Multiscale Hybrid-Mixed Method for Maxwell Equations in Time Domain

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[1] Joint work with S. Lanteri, R. Léger, C. Scheid and F. Valentin.

Model and motivation

Maxwell Equations

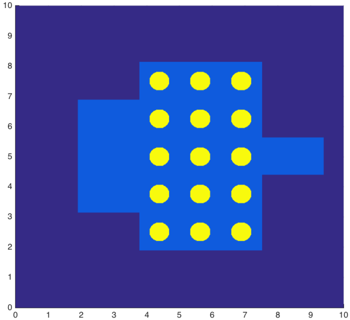
Find $\boldsymbol{e} = \boldsymbol{e}(\mathbf{x}, t)$ *and* $\boldsymbol{h} = \boldsymbol{h}(\mathbf{x}, t)$ *such that*

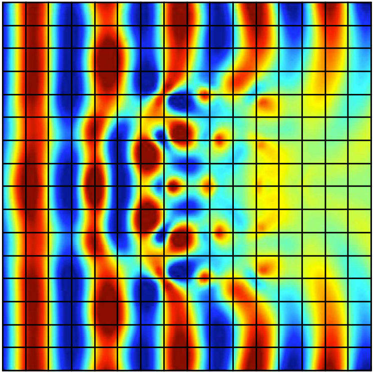
$$\left\{ \begin{array}{rcll} \varepsilon\, \partial_t \boldsymbol{e} - \nabla \times \boldsymbol{h} & = & \boldsymbol{f}, & \text{in } \Omega \times]0, T[\\ \mu\, \partial_t \boldsymbol{h} + \nabla \times \boldsymbol{e} & = & \mathbf{0}, & \text{in } \Omega \times]0, T[\\ \boldsymbol{e} \times \boldsymbol{n} & = & \mathbf{0}, & \text{in } \Gamma \times]0, T[\\ \boldsymbol{e} & = & \boldsymbol{e}_0, & \text{in } \Omega \times \{0\} \\ \boldsymbol{h} & = & \boldsymbol{h}_0, & \text{in } \Omega \times \{0\} \end{array} \right.$$

- μ is the symmetric tensor magnetic permeability
- ε is the symmetric tensor electric permitivity
- Initial field $\boldsymbol{e}_0$ and $\boldsymbol{h}_0$ and source $\boldsymbol{f}$ satisfy

$$\nabla \cdot \boldsymbol{f} = \nabla \cdot (\varepsilon\, \boldsymbol{e}_0) = \nabla \cdot (\mu\, \boldsymbol{h}_0) = 0, \in \Omega$$

Hypothesis: Multiscale behavior

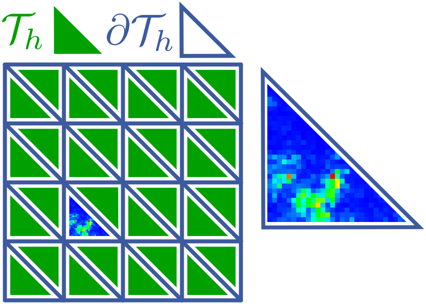


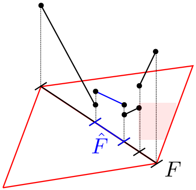


Settings

$$\mathbf{V} = \bigoplus_{K \in \mathcal{T}_H} \mathbf{V}_K$$

$$\boldsymbol{\Lambda}_h \subset \boldsymbol{\Lambda} = \prod_{K \in \mathcal{T}_H} \boldsymbol{H}^{-\frac{1}{2}}(\partial K)$$





polynomial $\lambda_h \in \Lambda_h$

$$\boldsymbol{\Lambda} = \left\{ \boldsymbol{\mu} \in \prod_{K \in \mathcal{T}_H} \boldsymbol{H}^{-\frac{1}{2}}(\partial K) : \boldsymbol{\mu} \text{ satisfy } \textit{some global continuity condition} \right\}$$

Outline

Methodology

Methodology

Multiscae problem

Find $u = u(\mathbf{x})$ *such that*

$$\begin{cases} \mathcal{L}u &= f, & \text{in } \Omega \\ u &= 0, & \text{in } \Gamma \end{cases}$$

- $\mathcal{L}$ and $f(\mathbf{x})$ may have *multi-scale* features
- $u = u(\mathbf{x})$ may have a *multi-scale* behavior

