE we use the direct solver MUMPS or the hybrid (direct/iterative) solver MaPHyS to solve the global linear system (24) (cf user's guides of MUMPS and MaPHyS for a complete description of these packages).

4 The HDG method combined to a Schwarz algorithm

A Schwarz-type domain decomposition method is applied for the solution of the system of 3D time-harmonic Maxwell's equations. The discrete system of the HDG method on each subdomain is then solved by sparse solvers. In HORSE we use the direct solver MUMPS or the hybrid (direct/iterative) solver MaPHyS (cf user's guides of MUMPS and MaPHyS for a complete description of these packages). It results a DD-HDG method for the solution of 3D time-harmonic Maxwell's equations. For a complete description of the method refer to [1].

5 The HDG method combined to a Schwarz algorithm accelerated by a Krylov subspace method

We consider the DD-HDG method presented in the previous section 4. The solution of the interface system in the domain decomposition framework is then accelerated by a Krylov subspace method. For a complete description of the method refer to [1].

References

[1] L. Li, S. Lanteri, and R. Perrussel. A hybridizable discontinuous galerkin method combined to a schwarz algorithm for the solution of 3d time-harmonic maxwell's equation. *J. Comput. Phys.*, 256:563–581, 2014.