Variational Hybrid-Mixed Formulation, Raviart-Thomas 1977

$$V = \bigoplus_{K \in \mathcal{T}_H} V_K$$

Hybrid Formulation

Find $(u, \lambda) \in V \times \Lambda$ *such that*

$$\begin{aligned} a(u, v) + (\lambda, v)_{\partial \mathcal{T}_H} &= (f, v)_{\mathcal{T}_H} \\ (u, \mu)_{\partial \mathcal{T}_H} &= 0 \end{aligned}$$

for all $(v, \mu) \in V \times \Lambda$

Localization process

Hybrid Formulation

Find $(u, \lambda) \in V \times \Lambda$ *such that*

$$\begin{aligned} a(u,v) + (\lambda, v)_{\partial \mathcal{T}_H} &= (f, v)_{\mathcal{T}_H} \\ (u, \mu)_{\partial \mathcal{T}_H} &= 0 \end{aligned}$$

for all $(v, \mu) \in V \times \Lambda$

$$a(u,v) = \sum_{K \in \mathcal{T}_H} a_K(u,v)$$

Local problem

Find $u \in V_K$ *such that*

$$a_K(u,v) = (f,v)_K - (\lambda, v)_{\partial K},$$

for all $v \in V_K$

Global and local problems for $u = u^\lambda + u^f$

Global problem

Find $\lambda \in \Lambda$

$$(u^\lambda, \mu)_{\partial \mathcal{T}_H} = -(u^f, \mu)_{\partial \mathcal{T}_H}$$

for all $\mu \in \Lambda$

Local problem for λ

Find $u^\lambda \in V_K$ *such that*

$$a_K(u^\lambda, v) = -(\lambda, v)_{\partial K}$$

for all $v \in V_K$

Local problem for f

Find $u^f \in V_K$ *such that*

$$a_K(u^f, v) = (f, v)_{\partial K}$$

for all $v \in V_K$

Global and local problems for $u = u^\lambda + u^f$

Global problem

Find $\lambda \in \Lambda$

$$(u^\lambda, \mu)_{\partial \mathcal{T}_H} = -(u^f, \mu)_{\partial \mathcal{T}_H}$$

for all $\mu \in \Lambda$

Local problem for λ

Find $u^\lambda \in V_K$ *such that*

$$\begin{cases} \mathcal{L} u^\lambda = 0, \text{ in } K \\ \gamma(u^\lambda) = \lambda, \text{ in } \partial K \end{cases}$$

Local problem for f

Find $u^f \in V_K$ *such that*

$$\begin{cases} \mathcal{L} u^f = f, \text{ in } K \\ \gamma(u^f) = 0, \text{ in } \partial K \end{cases}$$

$$Laplace: \ \gamma(u) = -\varepsilon \nabla u \cdot \boldsymbol{n}^K$$

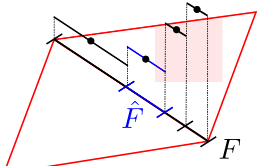
$$RAD: \ \gamma(u) = -\varepsilon \nabla u \cdot \boldsymbol{n}^K + \frac{1}{2}(\boldsymbol{\alpha} \cdot \boldsymbol{n}^K)\, u$$

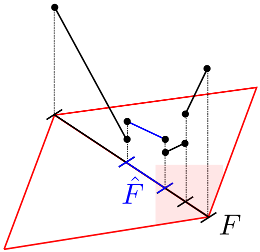
$$Maxwell: \ \boldsymbol{\gamma}(\boldsymbol{u}) = \boldsymbol{u} \times \boldsymbol{n}^K$$

Discrete Space $\Lambda_H \subset \Lambda$ — one level

$$\lambda \approx \lambda_H = \sum_{i=1}^{N_H} c_i \, \psi_i \; \Rightarrow \; u^{\lambda} \approx u^{\lambda_H}$$

$$\begin{aligned} u_H &= u^{\lambda_H} + u^f \\ &= \sum_{i=1}^{N_H} c_i \, \eta_i + u^f \end{aligned}$$



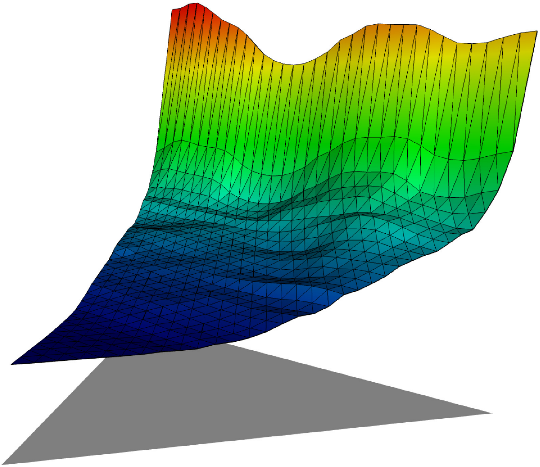


Basis functions

Find $\eta_i \in V_K$ *such that*

$$\begin{cases} \mathcal{L}\eta_i = 0 \\ \gamma(\eta_i) = \psi_i \end{cases}$$

Discrete Space $\Lambda_H \subset \Lambda$ — (reaction advection diffusion case)



Summary: Multiscale Hybrid Method

Basis functions

Find $\eta_i \in V_K$ *such that*

$$\begin{cases} \mathcal{L}\eta_i = 0, & \text{in } K \\ \gamma(\eta_i) = \psi_i, & \text{in } \partial K \end{cases}$$

Local problem for f

Find $u^f \in V_K$ *such that*

$$\begin{cases} \mathcal{L}u^f = f, & \text{in } K \\ \gamma(u^f) = 0, & \text{in } \partial K \end{cases}$$

Global problem

Find $c_1, \ldots, c_{N_H} \in \mathbb{R}$

$$\sum_{i=1}^{N_H} c_i \, (\eta_i, \phi_j)_{\partial \mathcal{T}_H} = -(u^f, \phi_j)_{\partial \mathcal{T}_H}$$

for all $\phi_1, \ldots, \phi_{N_H} \in \Lambda_h$

$$u_H = \sum_{i=1}^{N_H} c_i \, \eta_i + u^f$$

Summary: Multiscale Hybrid Method — two level

Basis functions

Find $\eta_i^h \in V_K^h$ *such that*

$$a_K(\eta_i^h, v_h) \quad = \quad -(\psi_i, v_h)_{\partial K}$$

for all $v_h \in V_K^h$

Local problem for f

Find $u_h^f \in V_K^h$ *such that*

$$a_K(u_h^f, v_h) \quad = \quad (f, v_h)_{\partial K}$$

for all $v_h \in V_K^h$

Global problem

Find $c_1, \ldots, c_{N_h} \in \mathbb{R}$

$$\sum_{i=1}^{N_h} c_i \, (\eta_i^h, \phi_j)_{\partial \mathcal{T}_H} \quad = \quad -(u^f, \phi_j)_{\partial \mathcal{T}_H}$$

for all $\phi_1, \ldots, \phi_{N_h} \in \Lambda_h$

$$u_{H,h} = \sum_{i=1}^{N_H} c_i \, \eta_i^h + u_h^f$$

The Maxwell case

The Maxwell case

Maxwell Equations

Find $\boldsymbol{e} = \boldsymbol{e}(\mathbf{x}, t)$ *and* $\boldsymbol{h} = \boldsymbol{h}(\mathbf{x}, t)$ *such that*

$$\left\{ \begin{array}{rcll} \varepsilon\,\partial_t \boldsymbol{e} - \nabla \times \boldsymbol{h} & = & \boldsymbol{f}, & \text{in } \Omega \times]0, T[\\ \mu\,\partial_t \boldsymbol{h} + \nabla \times \boldsymbol{e} & = & \mathbf{0}, & \text{in } \Omega \times]0, T[\\ \boldsymbol{e} \times \boldsymbol{n} & = & \mathbf{0}, & \text{in } \Gamma \times]0, T[\\ \boldsymbol{e} & = & \boldsymbol{e}_0, & \text{in } \Omega \times \{0\} \\ \boldsymbol{h} & = & \boldsymbol{h}_0, & \text{in } \Omega \times \{0\} \end{array} \right.$$

- μ is the symmetric tensor magnetic permeability
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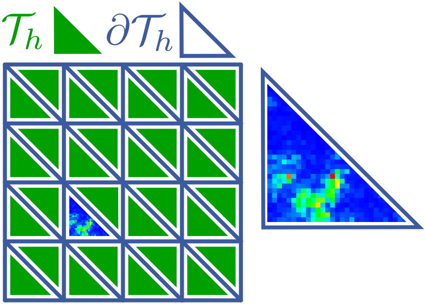
- Initial field $\boldsymbol{e}_0$ and $\boldsymbol{h}_0$ and source $\boldsymbol{f}$ satisfy

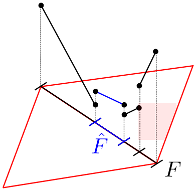
$$\nabla \cdot \boldsymbol{f} = \nabla \cdot (\varepsilon\, \boldsymbol{e}_0) = \nabla \cdot (\mu\, \boldsymbol{h}_0) = 0, \in \Omega$$

Settings

$$\mathbf{V} = \bigoplus_{K \in \mathcal{T}_H} \boldsymbol{H}(\mathbf{curl}; K)$$

$$\boldsymbol{\Lambda}_h \subset \boldsymbol{\Lambda} = \prod_{K \in \mathcal{T}_H} \boldsymbol{H}^{-\frac{1}{2}}(\partial K)$$





polynomial $\lambda_h \in \Lambda_h$

$$\boldsymbol{\Lambda} = \left\{ \boldsymbol{\mu} \in \prod_{K \in \mathcal{T}_H} \boldsymbol{H}^{-\frac{1}{2}}(\partial K) : \boldsymbol{\mu} = \boldsymbol{v} \times \boldsymbol{n}, \forall K \in \mathcal{T}_H, \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}; \Omega) \right\}$$

Variational Hybrid-Mixed Formulation

Hybrid Formulation

Find $(\boldsymbol{e}(t), \boldsymbol{h}(t), \boldsymbol{\lambda}(t)) \in \mathbf{V} \times \mathbf{V} \times \boldsymbol{\Lambda}$ *such that*

$$\begin{aligned} (\varepsilon\, \partial_t \boldsymbol{e}, \boldsymbol{v})_{\mathcal{T}_H} - (\boldsymbol{h}, \nabla \times \boldsymbol{v})_{\mathcal{T}_H} + (\boldsymbol{\lambda}, \boldsymbol{v})_{\partial \mathcal{T}_H} &= (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_H} \\ (\mu\, \partial_t \boldsymbol{h}, \boldsymbol{w})_{\mathcal{T}_H} + (\nabla \times \boldsymbol{e}, \boldsymbol{w})_{\mathcal{T}_H} &= 0 \\ (\boldsymbol{e}, \boldsymbol{\mu})_{\partial \mathcal{T}_H} &= 0 \end{aligned}$$

for all $(\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{\mu}) \in \mathbf{V} \times \mathbf{V} \times \boldsymbol{\Lambda}$

- continuity of $\boldsymbol{e}(t) \times \boldsymbol{n}$ is weakly imposed
- $\boldsymbol{e}(t)$ and $\boldsymbol{h}(t)$ solves the standard weak form
- $\boldsymbol{h}(t) \times \boldsymbol{n} = \boldsymbol{\lambda}(t)$
- $\boldsymbol{e}(t), \boldsymbol{h}(t) \in \boldsymbol{H}(\mathbf{curl}; \Omega)$

Localization process

Local problem

Find $(\boldsymbol{e}(t), \boldsymbol{h}(t)) \in \boldsymbol{H}(\mathbf{curl}; K) \times \boldsymbol{H}(\mathbf{curl}; K)$ *such that*

$$\begin{aligned} (\varepsilon\, \partial_t \boldsymbol{e}, \boldsymbol{v})_K - (\boldsymbol{h}, \nabla \times \boldsymbol{v})_K &= (\boldsymbol{f}, \boldsymbol{v})_K - (\boldsymbol{\lambda}, \boldsymbol{v})_{\partial K}, \\ (\mu\, \partial_t \boldsymbol{h}, \boldsymbol{w})_K + (\nabla \times \boldsymbol{e}, \boldsymbol{w})_K &= 0 \end{aligned}$$

for all $(\boldsymbol{v}, \boldsymbol{w}) \in \boldsymbol{H}(\mathbf{curl}; K) \times \boldsymbol{H}(\mathbf{curl}; K)$

$$\boldsymbol{e}(0) = \boldsymbol{e}_0 \text{ and } \boldsymbol{h}(0) = \boldsymbol{h}_0$$

Equivalent formulation

$$(\boldsymbol{e}, \boldsymbol{\mu})_{\partial \mathcal{T}_H} = 0, \ \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda}$$

Local problem

Find $(\boldsymbol{e}(t), \boldsymbol{h}(t)) \in \boldsymbol{H}(\mathbf{curl}; K) \times \boldsymbol{H}(\mathbf{curl}; K)$ *such that*

$$\begin{array}{rcll} \varepsilon\, \partial_t \boldsymbol{e} - \nabla \times \boldsymbol{h} & = & \boldsymbol{f}, & \text{in } K \\ \mu\, \partial_t \boldsymbol{h} + \nabla \times \boldsymbol{e} & = & \boldsymbol{0}, & \text{in } K \\ \boldsymbol{h} \times \boldsymbol{n} & = & \boldsymbol{\lambda}, & \text{in } \partial K \end{array}$$

$$\boldsymbol{e}(0) = \boldsymbol{e}_0 \text{ and } \boldsymbol{h}(0) = \boldsymbol{h}_0, \ \text{in } K$$

Fully discrete method = DG + Leap Frog

$$\boldsymbol{e}_{H,h}^{n+\frac{1}{2}} = \boldsymbol{e}_h^{\boldsymbol{\lambda}_H, n+\frac{1}{2}} + \boldsymbol{e}_h^{\boldsymbol{f}, n+\frac{1}{2}}$$

$$(\boldsymbol{e}_h^{\boldsymbol{\lambda}_H, n+\frac{1}{2}}, \boldsymbol{\mu}_H)_{\partial\mathcal{T}_H} = -(\boldsymbol{e}_h^{\boldsymbol{f}, n+\frac{1}{2}}, \boldsymbol{\mu}_H)_{\partial\mathcal{T}_H}, \ \forall \boldsymbol{\mu}_H \in \boldsymbol{\Lambda}_H$$

Local problem

Find $\boldsymbol{e}_h^{\boldsymbol{\lambda}_H, n+\frac{1}{2}} \in \mathbf{V}_K^h$ *such that*

$$\frac{1}{\Delta t}(\varepsilon\, \boldsymbol{e}_h^{\boldsymbol{\lambda}_H, n+\frac{1}{2}}, \boldsymbol{v}_h)_{\mathcal{T}_H^K} = -(\boldsymbol{\lambda}_H, \boldsymbol{v}_h)_{\partial K},$$

for all $\boldsymbol{v}_h \in \mathbf{V}_K^h$

$\mathbf{V}_K^h$ is the Discotinuous Galerkin discrete space

Fully discrete method = DG + Leap Frog

$$\boldsymbol{e}_{H,h}^{n+\frac{1}{2}} = \sum_{i=1}^{N_H} c_i^{n+\frac{1}{2}} \boldsymbol{\eta}_i^h + \boldsymbol{e}_h^{\boldsymbol{f},n+\frac{1}{2}}$$

$$\sum_{i=1}^{N_H} c_i^{n+\frac{1}{2}} (\boldsymbol{\eta}_i^h, \boldsymbol{\mu}_H)_{\partial\mathcal{T}_H} = -(\boldsymbol{e}_h^{\boldsymbol{f},n+\frac{1}{2}}, \boldsymbol{\mu}_H)_{\partial\mathcal{T}_H}, \ \forall \boldsymbol{\mu}_H \in \boldsymbol{\Lambda}_H$$

Local problem

Find $\boldsymbol{\eta}_i^h \in \mathbf{V}_K^h$ such that

$$\frac{1}{\Delta t}(\varepsilon\, \boldsymbol{\eta}_i^h, \boldsymbol{v}_h)_{\mathcal{T}_H^K} \quad = \quad -(\psi_i, \boldsymbol{v}_h)_{\partial K},$$

for all $\boldsymbol{v}_h \in \mathbf{V}_K^h$

Fully discrete method = DG + Leap Frog, $\boldsymbol{e}_{H,h}^{n+\frac{1}{2}} = \boldsymbol{e}_h^{\boldsymbol{\lambda}_H, n+\frac{1}{2}} + \boldsymbol{e}_h^{\boldsymbol{f}, n+\frac{1}{2}}$

$$\boldsymbol{e}_{H,h}^{n+\frac{1}{2}} = \sum_{i=1}^{N_H} c_i^{n+\frac{1}{2}} \boldsymbol{\eta}_i^h + \boldsymbol{e}_h^{\boldsymbol{f}, n+\frac{1}{2}}$$

$$\sum_{i=1}^{N_H} c_i^{n+\frac{1}{2}} (\boldsymbol{\eta}_i^h, \boldsymbol{\mu}_H)_{\partial\mathcal{T}_H} = -(\boldsymbol{e}_h^{\boldsymbol{f}, n+\frac{1}{2}}, \boldsymbol{\mu}_H)_{\partial\mathcal{T}_H}, \ \forall \boldsymbol{\mu}_H \in \boldsymbol{\Lambda}_H$$

Local problem

Find $\boldsymbol{e}_h^{\boldsymbol{f}, n+\frac{1}{2}} \in \mathbf{V}_K^h$ *such that*

$$\begin{aligned} \frac{1}{\Delta t}(\varepsilon\, \boldsymbol{e}_h^{\boldsymbol{f}, n+\frac{1}{2}}, \boldsymbol{v}_h)_{\mathcal{T}_H^K} &= \frac{1}{\Delta t}(\varepsilon\, \boldsymbol{e}_{H,h}^{n-\frac{1}{2}}, \boldsymbol{v}_h)_{\mathcal{T}_H^K} + (\boldsymbol{f}, \boldsymbol{v}_h)_{\mathcal{T}_H^K} \\ &+ (\boldsymbol{h}^n, \nabla \times \boldsymbol{v}_h)_{\mathcal{T}_H^K} - (\{\boldsymbol{h}^n\}, [\![\boldsymbol{v}_h]\!])_{\mathcal{E}_{0,h}^K} \end{aligned}$$

for all $\boldsymbol{v}_h, \mathbf{V}_K^h$

Fully discrete method = DG + Leap Frog, $\boldsymbol{e}_{H,h}^{n+\frac{1}{2}} = \boldsymbol{e}_h^{\boldsymbol{\lambda}_H, n+\frac{1}{2}} + \boldsymbol{e}_h^{\boldsymbol{f}, n+\frac{1}{2}}$

Local problem

Find $\boldsymbol{h}_h^n \in \mathbf{V}_K^h$ *such that*

$$\begin{aligned} \frac{1}{\Delta t}(\mu\, \boldsymbol{h}_h^n, \boldsymbol{w}_h)_{\mathcal{T}_H^K} &= \frac{1}{\Delta t}(\mu\, \boldsymbol{h}_h^{n-1}, \boldsymbol{w}_h)_{\mathcal{T}_H^K} \\ &- (\boldsymbol{e}_{H,h}^{n-\frac{1}{2}}, \nabla \times \boldsymbol{w}_h)_{\mathcal{T}_H^K} + (\{\boldsymbol{e}_{H,h}^{n-\frac{1}{2}}\}, [\![\boldsymbol{w}_h]\!])_{\mathcal{E}_h^K} \end{aligned}$$

for all $\boldsymbol{w}_h \in \mathbf{V}_K^h$

Numerical Experiments

Numerical Experiments

TM mode

Maxwell Equations

$$+$$

No dependence on z

$$\mathbf{h} = \begin{bmatrix} h_x \\ h_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ h_z \end{bmatrix} \quad \mathbf{e} = \begin{bmatrix} e_x \\ e_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ e_z \end{bmatrix}$$

$$e_x = e_y = h_z = 0$$

TM mode

Find (h_x, h_y, e_z) *such that*

$$\begin{cases} \mu\,\partial_t h_x + \partial_y e_z &= 0, & \text{in } \Omega \times (0,T) \\ \mu\,\partial_t h_y - \partial_x e_z &= 0, & \text{in } \Omega \times (0,T) \\ \epsilon\,\partial_t e_z - \partial_x h_y + \partial_y h_x &= j_z, & \text{in } \Omega \times (0,T) \\ e_z &= 0, & \text{in } \Gamma \times (0,T) \\ (h_x, h_y, e_z) &= (h_x^0, h_y^0, e_z^0), & \text{at } t = 0, \text{ in } \Omega \end{cases}$$

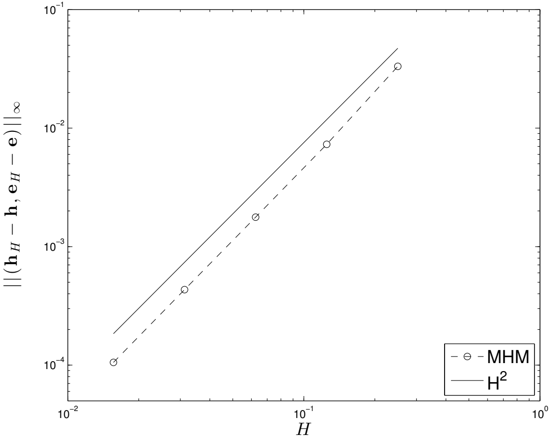
Exact Solution

$$h_x(x,y,t) = \frac{\sqrt{2}}{2}\cos(2\sqrt{2}\pi t)\sin(2\pi x)\cos(2\pi y)$$

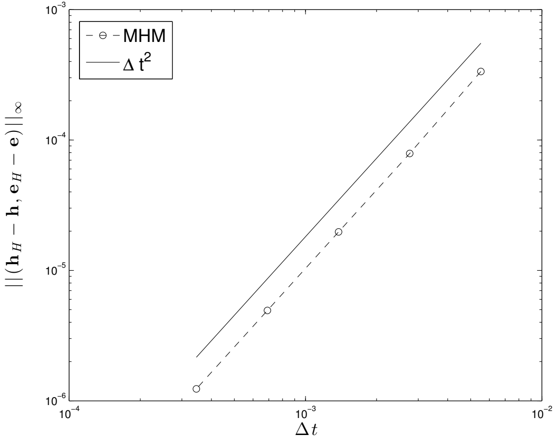
$$h_y(x,y,t) = -\frac{\sqrt{2}}{2}\cos(2\sqrt{2}\pi t)\cos(2\pi x)\sin(2\pi y)$$

$$e_z(x,y,t) = \sin(2\sqrt{2}\pi t)\sin(2\pi x)\cos(2\pi y)$$

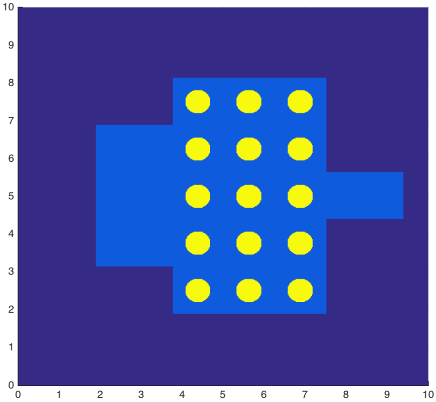
Convergence history for H ($l = 1$ and $k = 2$)



Convergence history for Δt

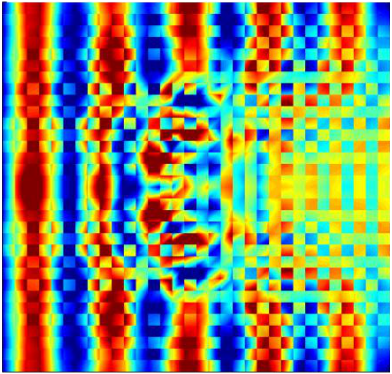


Waveguide device

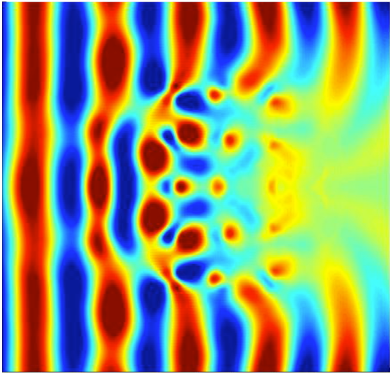


MHM results

DG $k = 2$,
1,024 elements,
9,216 d.o.f.



DG $k = 2$,
65,536 elements,
589,824 d.o.f.

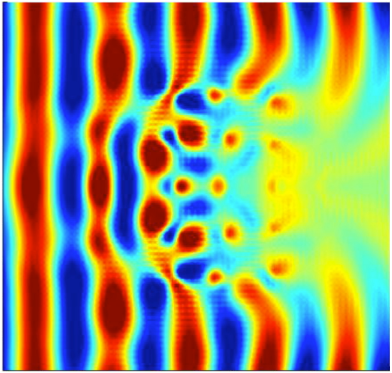


MHM results

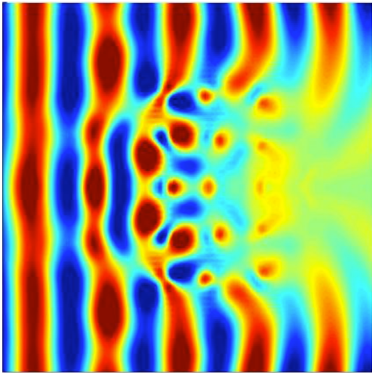
Video

MHM results

DG $k = 2$,
16,384 elements,
147,456 d.o.f.

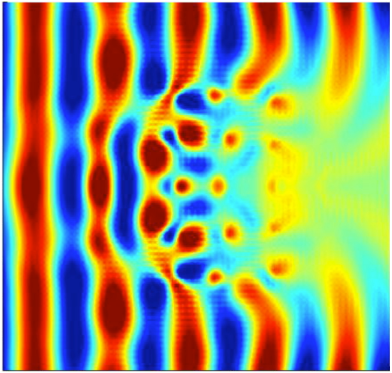


MHM $l = 1$ and $k = 2$,
64 elements,
4,608 + 147,456 d.o.f.



MHM results

DG $k = 2$,
16,384 elements,
147,456 d.o.f.



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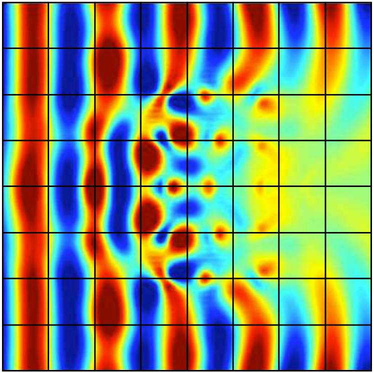

MHM results

DG $k = 2$,
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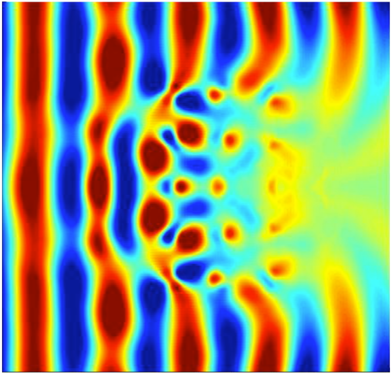
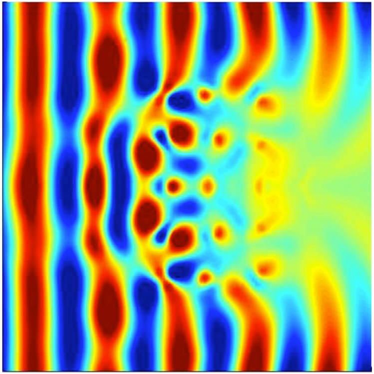
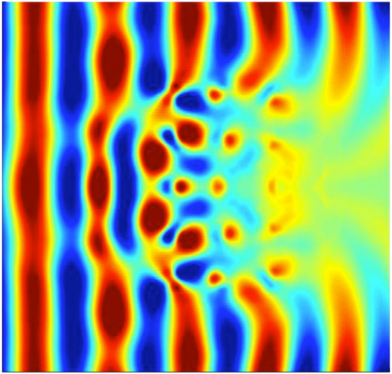


MHM $l = 1$ and $k = 2$,
256 elements,
8,704 + 147,456 d.o.f.

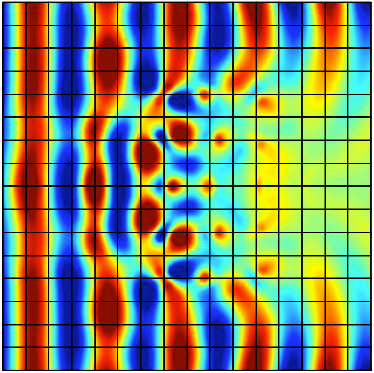


MHM results

DG $k = 2$,
65,536 elements,
589,824 d.o.f.



MHM $l = 1$ and $k = 2$,
256 elements,
8,704 + 147,456 d.o.f.



Conclusions

- A new multiscale finite element method for Maxwell equations
- Parallel numerical algorithm
- Implementation for a cluster in progress
- Error analysis in progress

Thanks slide

Thank you!